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On Two Parameter Identification Problems Arising from a Special Form of Computerized Tomography

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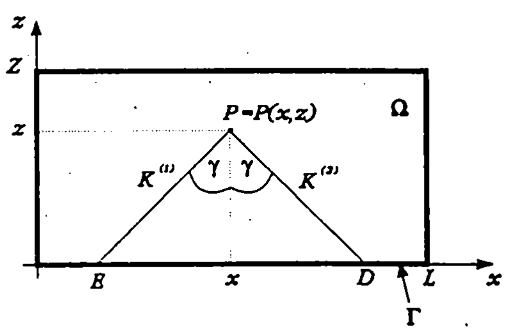
Abstract. The paper investigates a linear and a nonlinear operator problem for the identification of two parameter functions occurring in the Albedo operator in a special model of computerized tomography. It is shown that the linear inverse operator problem turns out to be well-posed, whereas the nonlinear problem under consideration becomes ill-posed.

Keywords: *Inverse problems, ill-posed problems, nonlinear operator equations*

AMS subject classification: Primary 65J10, secondary 65J15, 65J20

1. Introduction

The operator problem that will be considered in this paper describes mathematically a special problem in computerized tomography using backscattered photons. In contrast to the classical tomography (see [9, 11, 12]) using X-rays the idea here is to irradiate an object by light in the near infra-red region, using a laser for instance. Simultaneously, a detector on the surface of the considered object measures the intensity of the stream of reflected photons leaving the medium. For further details of the physical and medical background of this form of computerized tomography we refer to [3, 13]. In our investigations we will concentrate on the once-scattered photons model that was deduced and investigated in [5]. The geometric correlations in the model are shown in the figure below.



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Under some supplementary reducing assumptions of physical nature (let the irradiation be δ-like and monochromatic and let the scattering be isotropic) we can describe the modeling process as follows:

The photons are irradiated under a fixed angle γ with a fixed intensity I_E at an emiter point E on the surface Γ. Some of these photons can be absorbed along the straight lines $K^{(1)}$ and $K^{(2)}$. After an act of scattering in the point $P \in \Omega$ the stream of photons leaves the object and an intensity I_D of this stream at the point $D \in \Gamma$ can be measured.

The operator $\mathcal{A}$ in the following model that is called *Albedo operator* and that is dependent on the parameter functions σ_s (parameter of scattering) and σ_T (parameter of absorption) describes the interrelation between the conditions of irradiation (intensity of irradiation I_E, angle of irradiation γ) on the one hand and the measurable intensity I_D on the other hand. Now, we can formulate our mathematical problem:

$$\big(\mathcal{A}(\sigma_s, \sigma_T) I_E\big)(P) = I_D(P) \qquad (P \in \Omega) \tag{1.1}$$

where the operator

$$\mathcal{A}(\sigma_s, \sigma_T) : L_\infty \longrightarrow L_2(\Omega, z^2) \tag{1.2}$$

is given by

$$\begin{gathered}\big(\mathcal{A}(\sigma_s, \sigma_T) I_E\big)(P) = I_E(P) \frac{\sigma_s(P)}{|PD| 4\pi \sin(2\gamma)} \exp\left(-\int_{K(P)} \sigma_T(P')\, ds\right) \\ \sigma_s \in L_2(\Omega),\ \sigma_T \in W_2^1(\Omega)\end{gathered} \tag{1.3}$$

($|PD| = \operatorname{dist}(P, D)$ being the distance between P and D), and where

$$\begin{aligned} \Omega &= \left\{P(x,z) \in \mathbb{R}^2 :\ (x,z) \in [0,L] \times (0,Z]\right\} \\ \Gamma &= \left\{P(x,0) \in \mathbb{R}^2 :\ x \in [0,L]\right\} \\ \bar{\Omega} &= \Omega \cup \Gamma. \end{aligned} \tag{1.4}$$

Owing to the practical background of the model (1.1) - (1.3) we will restrict the domains of $\sigma_s \in L_2(\Omega)$, $\sigma_T \in W_2^1(\Omega)$ and $I_E \in L_\infty(\Omega)$ to the sets

$$\begin{aligned} D_0 &= \left\{ f \in W_2^1(\Omega) \,\middle|\, \begin{array}{l} f(P) \geq 0 \text{ a.e. in } \Omega,\ f(x,z) = \\ f(x + kL, z) \ \ (P(x,z) \in \Omega, k \in \pm \mathbb{N}) \end{array} \right\} \\ D_1 &= \left\{ f \in L_2(\Omega) \middle|\ f(P) > 0 \text{ a.e. in } \Omega \right\} \\ D_2 &= \left\{ f \in L_\infty(\Omega) \middle|\ f(P) \geq c > 0 \text{ a.e. in } \Omega \right\}, \end{aligned}$$

that means $\sigma_s \in D_1$, $\sigma_T \in D_0$ and $I_E \in D_2$.

Remark 1.1. (i) The requirements for the functions σ_s and I_E in D_1 and D_2, respectively, are determined by their physical meaning, whereas the assumption that σ_T is in $W_2^1(\Omega)$ is founded by the mathematical technique in proofs essentially based on the Sobolev embedding theorem (Lemma 1.1).

(ii) Although we consider the model (1.1) - (1.4) in the rectangular domain Ω the functions σ_T belonging to D_0 are defined as functions periodic referring to x. Such an assumption guarantees the existence of all function values $\sigma_T(Q)$ along each of the possible curves $K(P)$, $P \in \Omega$. Therefore the model (1.1) - (1.4) is well-determined for all points $P \in \Omega$.

(iii) We have chosen for the sake of simplicity a rectangular domain. Hence, the geometric description of the investigated problem (especially of the curves $K(P)$) is possible in a simple way.

(iv) If we move the emiter E along the surface Γ, then the intensity of irradiation I_E can be a function of the location in most general case. This possibility is described by the dependency $I_E(P)$. In many interesting cases $I_E(P)$ will be a constant.

In the form of representation of the operator $\mathcal{A}$ by $\mathcal{A}(\sigma_s, \sigma_T)$ we will suggest that σ_s and σ_T are interpreted as parameter functions. By $E(x_E, 0) \in \mathbb{R}^2$ and $D(x_D, 0) \in \mathbb{R}^2$ we describe in the model (1.1) - (1.4) two points on the boundary Γ, which are uniquely defined only by the point $P(x,z) \in \Omega$ and the a-priori chosen fixed angle $\gamma \in (0, \frac{\pi}{2})$ in the following manner:

$$x_E = x - b\,z \qquad \text{and} \qquad x_D = x + b\,z \qquad (b = \tan\gamma).$$

Therefore, it holds

$$|PD| = \left((x - x_D)^2 + z^2\right)^{1/2} = z\left(1 + \tan^2\gamma\right)^{1/2} = \frac{1}{\cos\gamma} z. \tag{1.5}$$

Using this we can transform the operator (1.3) into the form

$$\left(\mathcal{A}(\sigma_s, \sigma_T) I_E\right)(P) = I_E(P) \frac{\sigma_s(P)}{8\pi z \sin\gamma} \exp\left(- \int_{K(P)} \sigma_T(P')\, ds \right). \tag{1.3*}$$

Furthermore we define by

$$K(P) = K^{(1)}(P) \cup K^{(2)}(P) \tag{1.6}$$

the union of the curves

$$K^{(1)}(P) = \left\{ Q(x^{(1)}, z^{(1)}) \in \bar{\Omega} \,\middle|\, x^{(1)} = x + b(z^{(1)} - z),\ z^{(1)} \in [0, z] \right\}$$

$$K^{(2)}(P) = \left\{ Q(x^{(2)}, z^{(2)}) \in \bar{\Omega} \,\middle|\, x^{(2)} = x - b(z^{(2)} - z),\ z^{(2)} \in [0, z] \right\}$$

where $P = P(x, z)$ and $b = \tan\gamma$. We consider the separable Hilbert space $L_2(\Omega, z^2)$ with the scalar product

$$(f, g)_{L_2(\Omega, z^2)} = \int_\Omega z^2 f(P) g(P)\, d\Omega \qquad (f, g \in L_2(\Omega, z^2)).$$

The symbol $L_2(\Omega, z^2)$ denotes the space of quadratically integrable functions $f = f(P)$ using the integral weighted by z^2, that means

$$L_2(\Omega, z^2) = \left\{ f \in L_2(\Omega) \,\middle|\, \int_\Omega z^2 |f(P)|^2 d\Omega < +\infty,\ P = P(x,z) \right\}.$$

The problem (1.1) - (1.4) is well-defined under the above assumptions. To prove this, we have at first to show the existence of the curve integral $\int_{K(P)} \sigma_T(P')\,ds$ in (1.3).

Lemma 1.1. *A function $\sigma_T \in W_2^1(\Omega)$ is equivalent to a function $\tilde{\sigma}_T \in L_2(\Omega_H)$, where Ω_H is the intersection of $\Omega \in I\!R^2$ with a one-dimensional hyperplane $H \in I\!R^2$. It holds*

$$\|\tilde{\sigma}_T\|_{L_2(\Omega_H)} \leq C \|\sigma_T\|_{W_2^1(\Omega)}$$

where $C < \infty$ is a constant independent of σ_T.

Proof. Because of the special form of $\Omega \in I\!R^2$ in (1.4) our domain is star-shaped with regard to any point $P \in \Omega$. Then the statement is an immediate conclusion of a well-known embedding theorem (see [1: p. 144]) ∎

The embedding operator $\mathcal{E}: W_2^1(\Omega) \longrightarrow L_2(\Omega_H)$ is completely continuous. Due to Lemma 1.1 by setting $\Omega_H = K(P)$ the existence and finiteness of the curve integral $\int_{K(P)} \sigma_T(P')\,ds$ is shown for an arbitrary curve $K(P)$ corresponding (1.6).

Another problem consists in the choise of the image space $L_2(\Omega, z^2)$ for the Albedo operator $\mathcal{A}$. In the following investigations we denote by $R(\mathcal{F}) \subset H_2$ the range of an operator $\mathcal{F}$ mapping between two Hilbert spaces H_1 and H_2 and by $R(\mathcal{F}, B) \subset H_2$ the range of $\mathcal{F}$ under consideration of a restricted domain $B \subset H_1$.

Lemma 1.2. *For the range $R(\mathcal{A})$ of the Albedo operator $\mathcal{A}$ in problem* (1.1) – (1.4) *the inclusion $R(\mathcal{A}) \subset L_2(\Omega, z^2)$ holds.*

Proof. Because of the equivalence of the forms (1.3) and (1.3*) we investigate the integral

$$\int_\Omega z^2 \left| I_E(P) \frac{\sigma_s(P)}{8\pi z \sin\gamma} \exp\left(- \int_{K(P)} \sigma_T(P')\,ds \right) \right|^2 d\Omega$$
$$= \frac{1}{(8\pi \sin\gamma)^2} \int_\Omega \left| I_E(P) \sigma_s(P) \exp\left(- \int_{K(P)} \sigma_T(P')\,ds \right) \right|^2 d\Omega.$$

In view of the inclusion $\sigma_T \in D_0$, it is possible to estimate the term

$$\left| \exp\left(- \int_{K(P)} \sigma_T(P')\,ds \right) \right|^2$$

in the following way:

$$0 < \left| \exp\left(- \int_{K(P)} \sigma_T(P')\,ds \right) \right|^2 = \exp\left(-2 \int_{K(P)} \sigma_T(P')\,ds \right) \leq 1. \tag{1.7}$$

Using this estimate and the Hölder inequality we can continue our estimation as

$$\frac{1}{(8\pi \sin\gamma)^2}\int_\Omega \left| I_E(P)\sigma_s(P)\exp\left(-\int_{K(P)} \sigma_T(P')\,ds\right)\right|^2 d\Omega$$
$$\leq \frac{1}{(8\pi\sin\gamma)^2}\|I_E\|^2_{L_\infty(\Omega)}\|\sigma_s\|^2_{L_2(\Omega)}$$

where $\|I_E\|_\infty = \text{ess}\,\sup_{P\in\Omega}|I_E(P)|$. Therefore

$$\|I_D\|_{L_2(\Omega,z^2)} = \|\mathcal{A}(\sigma_s,\sigma_T)I_E\|_{L_2(\Omega,z^2)} \leq \frac{1}{8\pi\sin\gamma}\|I_E\|_{L_\infty(\Omega)}\|\sigma_s\|_{L_2(\Omega)}$$

and the statement is proven ∎

Our aim in this work is not immediately the detailed examination of the problem (1.1) - (1.4) but the exposition of some peculiarities in the reconstruction problems for the parameter functions σ_s and σ_T in the model (1.1) - (1.4) from well-known or measured values of I_E, γ and I_D. Consequently we deal with two inverse problems. For the general definition of an inverse problem see, for example, [6, 8, 9, 14].

Problem (P1). Given a function $I_D \in R(\mathcal{F}_1, D_1)$, find a function $\sigma_s \in D_1 \subset L_2(\Omega)$ such that the *linear* equation

$$(\mathcal{F}_1(\sigma_T, I_E,\gamma)\sigma_s)(P) = I_D(P) \quad (P\in\Omega), \qquad \mathcal{F}_1 : D_1 \to L_2(\Omega, z^2) \tag{1.8}$$

with

$$(\mathcal{F}_1(\sigma_T, I_E,\gamma)\sigma_s)(P) = g_1(P)\,\sigma_s(P)$$

and

$$g_1(P) = I_E(P)\frac{1}{8\pi z\sin\gamma}\exp\left(-\int_{K(P)}\sigma_T(P')\,ds\right)$$

is satisfied, where $\sigma_T \in D_0$, $I_E \in D_2$ and $\gamma \in (0,\frac{\pi}{2})$ are known.

Problem (P2). Given a function $I_D \in R(\mathcal{F}_2, D_0)$, find a function $\sigma_T \in D_0 \subset W_2^1(\Omega)$ such that the *nonlinear* equation

$$(\mathcal{F}_2(\sigma_s, I_E,\gamma)\sigma_T)(P) = I_D(P) \quad (P\in\Omega), \qquad \mathcal{F}_2 : D_0 \to L_2(\Omega, z^2) \tag{1.9}$$

with

$$(\mathcal{F}_2(\sigma_s, I_E,\gamma)\sigma_T)(P) = g_2(P)\,\exp\left(-\int_{K(P)}\sigma_T(P')\,ds\right)$$

and

$$g_2(P) = I_E(P)\frac{\sigma_s(P)}{8\pi z\sin\gamma}$$

is satisfied, where $\sigma_s \in D_1$, $I_E \in D_2$ and $\gamma \in (0,\frac{\pi}{2})$ are known.

2. The well-posedness of problem (P1)

The aim of this section is the investigation of essential properties of the operator problem (P1). In the following the principal objective will be the proof of the following assertion:

The problem (P1) with the linear operator $\mathcal{F}_1 : D_1 \subset L_2(\Omega) \to R(\mathcal{F}_1, D_1) \subset L_2(\Omega, z^2)$ is well-posed in the sense of Hadamard's definition.

To show this assertion we have at first to prove the uniqueness of the solution of problem (P1) for any given right-hand side function $I_D \in R(\mathcal{F}_1, D_1)$.

Lemma 2.1. *If $\sigma_T \in D_0 \subset W_2^1(\Omega)$ is a given function, then there exists a constant $M(\sigma_T) > 0$ such that*

$$\frac{1}{z} M(\sigma_T) \le g_1(P) \qquad \text{for all } P \in \Omega$$

where $\gamma \in (0, \frac{\pi}{2})$ and $I_E \in D_2$ are fixed.

Proof. Let $\hat{P} \in \Omega$ be an arbitrary, but fixed point. Then we know from Lemma 1.1 that $\sigma_T \in D_0 \subset W_2^1(\Omega)$ is equivalent to a non-negative function $\sigma_T \in L_2(K(\hat{P}))$. Using the Hölder inequality it further holds

$$\begin{aligned} \int_{K(\hat{P})} \sigma_T(P')\, ds &= \int_{K(\hat{P})} |\sigma_T(P')|\, ds \\ &\le \left(\int_{K(\hat{P})} |\sigma_T(P')|^2 ds \right)^{1/2} \left(\operatorname{meas} K(\hat{P})\right)^{1/2}. \end{aligned}$$

Due to Lemma 1.1 and the property $\operatorname{meas} K(\hat{P}) \le \operatorname{meas} \Omega =: M_1 < +\infty$ we can conclude that

$$\int_{K(\hat{P})} \sigma_T(P')\, ds \le M_1^{1/2} C \|\sigma_T\|_{W_2^1(\Omega)} < +\infty.$$

Therfore it holds

$$\exp\left(- \int_{K(\hat{P})} \sigma_T(P')\, ds \right) \ge \exp\left(M_2 \|\sigma_T\|_{W_2^1(\Omega)} \right) =: M(\sigma_T)$$

for any point $\hat{P} \in \Omega$ with $M_2 := M_1^{1/2} C$. Owing to the requirements in the lemma it is easy to see that

$$\frac{I_E(P)}{8\pi \sin\gamma} \ge \frac{c}{8\pi \sin\gamma} = M_3 > 0$$

and finally

$$g_1(P) \ge \exp\left(M_2 \|\sigma_T\|_{W_2^1(\Omega)} \right) M_3 z^{-1} = M(\sigma_T) z^{-1} > 0$$

holds ∎

Theorem 2.1. *For given functions $\sigma_T \in D_0$, $I_E \in D_2$ and angle $\gamma \in (0, \frac{\pi}{2})$ problem $(P1)$ is uniquely solvable for any data function $I_D \in R(\mathcal{F}_1, D_1)$.*

Proof. We consider two functions $I_D^{(1)} \in R(\mathcal{F}_1, D_1)$ and $I_D^{(2)} \in R(\mathcal{F}_1, D_1)$ with the property $I_D^{(1)}(P) = I_D^{(2)}(P)$ a.e. in Ω, where

$$\begin{aligned} I_D^{(1)}(P) &= g_1(P)\sigma_s^{(1)}(P) \\ I_D^{(2)}(P) &= g_1(P)\sigma_s^{(2)}(P) \end{aligned} \qquad (P \in \Omega).$$

Therefore we can conclude

$$0 = \left|I_D^{(1)}(P) - I_D^{(2)}(P)\right| = |g_1(P)| \left|\sigma_s^{(1)}(P) - \sigma_s^{(2)}(P)\right|$$

a.e. in Ω. Because of Lemma 2.1 it holds $g_1(P) > 0$ a.e. in Ω and $\sigma_s^{(1)}(P) = \sigma_s^{(2)}(P)$ a.e. in Ω ∎

After the proof for the injectivity of the operator $\mathcal{F}_1$ we now turn to the investigation of the continuity of the inverse operator $\mathcal{F}_1^{-1}$. That means, we have to make a statement on the kind of dependence of the solution $\sigma_s \in D_1$ from small uncertainties in $I_D \in R(\mathcal{F}_1, D_1)$ arising, for instance, from errors of measurements. This property will be described in the next theorem.

Theorem 2.2. *Under the assumptions of Theorem 2.1, there exists a constant $M < +\infty$, independent of I_D, such that*

$$\left\|\sigma_s^{(1)} - \sigma_s^{(2)}\right\|_{L_2(\Omega)} \le M \left\|I_D^{(1)} - I_D^{(2)}\right\|_{L_2(\Omega, z^2)}$$

holds, where

$$\begin{aligned} I_D^{(1)}(P) &= (\mathcal{F}_1\sigma_s^{(1)})(P) \\ I_D^{(2)}(P) &= (\mathcal{F}_1\sigma_s^{(2)})(P) \end{aligned} \qquad (P \in \Omega).$$

Proof. Due to the assumptions in Theorem 2.1 and the validity of Lemma 2.1 we can easily determine the inverse operator $\mathcal{F}_1^{-1}$. Applying this, the difference of solutions $\sigma_s^{(1)}(P) - \sigma_s^{(2)}(P)$ can be written in the form

$$\sigma_s^{(1)}(P) - \sigma_s^{(2)}(P) = \left(\mathcal{F}_1^{-1}(I_D^{(1)} - I_D^{(2)})\right)(P) = \frac{1}{g_1(P)}\left(I_D^{(1)}(P) - I_D^{(2)}(P)\right). \tag{2.1}$$

Using this we investigate the norm $\|\sigma_s^{(1)} - \sigma_s^{(2)}\|_{L_2(\Omega)}$:

$$\begin{aligned} \left\|\sigma_s^{(1)} - \sigma_s^{(2)}\right\|^2_{L_2(\Omega)} &= \int_\Omega \left|\sigma_s^{(1)}(P) - \sigma_s^{(2)}(P)\right|^2 d\Omega \\ &= \int_\Omega \left|\frac{1}{g_1(P)}\right|^2 \left|I_D^{(1)}(P) - I_D^{(2)}(P)\right|^2 d\Omega. \end{aligned}$$

Taking $I_E^{(1)}, I_E^{(2)} \in D_2$ and the boundness of the inverse operator $\mathcal{F}_1^{-1}$ (see Lemma 2.1) into account we obtain

$$\begin{aligned} \left\|\sigma_s^{(1)} - \sigma_s^{(2)}\right\|^2_{L_2(\Omega)} &\le \left(\frac{1}{M(\sigma_T)}\right)^2 \int_\Omega z^2 \left|I_D^{(1)}(P) - I_D^{(2)}(P)\right|^2 d\Omega \\ &= M^2 \left\|I_D^{(1)} - I_D^{(2)}\right\|^2_{L_2(\Omega, z^2)} \end{aligned}$$

and the statement is proven ∎

If we summerize the results of the Theorems 2.1 and 2.2, then we can conclude the following:

The problem (P1) for the identification of the function $\sigma_s \in D_1 \subset L_2(\Omega)$ from the given function $I_D \in R(\mathcal{F}_1, D_1) \subset L_2(\Omega, z^2)$, where $\sigma_T \in D_0$, $I_E \in D_2$ and $\gamma \in (0, \frac{\pi}{2})$ are known, is a well-posed problem. For any function $I_D \in R(\mathcal{F}_1, D_1)$ there exists an unique solution $\sigma_s \in D_1$ and this solution continuously depends on small perturbations in the data function $I_D \in R(\mathcal{F}_1)$.

3. The ill-posedness of problem (P2)

From a general point of view the inverse problem of identification of $\sigma_T \in D_0$ seems to involve large difficulties because of the necessary "differentiation" of the integral operator. The following investigations show the correctness of this hypothesis. Hence we will investigate in this section properties of the operator $\mathcal{F}_2$ in the problem (1.9) which can create instabilities in the solution of the inverse problem for identification of the function $\sigma_T \in D_0$. Before we can devote to the main property we have to formulate some statements being essential in the following.

Lemma 3.1. *For all functions $\sigma_T^{(1)} \in D_0 \subset W_2^1(\Omega)$ and $\sigma_T^{(2)} \in D_0 \subset W_2^1(\Omega)$ the estimate*

$$\begin{aligned} &\left|\exp\left(-\int_{K(P)} \sigma_T^{(1)}(P')\,ds\right) - \exp\left(-\int_{K(P)} \sigma_T^{(2)}(P')\,ds\right)\right| \\ &\qquad \le \left|\int_{K(P)} \left(\sigma_T^{(1)}(P') - \sigma_T^{(2)}(P')\right) ds\right| \qquad (P \in \Omega) \end{aligned} \tag{3.1}$$

holds.

Proof. Setting

$$m(P) = \min\left\{\int_{K(P)} \sigma_T^{(1)}(P')\,ds\ ,\ \int_{K(P)} \sigma_T^{(2)}(P')\,ds\right\}$$

$$M(P) = \max\left\{\int_{K(P)} \sigma_T^{(1)}(P')\,ds\ ,\ \int_{K(P)} \sigma_T^{(2)}(P')\,ds\right\}$$

and using the Taylor expansion of the function $\exp(-x)$ for $x \in [m(P), M(P)]$ it is clear that a value $\xi(P) \in [m(P), M(P)]$ with

$$\begin{aligned} &\left|\exp\left(-\int_{K(P)} \sigma_T^{(1)}(P')\,ds\right) - \exp\left(-\int_{K(P)} \sigma_T^{(2)}(P')\,ds\right)\right| \\ &\qquad = |\exp(-\xi(P))| \left|\int_{K(P)} \left(\sigma_T^{(1)}(P') - \sigma_T^{(2)}(P')\right) ds\right| \end{aligned} \tag{3.2}$$

exists. From the non-negativity of the functions $\sigma_T^{(1)}$ and $\sigma_T^{(2)}$ it follows that $0 \le m(P) \le M(P)$ and therefore $0 \le \xi(P)$ for all $P \in \Omega$. So it holds $|\exp(-\xi(P))| \le 1$ for all $\xi(P) \in [m(P), M(P)]$. Using this in (3.2) we obtain inequality (3.1) ∎

Theorem 3.1. *The operator $\mathcal{F}_2 : D_0 \subset W_2^1(\Omega) \longrightarrow L_2(\Omega, z^2)$ defined in* (1.8) *is Fréchet differentiable in any point $\sigma_T^{(0)} \in \operatorname{int} D_0$ with the Fréchet derivative*

$$(\mathcal{F}'_2(\sigma_T^{(0)})h)(P) = -g_2(P)\exp\left(-\int_{K(P)} \sigma_T^{(0)}(P')\,ds\right)\int_{K(P)} h(P')\,ds$$

$$\mathcal{F}'_2(\sigma_T^{(0)}) \in \mathcal{L}(W_2^1(\Omega), L_2(\Omega, z^2)).$$

This derivative $\mathcal{F}'_2(\sigma_T^{(0)})$ is a compact linear operator.

Proof. Let $\sigma_T^{(0)} \in \operatorname{int} D_0$ be an arbitrary function and $B_\delta(\sigma_T^{(0)}) = \{\sigma_T \in D_0 : \|\sigma_T - \sigma_T^{(0)}\|_{W_2^1(\Omega)} \leq \delta\}$ a neighbourhood of $\sigma_T^{(0)}$. For any function $\sigma_T \in B_\delta(\sigma_T^{(0)})$ we consider the difference

$$\begin{aligned}(\mathcal{F}_2\sigma_T)(P) &- (\mathcal{F}_2\sigma_T^{(0)})(P) \\ &= g_2(P)\left\{\exp\left(-\int_{K(P)} \sigma_T(P')\,ds\right) - \exp\left(-\int_{K(P)} \sigma_T^{(0)}(P')\,ds\right)\right\}.\end{aligned}$$

Introducing

$$m(P) = \min\left\{\int_{K(P)} \sigma_T(P')\,ds,\ \int_{K(P)} \sigma_T^{(0)}(P')\,ds\right\}$$

$$M(P) = \max\left\{\int_{K(P)} \sigma_T(P')\,ds,\ \int_{K(P)} \sigma_T^{(0)}(P')\,ds\right\}$$

it is clear that there exists a value $\xi(P) \in [m(P), M(P)]$ such that for the Taylor expansion of $\exp(-x)$ in $[m(P), M(P)]$ there holds

$$\begin{aligned}\exp&\left(-\int_{K(P)} \sigma_T(P')\,ds\right) - \exp\left(-\int_{K(P)} \sigma_T^{(0)}(P')\,ds\right) \\ &= -\exp\left(-\int_{K(P)} \sigma_T^{(0)}(P')\,ds\right)\int_{K(P)} \left(\sigma_T(P') - \sigma_T^{(0)}(P')\right)ds \\ &\qquad + \frac{1}{2}\exp(-\xi(P))\left(\int_{K(P)} \left(\sigma_T(P') - \sigma_T^{(0)}(P')\right)ds\right)^2\end{aligned}$$

and

$$\begin{aligned}(\mathcal{F}_2\sigma_T)(P) &- (\mathcal{F}_2\sigma_T^{(0)})(P) \\ &= -g_2(P)\exp\left(-\int_{K(P)} \sigma_T^{(0)}(P')\,ds\right)\int_{K(P)} \left(\sigma_T(P') - \sigma_T^{(0)}(P')\right)ds \\ &\qquad + \frac{1}{2}g_2(P)\exp(-\xi(P))\left(\int_{K(P)} \left(\sigma_T(P') - \sigma_T^{(0)}(P')\right)ds\right)^2\end{aligned}$$

for all $P \in \Omega$. Because of the non-negativity of the functions σ_T and $\sigma_T^{(0)}$ in D_0 we conclude that $|\exp(-\xi(P))| \le 1$ a.e. in Ω. On the other hand the properties of σ_s and I_E belonging to D_1 and D_2, respectively, guarantee the boundness of $g_2(P)$ a.e. in Ω, that means

$$|g_2(P)| \le K_1 \qquad \text{where } K_1 = K_1\big(\|I_E\|_{L_\infty(\Omega)},\ \|\sigma_s\|_{L_2(\Omega)},\ \operatorname{meas}\Omega,\ \gamma\big).$$

Using the Hölder inequality and Lemma 1.1 the integral $\int_{K(P)}(\sigma_T(P') - \sigma_T^{(0)}(P'))ds$ can be estimated by

$$\begin{aligned}\int_{K(P)} \left(\sigma_T(P') - \sigma_T^{(0)}(P')\right) ds &\le \int_{K(P)} \left|\sigma_T(P') - \sigma_T^{(0)}(P')\right| ds \\ &\le K_2^{1/2} \left\|\sigma_T - \sigma_T^{(0)}\right\|_{L_2(K(P))} \\ &\le K_2^{1/2} C \left\|\sigma_T - \sigma_T^{(0)}\right\|_{W_2^1(\Omega)},\end{aligned}$$

where $K_2 = \operatorname{meas}\Omega$, and therefore we obtain the result

$$(\mathcal{F}_2\sigma_T)(P) - (\mathcal{F}_2\sigma_T^{(0)})(P) = \mathcal{F}_2'(\sigma_T^{(0)})\left(\sigma_T - \sigma_T^{(0)}\right)(P) + o\left(\left\|\sigma_T - \sigma_T^{(0)}\right\|_{W_2^1(\Omega)}\right)$$

with

$$\begin{aligned}&\mathcal{F}_2'(\sigma_T^{(0)})\left(\sigma_T - \sigma_T^{(0)}\right)(P) \\ &\quad = -g_2(P)\exp\left(-\int_{K(P)} \sigma_T^{(0)}(P')\,ds\right) \int_{K(P)} \left(\sigma_T(P') - \sigma_T^{(0)}(P')\right) ds\end{aligned}$$

and $\mathcal{F}_2'(\sigma_T^{(0)}) \in \mathcal{L}\big(W_2^1(\Omega), L_2(\Omega, z^2)\big)$ for all $\sigma_T^{(0)} \in \operatorname{int} D_0$. The idea of the proof of compactness of the derivative $\mathcal{F}_2'(\sigma_T^{(0)})$ is the same as in the proof of Theorem 3.2 ∎

By some additional investigations we also can show that for any function $\sigma_T^{(0)} \in \operatorname{int} D_0$ and for an arbitrary value $\rho > 0$ there exists a neighbourhood $B_\rho(\sigma_T^{(0)}) = \{\sigma_T \in D_0 : \|\sigma_T - \sigma_T^{(0)}\|_{W_2^1(\Omega)} \le \rho\}$ and a constant $q_\rho > 0$ such that

$$\begin{aligned}&\left\|\mathcal{F}_2(\sigma_T) - \mathcal{F}_2(\sigma_T^{(0)}) - \mathcal{F}_2'(\sigma_T^{(0)})(\sigma_T - \sigma_T^{(0)})\right\|_{L_2(\Omega,z^2)} \\ &\qquad \le q_\rho \left\|\mathcal{F}_2'(\sigma_T^{(0)})(\sigma_T - \sigma_T^{(0)})\right\|_{L_2(\Omega,z^2)}.\end{aligned} \tag{3.3}$$

Theorem 3.2. *The nonlinear operator $\mathcal{F}_2$: $D_0 \subset W_2^1(\Omega) \to R(\mathcal{F}_2, D_0) \subset L_2(\Omega, z^2)$ in equation (1.9) is compact.*

Proof. To show the property of compactness we choose an arbitrary bounded set B of functions $\sigma_T \in D_0 \subset W_2^1(\Omega)$:

$$B = \left\{\sigma_T \in D_0 \subset W_2^1(\Omega) \,\middle|\, \|\sigma_T\|_{W_2^1(\Omega)} \le M < +\infty\right\}.$$

It is clear that B as a subset of the Hilbert space $W_2^1(\Omega)$ is weakly compact in $W_2^1(\Omega)$. Therefore we can select an infinite subsequence $\{\sigma_T^{(n_k)}\}_{k\in\mathbb{N}}$ from any infinite subset $\{\sigma_T^{(n)}\}_{n\in\mathbb{N}} \subset B$ tending to a function $\sigma_T^{(0)} \in W_2^1(\Omega)$ for $k \to \infty$ in a weak sense.

(1) From the weak convergence of $\{\sigma_T^{(n_k)}\}_{k\in\mathbb{N}} \subset B$ it follows that

$$\|\sigma_T^{(0)}\|_{W_2^1(\Omega)} \le \limsup_{k\to\infty} \|\sigma_T^{(n_k)}\|_{W_2^1(\Omega)} \le M < +\infty$$

holds (see [7: p. 332]).

(2) Owing to the compactness of the embedding operator $\mathcal{E}: W_2^1(\Omega) \to L_2(\Omega)$ and the property that the image of a weak convergent sequence applying a compact operator is a strong convergent sequence in the operator range (see [2: p. 236]) we obtain that $\{\sigma_T^{(n_k)}\}_{k\in\mathbb{N}} \subset W_2^1(\Omega)$ is a strong convergent subsequence in $L_2(\Omega)$. Following the theorem in [10: p.185], there exists a subsequence of $\{\sigma_T^{(n_k)}\}_{k\in\mathbb{N}}$ almost everywhere converging to $\sigma_T^{(0)}$. From the non-negativity of the functions $\sigma_T^{(n_k)}$ we can conclude that $\sigma_T^{(0)} \ge 0$ almost everywhere holds.

The statements (1) and (2) prove that $\sigma_T^{(0)} \in B$. If we describe by $I_D^{(n_k)}$ and $I_D^{(0)}$ the image of $\sigma_T^{(n_k)}$ and $\sigma_T^{(0)}$, respectively, then by application of Lemma 3.1 for any real $\varepsilon > 0$ there exists an index $k_0(P)$ such that

$$\begin{aligned}
&\left|I_D^{(n_k)}(P) - I_D^{(0)}(P)\right| \\
&\quad = |g_2(P)| \left|\exp\left(-\int_{K(P)} \sigma_T^{(n_k)}(P')\,ds\right) - \exp\left(-\int_{K(P)} \sigma_T^{(0)}(P')\,ds\right)\right| \\
&\quad \le |g_2(P)| \left|\int_{K(P)} \left(\sigma_T^{(n_k)}(P') - \sigma_T^{(0)}(P')\right) ds\right| \\
&\quad \le |g_2(P)|\varepsilon \qquad \text{for all } k \ge k_0(P).
\end{aligned}$$

Therefore the subsequence $\{I_D^{(n_k)}\}_{k\in\mathbb{N}}$ is convergent to $I_D^{(0)}$ almost everywhere in Ω. The index $k_0(P)$ of course depends on the selected point P. Under consideration of the notation $\hat{I}_D^{(n_k)}$ for the function $\hat{I}_D^{(n_k)}(P) = z\, I_D^{(n_k)}(P)$ we investigate in the following the norm

$$\begin{aligned}
\left\|I_D^{(n_k)} - I_D^{(0)}\right\|^2_{L_2(\Omega,z^2)} &= \left\|\hat{I}_D^{(n_k)} - \hat{I}_D^{(0)}\right\|^2_{L_2(\Omega)} \\
&= \int_\Omega \left|\hat{I}_D^{(n_k)}(P) - \hat{I}_D^{(0)}(P)\right|^2 d\Omega.
\end{aligned} \tag{3.4}$$

Obviously we can give the estimation

$$\left|\hat{I}_D^{(n_k)}(P) - \hat{I}_D^{(0)}(P)\right|^2 \le \left(\left|\hat{I}_D^{(n_k)}(P)\right| + \left|\hat{I}_D^{(0)}(P)\right|\right)^2$$

almost everywhere in Ω. From the validity of (1.7) for $\sigma_T^{(n_k)} \in B$ as well as for $\sigma_T^{(0)} \in B$ we conclude

$$|\hat{I}_D^{(n_k)}(P)| \le |g_2(P)| \quad \text{and} \quad |\hat{I}_D^{(0)}(P)| \le |g_2(P)| \qquad \text{a.e. in } \Omega$$

and therefore it follows

$$\left|\hat{I}_D^{(n_k)}(P) - \hat{I}_D^{(0)}(P)\right|^2 \le 4|g_2(P)|^2 =: F(P) \qquad \text{for all } k \in \mathbb{N}. \tag{3.5}$$

If we take into account the integrability of the functions I_E and σ_s in Problem (P2), then it is easy to derive that the majorant $F(P)$ is a function summable in Ω. Owing to (3.4), the integrability of $F(P)$ and the pointwise convergence $\hat{I}_D^{(n_k)}(P) \to \hat{I}_D^{(0)}(P)$ for $k \to \infty$ the assumptions of the Lebesgue theorem (see [10: p.166]) are fulfilled. The use of this theorem proves that, for any $\varepsilon > 0$, there exists an index $k_0 > 0$ such that

$$\left\|I_D^{(n_k)}(P) - I_D^{(0)}(P)\right\|^2_{L_2(\Omega,z^2)} = \left\|\hat{I}_D^{(n_k)}(P) - \hat{I}_D^{(0)}(P)\right\|^2_{L_2(\Omega)} \le \varepsilon$$

for all $k \ge k_0$. Therefore we have defined a convergent subsequence $\{I_D^{(n_k)}\}_{k\in\mathbb{N}}$ in an arbitrary subset $\{I_D^{(n)}\}_{n\in\mathbb{N}}$ in the range $R(\mathcal{F}_2, B) \subset L_2(\Omega, z^2)$ of a bounded subset $B \subset D_0$, that means, $R(\mathcal{F}_2, B)$ is compact in $L_2(\Omega, z^2)$ and the theorem is proven ∎

Theorem 3.3. *The operator $\mathcal{F}_2 : D_0 \subset W_2^1(\Omega) \longrightarrow L_2(\Omega, z^2)$ is weakly closed.*

Proof. Let $\{\sigma_T^{(n)}\}_{n\in\mathbb{N}} \subset D_0$ be a sequence weakly convergent to a function $\sigma_T^{(0)}$ and let $\mathcal{F}_2(\sigma_T^{(n)})$ tend weakly to a function y_0, that means $\sigma_T^{(n)} \rightharpoonup \sigma_T^{(0)}$ and $\mathcal{F}_2(\sigma_T^{(n)}) \rightharpoonup y_0$ for $n \to \infty$. If we take into account statement (2) in the proof of Theorem 3.2, then we can conclude the inclusion $\sigma_T^{(0)} \in D_0$. Moreover we can estimate the norm $\|\mathcal{F}_2(\sigma_T^{(n)}) - \mathcal{F}_2(\sigma_T^{(0)})\|_{L_2(\Omega,z^2)}$ as follows:

$$\begin{aligned} &\left\|\mathcal{F}_2(\sigma_T^{(n)}) - \mathcal{F}_2(\sigma_T^{(0)})\right\|_{L_2(\Omega,z^2)} \\ &\quad\le \left\|\mathcal{F}_2(\sigma_T^{(n)}) - \mathcal{F}_2(\sigma_T^{(0)}) - \mathcal{F}'_2(\sigma_T^{(0)})(\sigma_T^{(n)} - \sigma_T^{(0)})\right\|_{L_2(\Omega,z^2)} \\ &\quad\quad + \left\|\mathcal{F}'_2(\sigma_T^{(0)})(\sigma_T^{(n)} - \sigma_T^{(0)})\right\|_{L_2(\Omega,z^2)}. \end{aligned} \tag{3.6}$$

Because of the weak convergence $\sigma_T^{(n)} \rightharpoonup \sigma_T^{(0)}$ we find a constant M such that

$$\|\sigma_T^{(n)}\|_{W_2^1(\Omega)} \le M \quad \text{for all } n \qquad \text{and} \qquad \|\sigma_T^{(0)}\|_{W_2^1(\Omega)} \le M.$$

Applying inequality (3.3) for the ball B_{2M} the estimation (3.6) can be continued by

$$\left\|\mathcal{F}_2(\sigma_T^{(n)}) - \mathcal{F}_2(\sigma_T^{(0)})\right\|_{L_2(\Omega,z^2)} \le (1 + q_{2M}) \left\|\mathcal{F}'_2(\sigma_T^{(0)})(\sigma_T^{(n)} - \sigma_T^{(0)})\right\|_{L_2(\Omega,z^2)}.$$

From the compactness of the linear operator $\mathcal{F}'_2(\sigma_T^{(0)})$ and the weak convergence $\sigma_T^{(n)} \rightharpoonup \sigma_T^{(0)}$ it follows that $\mathcal{F}_2(\sigma_T^{(n)}) \longrightarrow \mathcal{F}_2(\sigma_T^{(0)})$ in the strong sense and therefore $y_0 = \mathcal{F}_2(\sigma_T^{(0)})$ ∎

Due to Theorems 3.1 - 3.3 the assumptions of Proposition A.3 in the Appendix of paper [4] are fulfilled. That proposition characterizes the compactness, the continuity and the weak closedness of a nonlinear operator as sufficient conditions for ill-posedness of this nonlinear problem. It provides the following conclusion:

The problem of identifying the function σ_T from the given function I_D using the operator problem (P2) is an ill-posed problem in the pair of sets $(D_0, R(\mathcal{F}_2, D_0))$ in the sense that the stability property in the definition of a well-posed problem is injured.

Consequently small perturbations in the right-hand side of equation (1.9) can cause very large perturbations in the corresponding solution and can make the numerical solution process unstable.

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