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A Fully Discrete Approximation Method for the Exterior Neumann Problem of the Helmholtz Equation

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Dedicated to Professor G. C. Hsiao on the occasion of his 60-th birthday

Abstract. Considering an exterior domain with smooth closed boundary curve we introduce a fully discrete scheme for the solution of the acoustic boundary value problem of the Neumann type. We use a boundary integral formulation of the problem which leads to a hypersingular boundary integral equation. Our discretization scheme for the latter equation can be considered as a discrete version of the trigonometric collocation method and has arbitrarily high convergence rate, even exponential if the solution and the curve are analytic.

Keywords: *Helmholtz equation, exterior problems, hypersingular operators, discrete approximation methods*

AMS subject classification: 65R20, 35J25, 45L10

1. Introduction

In this paper we consider the numerical solution of the acoustic exterior boundary value problem of the Neumann type. We represent the solution by means of the combined single-double layer representation, introduced by Brakhage and Werner [3], Leis [14], and Panich [19] for the Dirichlet problem and by Leis [15] and Panich [19] for the Neumann problem. This approach leads to the solution of a hypersingular boundary integral equation. Considering smooth closed curves we propose a numerical solution method which has arbitrarily high convergence rate, even exponential if the solution and the curve are analytic. For other equations such results have been obtained by several authors, see, e.g., [1, 2, 9, 10, 17, 18, 21, 22]. In particular, in [13] a fully discretized trigonometric Galerkin method based on a modification of the fast Fourier transform for approximately solving boundary integral equations has been studied. (See also the books [5], [20] and the pioneering paper [7] for applications of trigonometric Fourier series to the numerical solution of integral equations.) Our fully discrete method can be considered as a further discretization of the trigonometric collocation method. The excellent convergence of the scheme follows from the good consistency property obtained when discretizing the boundary integral operator.

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The boundary integral approach applied here has been used for the numerical approximation of the solution by Kußmaul [11], but the numerical scheme in [11] differs from that one proposed here, and moreover no error analysis was given in the work of Kußmaul. The corresponding Dirichlet problem has been studied by Brakhage and Werner [3], Chapko and Kress [4], Greenspan and Werner [6], Kress and Sloan [10]. The solution of the potential equation with the Neumann condition was investigated by Kieser, Kleemann and Rathsfeld [8].

2. Exterior Neumann boundary value problem

Let Γ be a smooth closed Jordan curve in the two-dimensional space $\mathbb{R}^2$, and let Ω_e be the corresponding exterior domain with boundary Γ. We consider the solution of the following boundary value problem:

Find a function Φ in Ω_e such that ($0 \neq \kappa \in \mathbb{C}$ with $\operatorname{Im}\kappa \geq 0$; $r = |x|$)

$$\Delta\Phi + \kappa^2\Phi = 0 \qquad \text{in } \Omega_e \tag{2.1$_a$}$$

$$\partial_n \Phi\big|_\Gamma = g_\Gamma \qquad \text{on } \Gamma \tag{2.1$_b$}$$

$$\Phi(x) = O\left(\frac{1}{\sqrt{r}}\right) \qquad \text{for } r \to \infty \tag{2.1$_c$}$$

$$\left(\frac{\partial}{\partial r} - i\kappa\right)\Phi(x) = o\left(\frac{1}{\sqrt{r}}\right) \qquad \text{for } r \to \infty. \tag{2.1$_d$}$$

Above n denotes the unit normal to the boundary Γ, directed into the exterior domain Ω_e. The problem (2.1), when considered in proper spaces of functions, is uniquely solvable. In particular, the uniqueness is implied by the Sommerfeld radiation condition $(2.1)_d$. By Leis [15] and Panich [19] it is shown that the solution Φ of problem (2.1) can be found using the combined single-double layer representation

$$\Phi(x) = \int_\Gamma u_\Gamma(y)(\partial_{n_y} - i\eta)g(x,y)\,ds_y \qquad (x \in \Omega_e)$$

where $\eta = 1$, if $\operatorname{Re}\kappa > 0$ and $\eta = -1$, if $\operatorname{Re}\kappa < 0$. Here

$$g(x,y) = \frac{i}{4}H_0^{(1)}(\kappa|x-y|)$$

is the fundamental solution to the Helmholtz equation, and $H_0^{(1)}$ is the Hankel function of first kind and of order zero.

By the well-known continuity and jump relations of the classical potentials, one obtains the boundary integral equation

$$\left(H_\Gamma - \frac{i\eta}{2}I_\Gamma + i\eta D'_\Gamma\right)u_\Gamma = -g_\Gamma \tag{2.2}$$

where H_Γ is the hypersingular acoustic integral operator and D'_Γ is the adjoint to the double layer acoustic integral operator,

$$(H_\Gamma u_\Gamma)(x) = -\partial_{n_x} \int_\Gamma \partial_{n_y} g(x,y) u_\Gamma(y)\, ds_y$$

$$(D'_\Gamma u_\Gamma)(x) = \int_\Gamma \partial_{n_x} g(x,y) u_\Gamma(y)\, ds_y.$$

Considering equation (2.2) in the Sobolev spaces $H^\lambda(\Gamma)$ $(\lambda \in \mathbb{R})$ of functions defined on Γ, equation (2.2) is uniquely solvable; more precisely,

$$L_\Gamma := H_\Gamma - \frac{i\eta}{2} I_\Gamma + i\eta D'_\Gamma \tag{2.3}$$

defines an isomorphism $L_\Gamma : H^\lambda(\Gamma) \to H^{\lambda-1}(\Gamma)$ $(\lambda \in \mathbb{R})$. With definition (2.3), equation (2.2) reads

$$L_\Gamma u_\Gamma = -g_\Gamma. \tag{2.4}$$

For the following we write the integral operator H_Γ in a more convenient form by using the kernel representation

$$(H_\Gamma u_\Gamma)(x) = -\int_\Gamma \partial_{n_x}\partial_{n_y} g(x,y) u_\Gamma(y)\, ds_y$$

where, because of the strong singularity of the kernel at the diagonal, the integration is to be understood in the sense of Hadamard. Abbreviating $z = \kappa|x-y|$ and applying the differentiation formulas for the Hankel functions

$$\frac{d}{dz} H_0^{(1)}(z) = -H_1^{(1)}(z) \qquad \text{and} \qquad \frac{d}{dz}(zH_1^{(1)}(z)) = zH_0^{(1)}(z) \tag{2.5}$$

we obtain

$$\begin{aligned} -\partial_{n_x}\partial_{n_y} g(x,y) = {} & \frac{i}{4} z H_1^{(1)}(z) \partial_{n_x} \left(\frac{n_y \cdot (y-x)}{|y-x|^2} \right) \\ & - \frac{i}{4} z^2 H_0^{(1)}(z) \frac{n_y \cdot (y-x)}{|y-x|^2} \cdot \frac{n_x \cdot (y-x)}{|y-x|^2}. \end{aligned} \tag{2.6}$$

Here the latter term has logarithmic singularity, and the term $zH_1^{(1)}(z)$ is bounded. Thus, the leading singularity is determined by the factor

$$\partial_{n_x} \left(\frac{n_y \cdot (y-x)}{|y-x|^2} \right) = \partial_{n_x}\partial_{n_y} \log|x-y| \tag{2.7}$$

which behaves like $O(|x-y|^{-2})$. A more detailed analysis of the kernel is shown in the next section and the Appendix.

3. Numerical method

In order to define our numerical method we introduce a 1-periodic parametrization of the curve Γ: $x = x(t) = (x_1(t), x_2(t))$ such that $|x'(t)| > 0$ $(t \in \mathbb{R})$. Thereby we fix the orientation requiring that the normal vector at the point x is given by $n_x = n_{x(t)} = |x'(t)|^{-1}(x_2'(t), -x_1'(t))$. Denoting $u(t) = u_\Gamma(x(t))$, $f(t) = -g_\Gamma(x(t))$, the equation (2.4) is equivalent to

$$(Lu)(t) := \left(\left(H - \frac{i\eta}{2} I + i\eta D'\right) u\right)(t) = f(t) \qquad (t \in \mathbb{R}) \tag{3.1}$$

where

$$(Hu)(t) = \int_{-1/2}^{+1/2} h_\kappa(t,\tau) u(\tau)\, d\tau \qquad \text{and} \qquad (D'u)(t) = \int_{-1/2}^{+1/2} d'_\kappa(t,\tau) u(\tau)\, d\tau$$

$$\begin{aligned} h_\kappa(t,\tau) &= -\partial_{n_{x(t)}} \partial_{n_{x(\tau)}} g\big(x(t), x(\tau)\big)|x'(\tau)| \\ d'_\kappa(t,\tau) &= \partial_{n_{x(t)}} g\big(x(t), x(\tau)\big)|x'(\tau)|. \end{aligned} \tag{3.2}$$

In the Appendix the following expansions are shown:

$$\begin{aligned} h_\kappa(t,\tau) &= a_0(t)\big|x_0(t) - x_0(\tau)\big|^{-2} + a_1(t,\tau)\log\big|x_0(t) - x_0(\tau)\big| + b_1(t,\tau) \\ d'_\kappa(t,\tau) &= a_2(t,\tau)\log\big|x_0(t) - x_0(\tau)\big| + b_2(t,\tau) \end{aligned} \tag{3.3}$$

where $a_0(t) = -2\pi|x'(t)|^{-1}$, $x_0(t) = e^{i2\pi t}$ and a_1, a_2, b_1, b_2 are infinitely differentiable biperiodic functions depending on the parameter κ. We correspondingly decompose the operator L as

$$L = H_0 - \frac{i}{2} I + A + B$$

where

$$(H_0 u)(t) = -\frac{\pi}{2|x'(t)|} \int_{-1/2}^{+1/2} \frac{u(\tau)\, d\tau}{\sin^2 \pi(t-\tau)}$$

$$(Au)(t) = \int_{-1/2}^{+1/2} a(t,\tau) \log\big|x_0(t) - x_0(\tau)\big| u(\tau)\, d\tau$$

$$(Bu)(t) = \int_{-1/2}^{+1/2} b(t,\tau) u(\tau)\, d\tau$$

with $a = a_1 + i\eta a_2$ and $b = b_1 + i\eta b_2$. We consider the above operators in the Sobolev spaces H^λ $(\lambda \in \mathbb{R})$ of 1-periodic functions u endowed with norm

$$\|u\|_\lambda = \left(\sum_{k\in\mathbb{Z}} \big(\max(1,|k|)\big)^{2\lambda} |\hat{u}(k)|^2\right)^{1/2}$$

where $\hat{u}(k)$ are the Fourier coefficients of u. The operator H_0, the main part of L, is continuous from H^λ into $H^{\lambda-1}$ and has the Fourier representation

$$(H_0u)(t) = \pi|x'(t)|^{-1} \sum_{k\neq 0} |k|\hat{u}(k)e^{ik2\pi t}.$$

The operator A is continuous from H^λ into $H^{\lambda+1}$, and the operator B, being infinitely smoothing, defines a continuous mapping from H^λ into H^μ for all $\lambda, \mu \in \mathbb{R}$. Finally, $L : H^\lambda \to H^{\lambda-1}$ is an isomorphism for all $\lambda \in \mathbb{R}$.

Next we introduce the finite-dimensional space T^n $(n \in \mathbb{N})$ of trigonometric polynomials

$$v(t) = \sum_{k\in \Lambda_n} c_k e^{ik2\pi t} \qquad (c_k \in \mathbb{C})$$

where

$$\Lambda_n = \left\{k \in \mathbb{Z} : -\frac{n}{2} \leq k < \frac{n}{2}\right\}.$$

In addition, we consider the meshpoints jh $(h = \frac{1}{n}, j \in \mathbb{Z})$ and introduce the corresponding interpolation projection $Q_n : H^\lambda \to T^n$ $(\lambda > \frac{1}{2})$ such that

$$(Q_nu)(jh) = u(jh) \qquad (j \in \Lambda_n,\ u \in H^\lambda).$$

We approximate the operators A and B by the following operators A_n and B_n, respectively:

$$(A_nu)(t) = \int_{-1/2}^{+1/2} \log\left|x_0(t) - x_0(\tau)\right|\big(Q_n(a_t u)\big)(\tau)\,d\tau$$

$$(B_nu)(t) = \int_{-1/2}^{+1/2} \big(Q_n(b_t u)\big)(\tau)\,d\tau$$

where, for given t, a_t and b_t denote the partial functions $a_t(\tau) = a_t(t,\tau)$ and $b_t(\tau) = b(t,\tau)$. Now, our numerical method is obtained replacing L in equation (3.1) by the approximation

$$L_n = H_0 - \frac{i\eta}{2}I + A_n + B_n$$

and applying the collocation at the points jh $(j \in \Lambda_n)$. The approximate solution is sought in the space of trigonometric polynomials T^n. Thus we have the method

$$u_n \in T^n, \qquad Q_nL_nu_n = Q_nf. \tag{3.4}$$

Before going to our analysis of the method we discuss matrix forms for the problem (3.4). In order to find the trigonometric polynomial u_n one can use, e.g., the Fourier coefficients or the pointwise values as unknowns. These are related to each other by means of the discrete Fourier transformations as

$$u_n(jh) = \sum_{\nu\in\Lambda_n} \hat{u}_n(\nu)e^{i\nu 2\pi jh} \qquad \text{and} \qquad \hat{u}_n(\nu) = h\sum_{j\in\Lambda_n} u_n(jh)e^{-i\nu 2\pi jh}.$$

For determination of the coefficient matrix corresponding to the operator A_n, it is helpful to present A_n as

$$(A_n u)(t) = \big(VQ_n(a_t u)\big)(t)$$

where V is the logarithmic single layer operator

$$(Vu)(t) = \int_{-1/2}^{+1/2} \log\big|x_0(t) - x_0(\tau)\big| u(\tau)\, d\tau = -\sum_{k\neq 0} \frac{\hat{u}(k)}{2|k|} e^{ik2\pi t}.$$

One easily finds that (3.4) is equivalent to the system

$$\sum_{j\in\Lambda_n} L_{kj}^{(n)} u_n(jh) = f(kh) \qquad (k \in \Lambda_n)$$

with

$$L_{kj}^{(n)} = \frac{\pi}{|x'(kh)|}\rho_{1,n}(k-j) - \frac{i\eta}{2}\delta_{kj} - \frac{1}{2}\rho_{-1,n}(k-j)a(kh,jh) + hb(kh,jh)$$

$$\rho_{\alpha,n}(p) = h \sum_{0\neq\nu\in\Lambda_n} |\nu|^{\alpha} e^{i\nu 2\pi p h} \qquad (|p| \leq n)$$

where δ_{kj} is the Kronecker symbol. If the Fourier coefficients are used as unknowns, we obtain the equations

$$\sum_{l\in\Lambda_n} \hat{L}_{kl}^{(n)} \hat{u}_n(l) = f(kh) \qquad (k \in \Lambda_n)$$

$$\begin{aligned}\hat{L}_{kl}^{(n)} &= \left(\frac{\pi|l|}{|x'(kh)|} - \frac{i\eta}{2}\right) e^{il2\pi kh} \\ &\quad + \sum_{j\in\Lambda_n} \left(-\frac{1}{2}\rho_{-1,n}(k-j)a(kh,jh) + hb(kh,jh)\right) e^{il2\pi jh}.\end{aligned}$$

In the Appendix it is shown how to determine the values $a(kh,jh)$ and $b(kh,jh)$.

4. Stability and convergence

Here we prove stability and convergence results for the numerical method introduced in the previous section. Thereby we consider our method as a further discretization of the trigonometric collocation:

Find $u_n \in T^n$ such that

$$Q_n L u_n = Q_n f \tag{4.1}$$

where $f = Lu$. For this method the following result is true (see [1,17,18]).

Theorem 4.1. *Assume that $u \in H^{\mu}$ $(\mu > \frac{3}{2})$. Then for sufficiently large $n \geq n_0$, the equation (4.1) is uniquely solvable, and the asymptotic error estimate*

$$\|u - u_n\|_{\lambda} \leq cn^{\lambda-\mu}\|u\|_{\mu} \qquad (1 \leq \lambda \leq \mu) \tag{4.2}$$

is valid.

In the following analysis we discuss the order of the consistency which is obtained when replacing L by L_n. More precisely, we estimate the norm of the operator $u_n \to Q_n(L - L_n)u_n$ $(u_n \in T^n)$ with respect to certain Sobolev norms. It turns out that the order of the consistency is sufficiently good, and we are able to derive our results for the method (3.4) from those of (4.1). First we observe that

$$Q_n(L - L_n)u_n = Q_n(A - A_n)u_n + Q_n(B - B_n)u_n.$$

Since the operator B_n has a smooth kernel, the following result can be derived (see [22] and [23]).

Lemma 4.1. *Let $\lambda, \mu \in \mathbb{R}$ and $r > 0$. Then the error estimate*

$$\|Q_n(B - B_n)u_n\|_{\lambda} \leq cn^{-r}\|u_n\|_{\mu} \qquad (u_n \in T^n) \tag{4.3}$$

is valid.

To discuss the approximation of A by A_n, we make use of the approximation and the inverse properties,

$$\|(I - Q_n)u\|_{\lambda} \leq cn^{\lambda-\mu}\|u\|_{\mu} \qquad (u \in H^{\mu},\ 0 \leq \lambda \leq \mu,\ \mu > 1/2) \tag{4.4}$$

$$\|u_n\|_{\mu} \leq cn^{\mu-\lambda}\|u_n\|_{\lambda} \qquad (u_n \in T^n,\ \lambda \leq \mu). \tag{4.5}$$

First we prove the consistency property obtained when replacing A by A_n.

Lemma 4.2. *For any $\varepsilon > 0$, $p \geq \frac{1}{2} + \varepsilon$ the estimates*

$$\|(A - A_n)u\|_0 \leq cn^{-p+1/2+\varepsilon}\|u\|_p \qquad (u \in H^p) \tag{4.6}$$

$$\|(A - A_n)u\|_1 \leq cn^{-p+1/2+\varepsilon}\|u\|_{p+1} \qquad (u \in H^{p+1}) \tag{4.7}$$

are valid.

Proof. For any t we obtain

$$\begin{aligned} \big|((A - A_n)u)(t)\big| &\leq c \max_{\tau \in \mathbb{R}} \big|((I - Q_n)a_t u)(\tau)\big| \\ &\leq c\big\|(I - Q_n)a_t u\big\|_{1/2+\varepsilon} \\ &\leq cn^{-p+1/2+\varepsilon}\|u\|_p \end{aligned} \tag{4.8}$$

which implies (4.6). For (4.7) we differentiate to get

$$\partial_t\big((A - A_n)u\big)(t) = \big(\partial_t V(I - Q_n)a_t u\big)(t) + \big(V(I - Q_n)\partial_t a_t u\big)(t) \tag{4.9}$$

$$\frac{d}{dt}(Vu)(t) = -i\pi(Su)(t) + i\pi\hat{u}(0) \tag{4.10}$$

where

$$(Su)(t) = 2\int_{-1/2}^{+1/2} \frac{u(\tau)x_0(\tau)\,d\tau}{x_0(\tau)-x_0(t)} = \sum_{k\geq 0}\hat{u}(k)e^{ik2\pi t} - \sum_{k<0}\hat{u}(k)e^{ik2\pi t}. \tag{4.11}$$

Abbreviating $v(t,\tau) = \big((I-Q_n)a_t u\big)(\tau)$ we have by (4.11)

$$\big(S(I-Q_n)a_t u\big)(t) = 2\int_{-1/2}^{+1/2} \frac{\big(v(t,\tau)-v(t,t)\big)x_0(\tau)\,d\tau}{x_0(\tau)-x_0(t)} + v(t,t). \tag{4.12}$$

From (4.10) and (4.12) we further obtain

$$\begin{aligned}\big|(\partial_t V(I-Q_n)a_t u)(t)\big| &\leq c\max_{\tau\in\mathbb{R}}\big(|\partial_\tau v(t,\tau)| + |v(t,\tau)|\big)\\ &\leq c\|v(t,\cdot)\|_{3/2+\varepsilon}\\ &\leq cn^{-p+1/2+\varepsilon}\|u\|_{p+1}\end{aligned}$$

which, by using (4.9) and (4.8) with $\partial_t a$ instead of a, gives

$$\|\partial_t(A-A_n)u\|_0 \leq cn^{-p+1/2+\varepsilon}\|u\|_{p+1}. \tag{4.13}$$

The assertion (4.7) follows from (4.6) and (4.13) ∎

Next we have

Theorem 4.2 (Consistency). *For any $\varepsilon > 0$, $p \geq \frac{1}{2}+\varepsilon$ and $\lambda \geq 0$, there holds*

$$\|Q_n(A-A_n)u_n\|_\lambda \leq cn^{-p+1/2+\varepsilon}\|u_n\|_{\lambda+p} \qquad (u_n \in T^n) \tag{4.14}$$

$$\|Q_n(L-L_n)u_n\|_\lambda \leq cn^{-p+1/2+\varepsilon}\|u_n\|_{\lambda+p} \qquad (u_n \in T^n). \tag{4.15}$$

Proof. From (4.4) and (4.7) we obtain

$$\|Q_n(A-A_n)u_n\|_1 \leq c\|(A-A_n)u_n\|_1 \leq cn^{-p+1/2+\varepsilon}\|u_n\|_{1+p} \tag{4.16}$$

and using (4.4) - (4.6) (with p replaced by $p+1$) and (4.7) we obtain

$$\begin{aligned}\|Q_n(A-A_n)u_n\|_0 &\leq \|(Q_n-I)(A-A_n)u_n\|_0 + \|(A-A_n)u_n\|_0\\ &\leq c\Big(n^{-1}\|(A-A_n)u_n\|_1 + n^{-p-1/2+\varepsilon}\|u_n\|_{p+1}\Big)\\ &\leq cn^{-p-1/2+\varepsilon}\|u_n\|_{p+1}\\ &\leq cn^{-p+1/2+\varepsilon}\|u_n\|_p.\end{aligned} \tag{4.17}$$

For $\lambda \geq 0$ we get from (4.5) and (4.17) (with p replaced by $\lambda + p$)

$$\|Q_n(A-A_n)u_n\|_\lambda \leq cn^\lambda\|Q_n(A-A_n)u_n\|_0 \leq cn^{-p+1/2+\varepsilon}\|u_n\|_{\lambda+p} \tag{4.18}$$

which proves (4.14). The other assertion follows from (4.3) and (4.14) ∎

Next we prove the stability of the method (3.4).

Theorem 4.3 (Stability). *Assume $u \in H^\mu$ $(\mu > \frac{3}{2})$. Then the equation (3.4) is uniquely solvable if $n \geq n_0$, and we have*

$$\|u_n\|_\mu \leq c\|u\|_\mu. \tag{4.19}$$

Proof. Assume that $u_n \in T^n$ is a solution of equation (3.4), $Q_n L_n u_n = Q_n L u$ or, equivalently,

$$Q_n L u_n = Q_n L\Big(u + L^{-1}\big(Q_n(L - L_n)u_n\big)\Big). \tag{4.20}$$

Using the stability of the trigonometric collocation obtained from (4.2) and (4.20),we have by the mapping properties of L and (4.15), $0 < \varepsilon < \frac{1}{2}$,

$$\begin{aligned}\|u_n\|_\mu &\leq c\big\|u + L^{-1}\big(Q_n(L - L_n)u_n\big)\big\|_\mu \\ &\leq c\Big(\|u\|_\mu + \|Q_n(L - L_n)u_n\|_{\mu-1}\Big) \\ &\leq c\Big(\|u\|_\mu + n^{\varepsilon-1/2}\|u_n\|_\mu\Big)\end{aligned}$$

which gives (4.19), if n is large enough. Having (4.19), the unique solvability of the finite-dimensional problem (3.4) also follows ∎

Finally we have our main result.

Theorem 4.4 (Convergence). *Assume $u \in H^\mu$ $(\mu > \frac{3}{2})$. Then for the solution u_n of the problem (3.4) there holds for $n \geq n_0$*

$$\|u - u_n\|_\lambda \leq c n^{\lambda-\mu}\|u\|_\mu \qquad (1 \leq \lambda \leq \mu). \tag{4.21}$$

Proof. The continuity of the operator $L^{-1} : H^{\lambda-1} \to H^{\lambda}$ together with (4.4) and (3.4) gives

$$\begin{aligned}\|u - u_n\|_\lambda &\leq c\|L(u - u_n)\|_{\lambda-1} \\ &\leq c\Big(\|(I - Q_n)L(u - u_n)\|_{\lambda-1} + \|Q_n L(u - u_n)\|_{\lambda-1}\Big) \\ &\leq c\Big(n^{\lambda-\mu}\|L(u - u_n)\|_{\mu-1} + \|Q_n(L - L_n)u_n\|_{\lambda-1}\Big).\end{aligned} \tag{4.22}$$

The continuity of the operator $L : H^\mu \to H^{\mu-1}$ and the stability (4.19) further yield

$$\|L(u - u_n)\|_{\mu-1} \leq c\|u - u_n\|_\mu \leq c\|u\|_\mu \tag{4.23}$$

and the consistency (4.15) with (4.19) implies

$$\|Q_n(L - L_n)u_n\|_{\lambda-1} \leq c n^{\lambda-\mu}\|u_n\|_\mu \leq c n^{\lambda-\mu}\|u\|_\mu. \tag{4.24}$$

The assertion now follows from (4.22) - (4.24) ∎

Having the optimal order estimates (4.21), one can show that the convergence is in fact exponential, if the solution and the boundary curve are analytic, see [9: p. 162] and [22].

5. Appendix

Here we derive the decompositions

$$\begin{aligned} h_\kappa(t,\tau) &= -2\pi|x'(t)|^{-1}|x_0(t)-x_0(\tau)|^{-2} \\ &\quad + a_1(t,\tau)\log|x_0(t)-x_0(\tau)| + b_1(t,\tau) \\ d'_\kappa(t,\tau) &= a_2(t,\tau)\log|x_0(t)-x_0(\tau)| + b_2(t,\tau) \end{aligned} \tag{A1}$$

which were used in Section 3. We obtain by (3.2), (2.6) and (2.7)

$$h_\kappa(t,\tau) = \frac{\pi i}{2}\left(zH_1^{(1)}(z)h(t,\tau) - z^2H_0^{(1)}(z)k(t,\tau)\right) \tag{A2}$$

where $z = \kappa|x(t)-x(\tau)|$ and

$$h(t,\tau) = \frac{1}{2\pi}\partial_{n_{x(t)}}\partial_{n_{x(\tau)}}\log|x(t)-x(\tau)|\,|x'(\tau)|$$

$$k(t,\tau) = \frac{1}{2\pi}\frac{n_{x(\tau)}\cdot\big(x(t)-x(\tau)\big)}{|x(t)-x(\tau)|^2}\cdot\frac{n_{x(t)}\cdot\big(x(t)-x(\tau)\big)|x'(\tau)|}{|x(t)-x(\tau)|^2}.$$

The latter function is infinitely differentiable, whereas the function $h(t,\tau)$ is the kernel of the hypersingular potential operator and decomposes as (see [8])

$$h(t,\tau) = h_0(t,\tau) + h_c(t,\tau) \tag{A3}$$

where the main part $h_0(t,\tau)$ has strong singularity,

$$h_0(t,\tau) = -2\pi|x'(t)|^{-1}|x_0(t)-x_0(\tau)|^{-2},$$

and the remaining part is smooth,

$$\begin{aligned} h_c(t,\tau) &= \frac{1}{\pi}\frac{\big(n_{x(t)}\cdot(x(t)-x(\tau))\big)\big(n_{x(\tau)}\cdot(x(t)-x(\tau))\big)|x'(\tau)|}{|x(t)-x(\tau)|^4} \\ &\quad - \frac{1}{2\pi}\frac{x'(t)\cdot x'(\tau)|x_0(t)-x_0(\tau)|^2 - x_0'(t)\cdot x_0'(\tau)|x(t)-x(\tau)|^2}{|x(t)-x(\tau)|^2|x_0(t)-x_0(\tau)|^2|x'(t)|} \\ &\quad + \pi|x'(t)|^{-1}. \end{aligned} \tag{A4}$$

For the following we recall the expansions of the Hankel functions

$$H_\nu^{(1)}(z) = J_\nu(z) + iN_\nu(z) \qquad (\nu \in \mathbb{N})$$

where J_ν and N_ν are the Bessel and Neumann functions

$$J_\nu(z) = \sum_{m=0}^{\infty}\frac{(-1)^m(\frac{z}{2})^{\nu+2m}}{m!(\nu+m)!}$$

$$\begin{aligned} N_\nu(z) &= \frac{2}{\pi}\left(\gamma+\log\frac{z}{2}\right)J_\nu(z) - \frac{1}{\pi}\sum_{m=0}^{\nu-1}\frac{(\nu-m-1)!}{m!}\left(\frac{z}{2}\right)^{2m-\nu} \\ &\quad - \frac{1}{\pi}\sum_{m=0}^{\infty}(-1)^m\frac{(k_{m+\nu}+k_m)}{m!(\nu+m)!}\left(\frac{z}{2}\right)^{2m+\nu} \end{aligned}$$

where $k_0 = 0$, $k_l = 1 + \frac{1}{2} + \cdots + \frac{1}{l}$ $(l \in \mathbb{N})$, $\sum_{m=0}^{\nu-1} = 0$ for $\nu = 0$, and γ is the Euler constant [16: pp. 65 - 69]. By the above formulas one easily derives

$$\frac{\pi i}{2} z H_1^{(1)}(z) = 1 - z J_1(z) \log z + \beta z^2 + z^4 E_1(z) \tag{A5}$$

where E_1 is an analytic function in the whole complex plane, and β denotes the constant

$$\beta = \frac{1}{4}(1 + i\pi + 2\log 2 - 2\gamma).$$

By (A2) and (A3) we obtain

$$\frac{\pi i}{2} z H_1^{(1)}(z) h(t,\tau) = h_0(t,\tau) + a_{11}(t,\tau) \log |x_0(t) - x_0(\tau)| + b_{11}(t,\tau) \tag{A6}$$

$$a_{11}(t,\tau) = -z J_1(z) h(t,\tau) \tag{A7}$$

$$\begin{aligned} b_{11}(t,\tau) = \beta z^2 h(t,\tau) - z J_1(z) h(t,\tau) \log \frac{\kappa |x(t) - x(\tau)|}{|x_0(t) - x_0(\tau)|} \\ + h_c(t,\tau) + |x_0(t) - x_0(\tau)|^2 F_1(t,\tau) \end{aligned} \tag{A8}$$

where $F_1(t,\tau)$ is a smooth biperiodic function. In particular, for the diagonal values one has

$$a_{11}(t,t) = \frac{\kappa^2}{4\pi} |x'(t)|$$

$$b_{11}(t,t) = \frac{\kappa^2}{4\pi} \left(\log \frac{\kappa |x'(t)|}{2\pi} - 2\beta \right) |x'(t)| + h_c(t,t).$$

Note that if $t \not\equiv \tau \pmod 1$, the values $a_{11}(t,\tau)$ are obtained by a direct substitution from (A7), and the values $b_{11}(t,\tau)$ can be computed using (A4). Some additional effort is needed to calculate explicitly the diagonal values $h_c(t,t)$ from (A4). However, it is straightforward and the (rather complicated) formula is not shown here.

The discussion of the second term in (A2) is easier. We have

$$-\frac{\pi i}{2} z^2 H_0^{(1)}(z) = z^2 J_0(z) \log z + z^2 E_2(z)$$

where E_2 is an entire function. From this we further obtain

$$-\frac{\pi i}{2} z^2 H_0^{(1)}(z) k(t,\tau) = a_{12}(t,\tau) \log |x_0(t) - x_0(\tau)| + b_{12}(t,\tau) \tag{A9}$$

where a_{12} and b_{12} are smooth functions such that

$$a_{12}(t,\tau) = z^2 J_0(z) k(t,\tau) \quad \text{and} \quad a_{12}(t,t) = b_{12}(t,t) = 0.$$

The non-diagonal values of $b_{12}(t,\tau)$ should be computed from (A9). To conclude, we have determined the first decomposition (A1) with $a_1 = a_{11} + a_{12}$, $b_1 = b_{11} + b_{12}$.

Now, we turn to consider the kernel of D'. Using (2.5) and (3.2), we obtain

$$d'_\kappa(t,\tau) = \frac{\pi i}{2} z H_1^{(1)}(z) d'(t,\tau)$$
$$d'(t,\tau) = -\frac{1}{2\pi} \frac{n_{x(t)} \cdot (x(t) - x(\tau)) |x'(\tau)|}{|x(t) - x(\tau)|^2}$$

where $d'(t,\tau)$ is the kernel of the adjoint to the double layer potential operator. Using (A5) again, we get

$$d'_\kappa(t,\tau) = d'(t,\tau) - zJ_1(z)d'(t,\tau)\log|x_0(t) - x_0(\tau)| + |x_0(t) - x_0(\tau)|^2 F_2(t,\tau)$$

with a smooth function $F_2(t,\tau)$. Thus the second equation (A1) holds with

$$a_2(t,\tau) = -zJ_1(z)d'(t,\tau)$$
$$b_2(t,\tau) = d'(t,\tau) + |x_0(t) - x_0(\tau)|^2 F_2(t,\tau).$$

In particular, $a_2(t,t) = 0$ and $b_2(t,t) = d'(t,t)$. The non-diagonal values of $b_2(t,\tau)$ should be calculated from (A1).

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