Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 13 (1994), No. 4, 567–576

Sequential Convergence in the Space of Absolutely Riemann Integrable Functions

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Abstract. The concept of R-convergence in the space of usual Riemann integrable functions was introduced by Dickmeis, Mevissen, Nessel and van Wickeren in 1986. They proved that this space, provided with R-convergence, is sequentially complete and that every function in the space can be approximated by a sequence of continuous functions. They gave several applications. This paper extends that concept to the space of absolutely Riemann integrable functions and it is shown that the results mentioned above still hold there.

Keywords: *Riemann integral, R-convergence, approximation by continuous functions*

AMS subject classification: 41A30

1. Introduction

Let X be a finite or infinite closed interval of $\mathbb{R}^m$ whose interior set $(\operatorname{Int} X)$ is non-empty. Furthermore, let $L^1(X)$, $B(X)$ and $C(X)$ denote the real Banach spaces of Lebesgue integrable, bounded, and continuously bounded functions on X, respectively. If X is bounded, we will also denote the real space of Riemann integrable functions on X by $R(X)$.

If we consider $R(X)$ as a subspace of $L^1(X)$, then it is not closed. In other words, $R(X)$ is not complete under the induced norm $\|\cdot\|_1$ of $L^1(X)$. On the other hand, if we induce the sup-norm $\|\cdot\|_X$ of $B(X)$ in $R(X)$, then it is not possible to approximate every Riemann integrable function by sequences of continuous functions. This means that $C(X)$ is not dense in $R(X)$.

Inspired by a work of Polya in 1933 (cf. [4]), Dickmeis, Mevissen, Nessel and van Wickeren (cf. [1]) introduced the following concept of convergence.

Definition 1.1: Let X be bounded. A sequence $\{f_n\}_{n\in\mathbb{N}} \subset B(X)$ is called *R-convergent* (i.e. *Riemann convergent*) to $f \in B(X)$ and denoted by $R-\lim_{n\to\infty} f_n = f$ if

$$\sup_{n\in\mathbb{N}} \|f_n\|_X < \infty \tag{1}$$

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This work was supported by the Deutsche Forschungsgemeinschaft under grant Gu 304/1-2

ISSN 0232-2064 / \$ 2.50

and

$$\overline{\int} \sup_{k \geq n} |f_k - f| \to 0 \qquad \text{as } n \in \mathbb{N} \tag{2}$$

where $\overline{\int}$ (and further $\underline{\int}$) denotes the usual upper (lower) integral.

Condition (1) ensures the existence of the upper integral in condition (2), and since $\overline{\int}|f|$ is a seminorm on $B(X)$, the R-convergence is linear.

The approach given in Definition 1.1 shows that the space of Riemann integrable functions as a linear subspace of $B(X)$ is sequentielly R-complete and that $C(X)$ is R-dense in it.

Of course, this definition cannot solve all deficiencies derived from the well-known fact that the collection of Jordan measurable sets does not form a σ-algebra. In Section 3 we will show some examples. However, the introduced concept is good enough to enable the authors from [1] to discuss several approximation problems successfully. See not only [1], but the references cited there, and also [3], for instance. We can search at least for two different generalizations of Definition 1.1. A first one deals with the abstract version of the Riemann integral that has been already accomplished by van Wickeren in [6]. The present paper discusses a second generalization: the extension to unbounded functions on unbounded domains.

2. Absolutely Riemann integrable functions

A first step is to define the functional spaces that will support the generalization of the R-convergence and, in the same way, the upper integral for unbounded functions on unbounded domains.

Let $\nu := \nu_m$ denote the Lebesgue measure in $\mathbb{R}^m$, and E the set of real functions f defined on $\mathbb{R}^m$ that satisfy the following condition.

Definition 2.1: The function f belongs to E if there exists an open set $A \subset \mathbb{R}^m$ such that

$$\nu(\mathbb{R}^m \setminus A) = 0 \tag{3}$$

and, if $K \subset A$ is a compact set,

$$\|f\|_K < \infty. \tag{4}$$

Let $A(f)$ denote the collection of all open sets A satisfying conditions (3) and (4) for a given $f \in E$. If $f, g \in E$, $A \in A(f)$, $B \in A(g)$ and $\alpha \in \mathbb{R}$, then $A \cap B$ is an open set that satisfies condition (3) with respect to $f + \alpha g$. Hence, E is a real linear space. We remark that a function in E is not necessarily measurable.

Definition 2.2: If A is a non-empty open set of $\mathbb{R}^m$, a *partition* P of A is a numerable collection of compact intervals $\{c_j | \, j \in \mathbb{N}\}$ that satisfies the following conditions:

$$\begin{aligned} &\operatorname{Int} c_j \neq \emptyset \quad \text{for all } j \in \mathbb{N} \\ &\operatorname{Int}(c_i \cap c_j) = \emptyset \quad \text{for all } i, j \in \mathbb{N} \text{ with } i \neq j \\ &A = \cup\{c_j | \, j \in \mathbb{N}\}. \end{aligned}$$

The collection of all partitions of A will be denoted by $P(A)$.

Of course $P(A)$ is not empty. In order to prove this, construct for every $x \in A$ a compact interval $I_x \subset A$ such that $x \in \operatorname{Int} I_x$. By the Lindelöf Theorem we obtain a numerable set of intervals $\{I_{x_n}\}$ that covers A and using it we can define a partition $P \in P(A)$ by induction.

Definition 2.3: A function $f \in E$ is *absolutely Riemann integrable* if there exist $A \in A(f)$ and $P \in P(A)$ such that

$$f \in R(c_j) \quad \text{for all } c_j \in P \tag{5}$$

and

$$\sum_{j \in \mathbb{N}} \int_{c_j} |f| < \infty. \tag{6}$$

The collection of all absolutely Riemann integrable functions will be denoted by $R^1 := R^1(\mathbb{R}^m)$.

It is well-known in the usual theory of Riemann integration that a bounded function f belongs to $R(X)$ if and only if the Lebesgue measure of the set of discontinuities of f in X is equal to zero (cf., e.g., [2, 5]). Of course, in that case $f \in L^1(X)$.

Theorem 2.4: *A real function f defined on $\mathbb{R}^m$ is in R^1 if and only if $f \in L^1(\mathbb{R}^m)$ and the Lebesgue measure of the set $D(f)$ of discontinuities of f in $\mathbb{R}^m$ is equal to zero.*

Proof: Let $f \in R^1$ be fixed. It follows from Definition 2.3 that $f \in L^1(\mathbb{R}^m)$. Denote the characteristic function of a set S by χ_S. We obtain from (5) that $\nu(D(\chi_{c_j} f)) = 0$ for every $j \in \mathbb{N}$. Then $\nu(D(f)) = 0$ since $D(f) \subset \bigcup_{j \in \mathbb{N}} D(\chi_{c_j} f)$. Now we recall that the *modulus of continuity* of a function g at $x \in X$ is defined for $\alpha > 0$ by the finite or infinite value

$$\omega(g, x, \alpha) := \sup \left\{ |g(y) - g(z)| \,\middle|\, y, z \in X, \ |x - y| < \alpha, \ |x - z| < \alpha \right\}$$

and the *oscillation* of g at x by

$$O(g, x) := \omega(g, x, 0) := \lim_{\alpha \to 0+} \omega(g, x, \alpha).$$

The function g is continuous at x if and only if $O(g, x) = 0$. So, for a fixed f on $\mathbb{R}^m$, we have

$$F := \left\{ x \in \mathbb{R}^m \,\middle|\, O(g, x) \geq 1 \right\} \subset D(f)$$

and, under the assumptions of the second part of the theorem, $\nu(F) = 0$.

Since we can prove that F is closed, the complement set A of F is open and satisfies condition (3). So, for every $x \in A$, there exists an open neighbourhood $V_x \subset A$ of x such that $O(f, y) < 1$ for every $y \in V_x$. If $K \subset A$ is a compact set, we can choose a covering $\{V_{x_1}, \ldots, V_{x_p}\}$ of K. This implies that f is bounded on K and condition (4) holds. So, we have proved that f belongs to E.

Furthermore, if $P \in P(A)$, then f is bounded on each c_j. Denote the boundary of a set S by ∂S. Since

$$D(\chi_{c_j} f) \subset D(f) \cup \partial c_j,$$

we have $\nu(D(\chi_{c_j} f)) = 0$ and condition (5) holds. Now condition (6) follows from $f \in L^1(\mathbb{R}^m)$ and then f meets the requirements of Definition 2.3 ∎

Remark 2.5: From Theorem 2.4 it follows that the Riemann integral

$$\int_{\mathbb{R}^m} f := \sum_{j\in\mathbb{N}} \int_{c_j} f$$

does not depend on the choice of $A \in A(f)$ and $P \in P(A)$ in Definition 2.3. It coincides, of course, with the Lebesgue integral of f. If f is only defined on a compact interval X, we also define $f(x) = 0$ for every $x \in \mathbb{R}^m \setminus X$, in order to construct the space $R^1(X)$. So we have the inclusions $R(X) \subset R^1(X) \subset R^1(\mathbb{R}^m)$.

We can perform the same extension when f is only defined in any Jordan measurable set or even on certain unbounded sets. However, since there exist open sets U that are not Jordan measurable, a reexamination of this theory with U instead of $\mathbb{R}^m$ will be necessary if we want to extend the Riemann integration to those sets. Other approaches are also possible. For an open but not Jordan measurable set see, e.g., [5].

We still need to define the upper integral for unbounded functions and unbounded domains.

Lemma 2.6: *Let $f \geq 0$ be a bounded function on a compact interval C. Extend the concept of a partition $P := \{c_j\}_{j\in\mathbb{N}}$ of C as in the Definition 2.2 with C instead of A. Then*

$$\overline{\int_C} f = \sum_{j\in\mathbb{N}} \overline{\int_{c_j}} f.$$

Proof: We have

$$\overline{\int_C} f = \sum_{j\leq k} \overline{\int_{c_j}} f + \overline{\int_C} f\chi_{\bigcup_{j>k} c_j}$$

but

$$\overline{\int_C} f\chi_{(\cup_{j>k}c_j)} \leq \|f\|_C\, \nu\left(\cup_{j>k}c_j\right) \to 0 \quad \text{as } k \to \infty.$$

Thus the statement is proved ∎

Lemma 2.7: *Let $f \geq 0$ be in E and $A, B \in A(f)$, $P := \{c_i\}_{i\in\mathbb{N}} \in P(A)$, $P' := \{c'_j\}_{j\in\mathbb{N}} \in P(B)$. Then, for a finite or infinite value, the equality*

$$\sum_{i\in\mathbb{N}} \overline{\int_{c_i}} f = \sum_{j\in\mathbb{N}} \overline{\int_{c'_j}} f$$

is true.

Proof: Define $c_{i,j} := c_i \cap c'_j$ for every $i, j \in \mathbb{N}$. Using Lemma 2.6 and since we deal now with positive values, it follows

$$\sum_{i\in\mathbb{N}} \overline{\int_{c_i}} f = \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} \overline{\int_{c_{i,j}}} f = \sum_{j\in\mathbb{N}} \sum_{i\in\mathbb{N}} \overline{\int_{c_{i,j}}} f = \sum_{j\in\mathbb{N}} \overline{\int_{c'_j}} f$$

and the statement is proved ∎

Lemma 2.7 enables us to define the upper integral of a positive function f in E independent on the set $A \in A(f)$ or the partition $P := \{c_i\}_{i \in \mathbb{N}} \in P(A)$. Moreover, Lemma 2.6 guarantees that it will be consistent with the normal definition of an upper integral in the case of positive bounded functions on bounded intervals.

Definition 2.8: Let $f \geq 0$ be given in E. The upper integral of f is the finite or infinite value

$$\overline{\int} f := \overline{\int}_{\mathbb{R}^m} f := \sum_{i \in \mathbb{N}} \overline{\int}_{c_i} f.$$

We will systematically use the following properties.

Proposition 2.9: *The upper integral of positive functions satisfies the following three conditions:*

(a) If $f, g \in E$ and $0 \leq f \leq g$, then $\overline{\int} f \leq \overline{\int} g$.

(b) $\overline{\int} |f|$ is a seminorm on E.

(c) If $f \in R^1$, then $\int_{\mathbb{R}^m} f = \overline{\int}_{\mathbb{R}^m} f$.

The proof of these properties can be carried out without any difficulties. Now we can give a trivial but useful equivalence of Definition 2.3.

Proposition 2.10: *Let $f \in E$ and $A \in A(f)$. Then $f \in R^1$ if and only if $f \in R(X)$ for every compact interval $X \subset A$ and $\overline{\int} |f| < \infty$.*

Definition 2.11: For a function $f \in E$, the *class* $[f]$ of f is defined by $[f] := \{g \in E | \ \overline{\int} |f - g| = 0\}$.

It follows from Condition (b) in Proposition 2.9 that the relation $f \sim g$ if and only if $g \in [f]$ is an equivalence relation in the linear space E.

Proposition 2.12: *If $f \in R^1$, then $[f] \subset R^1$. Moreover, if $g \in [f]$, then $\int f = \int g$.*

Proof: Let $g \in [f]$ be fixed. Furthermore, let $A \in A(f)$, $B \in A(g)$ and X be a compact interval in $A \cap B$. It follows that

$$0 \leq \left(\overline{\int}_X - \underline{\int}_X \right) (f - g) \leq 2 \overline{\int}_X |f - g| \leq 2 \overline{\int} |f - g| = 0.$$

So $(f - g) \in R(X)$ and then $g = f - (f - g) \in R(X)$. In particular, $\int_X f = \int_X g$.

Now, if $P := \{c_j\}_{j \in \mathbb{N}} \in P(A \cap B)$, we have

$$\overline{\int} |g| = \sum_{j \in \mathbb{N}} \overline{\int}_{c_j} |g| = \sum_{j \in \mathbb{N}} \overline{\int}_{c_j} |f| = \overline{\int} |f| < \infty.$$

So, the first part of the proposition follows from Proposition 2.10. Furthermore, we have

$$\left| \int f - \int g \right| \leq \overline{\int} |f - g| = 0.$$

Thus the statements are proved ∎

3. Convergent sequences

Let $G \subset R^1$ be a set of positive functions with the property that for all $g_1, g_2 \in G$ there exists $g_3 \in G$ such that $g_1 + g_2 \leq g_3$.

Definition 3.1: A sequence $\{f_n\}_{n\in I\!N} \subset E$ is called *G-convergent* to $f \in E$ and denoted by $G - \lim_{n\to\infty} f_n = f$ if the properties

$$|f_n| \leq g \qquad \text{for some } g \in G \text{ and all } n \in I\!N \tag{7}$$

and

$$\overline{\int} \sup_{k\geq n} |f_k - f| \to 0 \qquad \text{as } n \to \infty \tag{8}$$

are fulfilled.

Since $\overline{\int}|f|$ is a seminorm, the G-convergence has linear properties. Here we have to use the assumptions on G for the proof of property (7).

Example 3.2: Let X be bounded and denote $R := \{n\chi_X | n \in I\!N\}$. Take $G := R$. Then the G-convergence induced in $R(X)$ is the R-convergence in Definition 1.1.

Proposition 3.3: *If* $G - \lim_{n\to\infty} f_n = f$ *and* $\{f_n\}_{n\in I\!N} \subset R^1$, *then* $f \in R^1$.

Proof: Let X be a bounded interval where f is uniformly bounded. Then $f \in R(X)$ because

$$0 \leq \left(\overline{\int_X} - \underline{\int_X}\right) f \leq 2\overline{\int_X} \sup_{k\geq n} |f_k - f| \leq 2\overline{\int} \sup_{k\geq n} |f_k - f|$$

and the latter term tends to zero according to property (8).

On the other hand, for k large enough we have

$$\overline{\int}|f| \leq \overline{\int}|f - f_k| + \overline{\int}|f_k| \leq \overline{\int}|f - f_k| + \int g < \infty.$$

Thus all statements are proved ∎

Proposition 3.4: *Let* $\{f_n\}_{n\in I\!N} \subset E$ *and* $G - \lim_{n\to\infty} f_n = f$. *Then* $G - \lim_{n\to\infty} f_n = h$ *if and only if* $h \in [f]$.

Proof: Suppose that $h \in [f]$. Since

$$\sup_{k\geq n} |f_k - h| \leq \sup_{k\geq n} |f_k - f| + |f - h|$$

we have (8) for h instead of f. The inequality

$$|f - h| \leq \sup_{k\geq n} |f - f_k| + \sup_{k\geq n} |f_k - h|$$

leads to the other implication ∎

Definition 3.5: A sequence $\{f_n\}_{n\in\mathbb{N}} \subset E$ is a Cauchy G-sequence if it satisfies property (7) and the relation

$$\overline{\int} \sup_{j,k\geq n} |f_j - f_k| \to 0 \quad \text{as } n \to \infty. \tag{9}$$

is fulfilled.

Theorem 3.6: *A sequence $\{f_n\}_{n\in\mathbb{N}} \subset E$ is G-convergent if and only if it is a Cauchy G-sequence. So we say that E is G-complete.*

Proof: If $G - \lim_{n\to\infty} f_n = f$, then relation (9) follows from

$$\sup_{j,k\geq n} |f_j - f_k| \leq 2 \sup_{j\geq n} |f_j - f|.$$

Now suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy G-sequence and define the function f by $f(x) = \lim_{n\to\infty} \sup f_n(x)$. Since (7) holds, we have $|f| \leq g$ and, in particular, $f \in E$.

On the other hand, for every $k \geq n$,

$$|f_k(x) - f(x)| \leq \sup_{j\geq n} |f_k(x) - f_j(x)| \leq \sup_{i,j\geq n} |f_i(x) - f_j(x)|.$$

Hence, (8) follows from (9) ∎

Corollary 3.7: *R^1 is a G-complete space.*

Proof: Combine Theorem 3.6 and Proposition 3.3 ∎

Proposition 3.8: *The R-convergence (and then the G-convergence) is not metrizable.*

Proof: Following Proposition 3.4, we must search for a pseudometric d on $R(X) \times R(X)$ such that

$$d(f, h) = 0 \iff h \in [f]$$

and

$$R - \lim_{n\to\infty} f_n = f \iff d(f_n, f) \to 0.$$

If such a pseudometric existed and if $R - \lim_{n\to\infty} f_n = f$ (for instance, $f_n \equiv f$), then we should also have $R - \lim_{n\to\infty} h_n = f$ for any sequence $\{h_n\}_{n\in\mathbb{N}}$ such that $h_n \in [f_n]$ for every $n \in \mathbb{N}$. In fact

$$d(h_n, f) \leq d(h_n, f_n) + d(f_n, f) = d(f_n, f) \to 0 \quad \text{as } n \to \infty.$$

However, that is not always possible as it will be shown in the following example.

Example 3.9: Suppose $R - \lim_{n\to\infty} f_n = f$ in $R(X)$. Let $\{x_n\}_{n\in\mathbb{N}}$ be the set of points in X whose coordinate components are rational numbers. If $M > 0$ is a uniform bound for the norms $\|f_n\|_X$, $n \in \mathbb{N}$ and $\|f\|$, define $h_n(x_n) = 2M$ and $h_n(x) = f_n(x)$ otherwise. Then (1) holds, but $\overline{\int} \sup_{k\geq n} |h_k - f| \geq M\nu(X) > 0$ and so (2) is not fulfilled.

Proposition 3.8 is neither a good result nor a real trouble. However, the above example clearly shows a disadvantage of the R-convergence and, thus, of the general G-convergence. Note that we have only changed every f_n at one point!

However, Definition 1.1 and later Definition 3.1 have been conceived for functions and not for classes of functions as it is the case in $L^1(\mathbb{R}^m)$. It must be so because of the well-known intrinsic deficiencies of the Riemann integration as we have pointed out in the comments to Definition 1.1.

In a personal letter, Prof. Nessel kindly explained to one of the authors that he had already understood the above deficiency of the R-convergence. All the time, in the very beginning their intention only was to give an appropriate concept of convergence in order to establish Polya's work about quadrature formulas for Riemann integrable functions by an application of Banach-Steinhaus type theorems, as it was successfully treated in [1]. Other applications to approximation theory came later. However, perhaps it is our main interest that the concept of R-convergence brings to light some links of the relevant but historically intrincate integration theory.

4. Approximation by continuous functions

This paper would be incomplete without the following result, which is a very important feature in dealing with approximation problems.

Theorem 4.1: *Let $f \in R^1$. Suppose $|f| \le g$ for some $g \in G$. Then there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of continuous and absolutely Riemann integrable functions such that $G - \lim_{n\to\infty} f_n = f$.*

Proof: The case of $f \equiv 0$ is trivial. Otherwise, consider the positive variations

$$f^+ = \frac{|f|+f}{2} \qquad \text{and} \qquad f^- = \frac{|f|-f}{2}$$

which are Riemann integrable. Furthermore, $f^+ + f^- = |f| \le g$, $f = f^+ - f^-$ and the G-convergence is linear. So we only need to prove the theorem for a positive f.

Let $A \in A(f)$ and $P := \{c_j | j \in \mathbb{N}\} \in P(A)$. Furthermore, fix any $n \in \mathbb{N}$. For each $j \in \mathbb{N}$, let $P^j := \{c_j^i | i = 1, 2, \dots, I(j)\}$ be a finite partition of c_j, as we pointed out in Lemma 2.6, such that

$$\int_{c_j} f - \sum_{1\le i\le I(j)} m_j^i \nu(c_j^i) \le 2^{-j-n} \tag{10}$$

where $m_j^i := \inf\{f(x) | x \in c_j^i\}$. Define $h_n \equiv 0$ if each m_j^i is zero. In that case $h_n \le f \le g$ and

$$\overline{\int} f - h_n = \int f = \sum_{j\in\mathbb{N}} \int_{c_j} f \le \sum_{j\in\mathbb{N}} 2^{-j-n} = 2^{-n} \tag{11}$$

where the inequality is deduced from (10). Consider a non-empty compact interval $x_j^i \subset c_j^i$, with $\partial x_j^i \cap \partial c_j^i = \emptyset$ and

$$\nu(c_j^i \setminus x_j^i) \le (m_j^i)^{-1} 2^{-j-i-n} \tag{12}$$

whenever $m_j^i \neq 0$. Using the Urysohn Lemma, there exists a continuous function h_j^i with values in the real closed interval $[0,1]$ such that

$$h_j^i/_{x_j^i} \equiv 1 \qquad \text{and} \qquad h_j^i/_{\mathbb{R}^m \setminus c_j^i} \equiv 0.$$

Define $h_j^i \equiv 0$ whenever $m_j^i = 0$ and

$$h_n(x) := \sum_{j \in \mathbb{N}} \sum_{1 \le i \le I(j)} m_j^i h_j^i(x).$$

For a given $x \in \mathbb{R}^m$ only one function h_j^i could be different from zero at x. So h_n is well-defined and continuous. Therefore, if $h_n(x) \neq 0$, then there exists one and only one c_j^i such that $x \in \operatorname{int} c_j^i$, and in that case we obtain

$$h_n(x) \le m_j^i \le f(x) \le g(x).$$

Using (10) and (12), we have

$$\begin{aligned}
\overline{\int} f - h_n &= \int f - \int h_n \\
&= \sum_{j \in \mathbb{N}} \left(\int_{c_j} f - \int_{c_j} h_n \right) \\
&\le \sum_{j \in \mathbb{N}} \left[\left(\int_{c_j} f - \sum_{1 \le i \le I(j)} m_j^i \nu(c_j^i) \right) + \left(\sum_{1 \le i \le I(j)} m_j^i \nu(c_j^i) - \int_{c_j} h_n \right) \right] \qquad (13) \\
&\le \sum_{j \in \mathbb{N}} 2^{-n-j} + \sum_{j \in \mathbb{N}} \sum_{1 \le i \le I(j)} m_j^i \nu(c_j^i \setminus x_j^i) \\
&\le 2^{-n} + \sum_{j \in \mathbb{N}} 2^{-j-n} \\
&\le 2^{-n+1}.
\end{aligned}$$

Now we define the sequence of continuous functions $\{f_n\}_{n \in \mathbb{N}}$ by induction. Construct h_1 and define $f_1 := h_1$. If f_n has been defined, construct h_{n+1} and define

$$f_{n+1}(x) := \max\{f_n(x), h_{n+1}(x)\}.$$

Since $h_n \le f$, we have $f_n \le f_{n+1} \le f \le g$ and, using (11) and (13), we finally obtain

$$\overline{\int} \sup_{k \ge n}(f - f_n) = \int f - f_n \le \int f - h_n \le 2^{-n+1} \to 0 \quad \text{as } n \to \infty.$$

Thus the theorem is proved ∎

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Received 10.02.1994