On the Existence of Holomorphic Functions Having Prescribed Asymptotic Expansions

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Abstract. A generalization of some results of T. Carleman in [1] is developped. The practical form of it states that if the non-empty subset $D$ of the boundary $\partial \Omega$ of a domain $\Omega$ of $\mathbb{C}$ has no accumulation point and if the connected component in $\partial \Omega$ of every $u \in D$ has more than one point, then $D$ is regularly asymptotic for $\Omega$, i.e. for every family $\{c_{u,n} : u \in D, n \in \mathbb{N}_0\}$ of complex numbers, there is a holomorphic function $f$ on $\Omega$ which at every $u \in D$ has $\sum_{n=0}^{\infty} c_{u,n}(z - u)^n$ as asymptotic expansion at $u$.

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1. Generalities

a) About the vector spaces. All the vector spaces we consider are over the field $\mathbb{C}$ of the complex numbers. If $A$ is a subset of a vector space, span$A$ denotes its linear hull. If $I$ is a set, $\omega(I)$ denotes as usual the vector space $\mathbb{C}^I$ endowed with the product topology. A subset $\{v_i : i \in I\}$ of the algebraic dual of a vector space $E$ has the interpolation property if, for every family $\{c_i : i \in I\}$ of $\mathbb{C}$, there is $x \in E$ such that $(x, v_i) = c_i \text{ for every } i \in I$.

For future reference, let us mention the following result of M. Eidelheit, a generalization of which to the case when $E$ is a B-complete space can be found as Theorem 1 of [4].

Theorem 1.1 (Interpolation): A subset $\{v_i : i \in I\}$ of the topological dual $E'$ of a Fréchet space $E$ has the interpolation property if and only if its elements are linearly independent and such that, for every equicontinuous subset $B$ of $E'$, the vector space span$\{v_i : i \in I\} \cap \text{span } B$ has finite dimension.

b) About the asymptotic expansions. We begin with the following

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Definition: A holomorphic function $f$ on a non-void domain $\Omega$ of $\mathbb{C}$ has an asymptotic expansion at $u \in \partial \Omega$ if the limits

$$a_0 = \lim_{z \in \Omega, z \to u} f(z)$$

and, for every $n \in \mathbb{N}$,

$$a_n = \lim_{z \in \Omega, z \to u} \frac{f(z) - \sum_{j=0}^{n-1} a_j (z - u)^j}{(z - u)^n}$$

exist and are finite. In such a case,

a) we say that the series $\sum_{n=0}^{\infty} a_n (z - u)^n$ is the asymptotic expansion of $f$ at $u$

b) we write $f(z) \approx \sum_{n=0}^{\infty} a_n (z - u)^n$ at $u$

c) we use the notations

$$f^{[n]}(u) = a_n$$

for all $n \in \mathbb{N}_0$

$$f^{[0]}(z, u) = f(z)$$

for all $z \in \Omega$

$$f^{[n]}(z, u) = \frac{f^{[n-1]}(z, u) - f^{[n-1]}(u)}{z - u}$$

for all $z \in \Omega, n \in \mathbb{N}$.

(So, in fact, we have

$$f^{[n]}(z, u) = \frac{f(z) - \sum_{j=0}^{n-1} a_j (z - u)^j}{(z - u)^n}$$

for all $z \in \Omega, n \in \mathbb{N}$

as well as

$$\lim_{z \in \Omega, z \to u} f^{[n]}(z, u) = f^{[n]}(u)$$

for all $n \in \mathbb{N}_0$.)

c) Notations. Unless explicitly stated, throughout this paper, we use the following notations:

a) $\Omega$ is a non-void domain of $\mathbb{C}$.

b) $\{K_m : m \in \mathbb{N}\}$ is a compact cover of $\Omega$ such that $(K_1)^c \neq \emptyset$ and $K_m \subset (K_{m+1})^c$ for every $m \in \mathbb{N}$.

c) $\mathcal{H}(\Omega)$ is the Fréchet-Montel space of holomorphic functions on $\Omega$, endowed with the compact-open topology (i.e., for instance, with the countable system of norms $\{\| \cdot \|_{K_m} : m \in \mathbb{N}\}$).

d) $D$ is a non-void subset of $\partial \Omega$.

e) $A(\Omega; D)$ is the set of the elements of $\mathcal{H}(\Omega)$ which have an asymptotic expansion at every point of $D$. Of course, it is a vector subspace of $\mathcal{H}(\Omega)$. We endow it canonically with the topology induced by $\mathcal{H}(\Omega)$. Let us insist on this fact: from now on, $A(\Omega; D)$ is a topological vector subspace of $\mathcal{H}(\Omega)$.

f) The notation $T$ refers to the linear map

$$T : A(\Omega; D) \to \omega(D \times \mathbb{N}_0), \quad f \mapsto \left( f^{[n]}(u) \right)_{(u, n) \in D \times \mathbb{N}_0}$$

g) For every $u \in D$ and $n \in \mathbb{N}_0$, the notation $\eta_{u, n}$ refers to the linear functional

$$\eta_{u, n} : A(\Omega; D) \to \mathbb{C}, \quad f \mapsto f^{[n]}(u).$$

As the set $\mathcal{P}(\Omega)$ of the restrictions to $\Omega$ of the polynomials is a vector subspace of $A(\Omega; D)$, the next result proves that the linear functionals $\eta_{u, n}$ for $u \in D$ and $n \in \mathbb{N}_0$ are linearly independent on any vector subspace of $A(\Omega; D)$ containing $\mathcal{P}(\Omega)$. 
Lemma 1.2: For every subset $D$ of $\partial \Omega$, \{\eta_{u,n} : u \in D, n \in \mathbb{N}_0\} is a set of linearly independent linear functionals on $\mathcal{P}(\Omega)$.

Proof: If it is not the case, there are a finite subset $D'$ of $D$, an integer $N \in \mathbb{N}_0$ and elements $c_{u,n}$ of $\mathcal{C}$ for $u \in D'$ and $n \in \{0, \ldots, N\}$ such that $(P, \sum_{u \in D'} \sum_{n=0}^N c_{u,n} \eta_{u,n}) = 0$ for every $P \in \mathcal{P}(\Omega)$, although the coefficients are not all equal to 0. So we may suppose the existence of $u_0 \in D'$ and $n_0 \in \{0, \ldots, N\}$ such that $c_{u_0,n_0} \neq 0$ and $c_{u_0,n} = 0$ for every $n \in \{n_0 + 1, \ldots, N\}$. Then the consideration of the polynomial

$$P(z) = (z - u_0)^{n_0} \prod_{u \in D' \setminus \{u_0\}} (z - u)^{N+1}$$

leads immediately to a contradiction. 

d) About the regularly asymptotic sets. We begin with the following:

Definition. The set $D$ is regularly asymptotic for $\Omega$ if, for every family

$$\{c_{u,n} : u \in D, n \in \mathbb{N}_0\}$$

of $\mathcal{C}$, there is a function $f \in A(\Omega; D)$ such that $f(z) \approx \sum_{n=0}^\infty c_{u,n}(z - u)^n$ at $u$ for every $u \in D$. So $D$ is regularly asymptotic for $\Omega$ if and only if the linear map $T$ is surjective which happens if and only if the set $\{\eta_{u,n} : u \in D, n \in \mathbb{N}_0\}$ has the interpolation property on $A(\Omega; D)$.

In such a case, it is clear that no element of $D$ can be an isolated point of $\partial \Omega$. The next result gives another restriction on such sets $D$.

Proposition 1.3: If $D$ is regularly asymptotic for $\Omega$, then it has no accumulation point. Hence it is countable.

Proof: If it is not the case, there is a sequence $(u_m)_{m \in \mathbb{N}}$ of distinct points of $D$ converging to some $u_0 \in D$ with $u_m \neq u_0$ for every $m \in \mathbb{N}$. As $D$ is regularly asymptotic for $\Omega$, there is then $f \in A(\Omega; D)$ such that

$$f(z) \approx \sum_{n=0}^\infty c_{m,n}(z - u_m)^n \quad \text{at } u_m \text{ for all } m \in \mathbb{N}_0$$

with

$$c_{m,n} = \begin{cases} 1 & \text{if } m \in \mathbb{N} \\ 0 & \text{if } m = 0 \end{cases} \quad \text{for all } n \in \mathbb{N}_0.$$

This leads directly to the following contradiction:

$$\lim_{z \in \Omega, z \to u_m} f(z) = f(0)(u_m) = 1 \quad \text{for all } m \in \mathbb{N}$$

and

$$\lim_{z \in \Omega, z \to u_0} f(z) = f(0)(u_0) = 0.$$ 

Thus the statement is proven.
Remark. However these restrictions on \( D \) are not sufficient to ensure that \( D \) is regularly asymptotic for \( \Omega \). If we set \( \Omega = \mathbb{C} \setminus \{ 0 \} \), it is clear that \( D = \{ 0 \} \) has no isolated point of \( \partial \Omega \) and that \( D \) has no accumulation point. However, for every \( f \in A(\Omega; D) \), there is a neighbourhood \( U \) of 0 such that \( f \) is holomorphic and bounded on \( U \setminus \{ 0 \} \). Hence \( f \) must have a holomorphic extension to \( \mathbb{C} \setminus \{ 0 \} \).

e) Aim of this paper. In [1], T. Carleman has proved the following:

a) Every finite subset \( D \) of the boundary of a bounded, convex and open subset \( \Omega \) of \( \mathbb{C} \) is regularly asymptotic for \( \Omega \); for every family \( \{ c_{u,n} : u \in D, n \in \mathbb{N}_0 \} \) of \( \mathbb{C} \), there are infinitely many functions \( f \in A(\Omega; D) \) such that \( f(z) \approx \sum_{n=0}^{\infty} c_{u,n}(z-u)^n \) at \( u \) for every \( u \in D \) (cf. [1: p. 31]).

b) \( \{ 0 \} \) is regularly asymptotic for the open subset

\[ \Omega = \{ z \in \mathbb{C} : \| z \| < R \} \setminus \{ (x,0) : x \neq 0 \} \]

of \( \mathbb{C} \) (cf. [1: p. 37]).

We are going to generalize these results. We will first introduce some definitions and a basic result when \( D \) is a singleton, then consider the case when \( D \) is finite, next discuss the case when \( P \) is countable and finally state the generalizations we have obtained.

2. If \( D \) is a singleton

Let \( u \) be a point of \( \partial \Omega \). For every \( r \in \mathbb{N} \), we set

\[ A_r(\Omega; \{ u \}) = \left\{ f \in A(\Omega; \{ u \}) : \sup_{z \in \Omega, |z-u| \leq 1/r} |f(z)| < \infty \right\}. \]

Clearly \( A_r(\Omega; \{ u \}) \) is a vector subspace of \( A(\Omega; \{ u \}) \) and

\[ \bigcup_{r=1}^{\infty} A_r(\Omega; \{ u \}) = A(\Omega; \{ u \}). \]

Moreover an easy recursion on \( j \in \mathbb{N} \) establishes that the boundedness of \( f \in A(\Omega; \{ u \}) \) on \( \{ z \in \Omega : |z-u| \leq 1/r \} \) implies for every \( j \in \mathbb{N} \) the boundedness of \( f^{[j]}(\cdot, u) \) on the same set. Therefore, for every \( n \in \mathbb{N} \),

\[ p_{u,r,n} : A_r(\Omega; \{ u \}) \to [0, +\infty), \quad f \mapsto \| f \|_{K_n} + \left\| \sum_{j=0}^{n} |f^{[j]}(\cdot, u)| \right\|_{\{ z \in \Omega : |z-u| \leq 1/r \}} \]

is a semi-norm on \( A_r(\Omega; \{ u \}) \); it is even a norm since \( \Omega \) is a domain and since \( (K_n)^o \neq \emptyset \) for every \( n \in \mathbb{N} \). So

\[ P_{u,r} = \{ p_{u,r,n} : n \in \mathbb{N} \} \]
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is a countable system of semi-norms on \( A_r(\Omega; \{u\}) \), endowing this space with a metrizable locally convex topology which is finer than the one induced by \( \mathcal{H}(\Omega) \). In fact more can be said: we will prove in the next section that \( (A_r(\Omega; \{u\}), P_{u,r}) \) is a Fréchet space.

The sets

\[
V_{u,r,n} = \left\{ f \in A_r(\Omega; \{u\}) : p_{u,r,n}(f) \leq \frac{1}{n} \right\} \quad (n \in \mathbb{N})
\]

constitute a fundamental basis of the neighbourhoods of the origin in \( (A_r(\Omega; \{u\}), P_{u,r}) \).

So, for every sequence \( \mu = (m_n)_{n \in \mathbb{N}} \) of \( \mathbb{N} \),

\[
B_{u,r,\mu} = \bigcap_{n=1}^{\infty} m_nV_{u,r,n}
\]

is an absolutely convex and bounded subset of that space but more can be said here too.

**Proposition 2.1:** For every \( u \in \partial \Omega \), \( r \in \mathbb{N} \) and \( \mu \in \mathbb{N}^\mathbb{N} \), the set \( B_{u,r,\mu} \) is an absolutely convex and compact subset of \( \mathcal{H}(\Omega) \).

**Proof:** As \( B_{u,r,\mu} \) is an absolutely convex and bounded subset of the Fréchet-Montel space \( \mathcal{H}(\Omega) \), it is enough to prove that \( B_{u,r,\mu} \) is sequentially closed. Let \( (f_k)_{k \in \mathbb{N}} \) be a sequence of \( B_{u,r,\mu} \) converging in \( \mathcal{H}(\Omega) \) to \( f \). First we establish that \( f \) has an asymptotic expansion at \( u \) with

\[
f^{[j]}(u) = \lim_{k \to \infty} f^{[j]}_k(u)
\]

for every \( j \in \mathbb{N}_0 \). For every \( k \in \mathbb{N} \), as we have \( f_k \in B_{u,r,\mu} \), we get at once \( |f^{[0]}_k(u)| \leq m_1 \).

So from every subsequence of \( (f^{[0]}_k(u))_{k \in \mathbb{N}} \), we can extract a subsequence converging to some \( a_0 \in \mathbb{C} \). As we also have

\[
\left| f^{[1]}_k(z, u) \right| = \left| \frac{f^{[0]}_k(z, u) - f^{[0]}_k(u)}{z - u} \right| \leq m_1
\]

for every \( k \in \mathbb{N} \) and \( z \in \Omega \) such that \( |z - u| \leq 1/r \), we get

\[
\left| \frac{f(z) - a_0}{z - u} \right| \leq m_1
\]

for every \( z \in \Omega \) such that \( |z - u| \leq 1/r \), hence \( \lim_{z \in \Omega, z \to u} f(z) = a_0 \). The conclusion is then direct by use of a recursion.

It is then an easy matter to check that \( f \) belongs to \( B_{u,r,\mu} \). Hence the conclusion
3. If $D$ is finite

If $D = \{u_1, \ldots, u_J\}$ is finite, we use the following notations:

a) For every $r \in \mathbb{N}$,

\[
A_r(\Omega; D) = \bigcap_{u \in D} A_r(\Omega; \{u\})
\]

\[
= \left\{ f \in A(\Omega; D) : \sup_{z \in \Omega, |z-u| \leq 1/r} |f(z)| < \infty \text{ for all } u \in D \right\}.
\]

So $A_r(\Omega; D)$ is a vector subspace of $A(\Omega; D)$ and

\[
\bigcup_{r=1}^{\infty} A_r(\Omega; D) = A(\Omega; D).
\]

b) For every $r, n \in \mathbb{N}$,

\[
p_{D,r,n} : A_r(\Omega; D) \to [0, +\infty), \quad f \mapsto \sup\{p_{u,r,n}(f) : u \in D\}
\]

is a norm on $A_r(\Omega; D)$ and

\[
P_{D,r} = \{p_{D,r,n} : n \in \mathbb{N}\}
\]

is a countable system of norms on $A_r(\Omega; D)$ endowing it with a finer locally convex topology than the one induced by $\mathcal{H}(\Omega)$. From now on, if $D$ is finite and if $r > 0$, the notation $A_r(\Omega; D)$ will refer to the locally convex space $(A_r(\Omega; D), P_{D,r})$. So

\[
V_{D,r,n} = \left\{ f \in A_r(\Omega; D) : p_{D,r,n}(f) \leq \frac{1}{n} \right\} \quad (n \in \mathbb{N})
\]

is a fundamental sequence of neighbourhoods of the origin in $A_r(\Omega; D)$.

c) For every sequence $\rho = (r_n)_{n \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$,

\[
B_{D,\rho} = \bigcap_{u \in D} B_{u,r_1,\rho_1}
\]

where $\rho_1$ is the sequence $(r_{n+1})_{n \in \mathbb{N}}$. Of course, $B_{D,\rho}$ is an absolutely convex and compact subset of $A(\Omega; D)$; moreover it is a bounded subset of the space $A_{r_1}(\Omega; D)$.

**Proposition 3.1:** If $D$ is finite, then, for every $r \in \mathbb{N}$, $A_r(\Omega; D)$ is a Fréchet space.

**Proof:** Let $(f_k)_{k \in \mathbb{N}}$ be a Cauchy sequence. As it clearly is a Cauchy sequence in $\mathcal{H}(\Omega)$, it converges in $\mathcal{H}(\Omega)$ to some $f$. Let us prove now that $f$ belongs to $A_r(\Omega; D)$. Let us fix $u \in D$. Let us also fix $j \in \mathbb{N}_0$. On one hand, for every $k \in \mathbb{N}$, the
function $f_k^j(\cdot,u)$ on $\Omega$ has a finite limit $f_k^j(u)$ at $u$. On the other hand, the sequence $\left( f_k^j(\cdot,u) \right)_{k \in \mathbb{N}}$ is uniformly Cauchy on $\left\{ z \in \Omega : |z - u| \leq 1/r \right\}$. Therefore the two limits

$$\lim_{k \to \infty} \lim_{z \in \Omega, z \to u} f_k^j(z,u) = \lim_{k \to \infty} f_k^j(u)$$

and

$$\lim_{z \in \Omega, z \to u} \lim_{k \to \infty} f_k^j(z,u)$$

exist, are finite and are equal. A direct recursion on $j \in \mathbb{N}_0$ establishes then that $f$ has an asymptotic expansion at $u$.

It is finally a standard matter to prove that the sequence $(f_k)_{k \in \mathbb{N}}$ converges in $A_r(\Omega; D)$ to $f$.

The previous result is also a consequence of the following considerations which will prove to be very fruitful (we refer the reader to [3] for the definition and the properties of the quasi-LB spaces and representations).

**Proposition 3.2:** If $D$ is finite, then the family $\{B_{D,\rho} : \rho \in \mathbb{N}^\infty\}$ is a quasi-LB representation of the space $A(\Omega; D)$, made of absolutely convex and compact sets.

**Proof:** By Proposition 2.1, for every $\rho \in \mathbb{N}^\infty$, $B_{D,\rho}$ is an absolutely convex and compact subset of $\mathcal{H}(\Omega)$ hence of $A(\Omega; D)$ since it is a subset of $A(\Omega; D)$. It is also clear that, for every $\rho, \sigma \in \mathbb{N}^\infty$ such that $\rho \leq \sigma$ (i.e. $r_n \leq s_n$ for every $n \in \mathbb{N}$ if $\sigma = (s_n)_{n \in \mathbb{N}}$), we have $B_{D,\rho} \subseteq B_{D,\sigma}$.

To conclude, it is then enough to check that $\bigcup_{\rho \in \mathbb{N}^\infty} B_{D,\rho}$ is equal to $A(\Omega; D)$. This is a direct matter: every $f \in A(\Omega; D)$ has an asymptotic expansion at every element of $D$ so it is a bounded function on $\bigcup_{u \in D} \left\{ z \in \Omega : |z - u| \leq 1/r \right\}$ for some $r \in \mathbb{N}$. Moreover, for every $n \in \mathbb{N}_0$, there is $s_n \in \mathbb{N}$ such that $f \in s_n V_{D,r,n}$. Therefore $f$ belongs to $B_{D,\rho}$ with $r_1 = r$ and $r_{n+1} = s_n$ for every $n \in \mathbb{N}$.

Now, as in [3], for every $\rho = (r_n)_{n \in \mathbb{N}} \in \mathbb{N}^\infty$, we may introduce successively:

a) For every $n \in \mathbb{N}$, the set

$$B_{D,r_1,\ldots,r_n} = \bigcup \left\{ B_{D,\sigma} : \sigma = (s_n)_{n \in \mathbb{N}} \in \mathbb{N}^\infty ; s_1 = r_1, \ldots, s_n = r_n \right\}.$$ 

In fact, it is a direct matter to check that in our case

$$B_{D,r_1} = A_{r_1}(\Omega; D)$$

and, for every $n = 2, 3, \ldots$,

$$B_{D,r_1,\ldots,r_n} = (r_2 V_{D,r_1,1}) \cap \ldots \cap (r_n V_{D,r_1,\ldots,r_{n-1}}).$$

b) For every $n \in \mathbb{N}$, the linear hull $F_{D,r_1,\ldots,r_n}$ of $B_{D,r_1,\ldots,r_n}$. Of course, in our case, we get at once $F_{D,r_1,\ldots,r_n} = A_{r_n}(\Omega; D)$ for every $n \in \mathbb{N}$.

c) The vector space $F_{D,\rho} = \bigcap_{n=1}^\infty F_{D,r_1,\ldots,r_n}$, i.e. $F_{D,\rho} = A_{r_1}(\Omega; D)$ in our case.

At this point, [3] provides another way to prove that, for every $r \in \mathbb{N}$, $A_r(\Omega; D)$ is a Fréchet space.
Definition: If $D$ is finite, we have

$$A(\Omega; D) = \bigcup_{r=1}^{\infty} A_r(\Omega; D)$$

and, for every $r, s \in \mathbb{N}$ such that $r < s$, the canonical injection from $A_r(\Omega; D)$ into $A_s(\Omega; D)$ is continuous. Therefore we may endow $A(\Omega; D)$ with an (LF)-topology $\tau$ by considering the inductive limit of the sequence $(A_r(\Omega; D))_{r \in \mathbb{N}}$ of Fréchet spaces. Of course, $\tau$ is finer than the topology of $A(\Omega; D)$.

Notations: If $D = \{u_1, \ldots, u_J\}$ is a finite subset of $\partial \Omega$, then, in this section and unless specifically stated, we use the following notations:

a) For every $j \in \{1, \ldots, J\}$, the notation $T_j$ refers to the linear map $T_j : A(\Omega; D) \to \omega(N_0)$, $f \mapsto (f^{[n]}(u_j))_{n \in \mathbb{N}_0}$ which is continuous if we endow $A(\Omega; D)$ with the topology $\tau$ and $L_j$ is the vector subspace span$\{\eta_{uj} : n \in \mathbb{N}_0\}$ of the topological dual $(A(\Omega; D), \tau)'$.

b) $L = \text{span}(u_1, \ldots, u_J)$.

Now we are looking for necessary and sufficient conditions under which $D$ is regularly asymptotic for $\Omega$.

Definition: If $r$ belongs to $\mathbb{N}$, the finite subset $D$ of $\partial \Omega$ is $r$-regularly asymptotic for $\Omega$ if the restriction of the map $T$ to $A_r(\Omega; D)$ is also surjective onto $\omega(D \times \mathbb{N}_0)$.

Proposition 3.3: The finite subset $D$ of $\partial \Omega$ is regularly asymptotic for $\Omega$ if and only if there is $r \in \mathbb{N}$ such that $D$ is $r$-regularly asymptotic for $\Omega$.

Proof: The proof of Proposition 9 in [4] applies also to this case. 

Corollary 3.4: If $D$ is finite and is regularly asymptotic for $\Omega$, then the kernel of $T$ has $2^{\mathbb{N}_0}$ as algebraic dimension.

Proof: By the previous proposition, there is $r \in \mathbb{N}$ such that the restriction $S$ of $T$ to $A_r(\Omega; D)$ is a continuous and surjective linear map. To conclude, we just have to prove that the Fréchet subspace $\ker S$ of $A_r(\Omega; D)$ is not finite-dimensional. If it were finite-dimensional, it would have a topological complement $M$ and the restriction of $T$ to $M$ would appear as an isomorphism in between $M$ and $\omega(D \times \mathbb{N})$ although $M$ has a continuous norm.

The next result comes from [5]. We repeat it here, with proof, for the sake of completeness since the reference [5] is not readily accessible.

Theorem 3.5: The finite subset $D$ of $\partial \Omega$ is regularly asymptotic for $\Omega$ if and only if, for every $u \in D$ and $(c_n)_{n \in \mathbb{N}_0} \in \omega(\mathbb{N}_0)$, there is $f \in A(\Omega; D)$ such that $f(z) \approx \sum_{n=0}^{\infty} c_n(z - u)^n$ at $u$.

Proof: The condition is trivially necessary. The condition is also sufficient. Indeed, if $D$ is a singleton, the result is trivial. So let us consider the case $D = \{u_1, \ldots, u_J\}$ with an integer $J \geq 2$. 
For every \( j \in \{1, \ldots, J\} \), \( T_j \) is clearly a surjective, linear and continuous map. So the Fréchet space \( \omega(\mathbb{N}_0) \) is equal to \( \bigcup_{j=1}^{\infty} T_j \mathcal{A}_r(\Omega; D) \) hence there is \( r_j \in \mathbb{N} \) such that \( T_j \mathcal{A}_r(\Omega; D) \) is a second category vector subspace of \( \omega(\mathbb{N}_0) \). As a surjective, linear and continuous map from the Fréchet space \( \mathcal{A}_r(\Omega; D) \) onto its image, the restriction of \( T_j \) is a topological homomorphism. Therefore \( T_j \mathcal{A}_r(\Omega; D) \) is a Fréchet space hence is equal to \( \omega(\mathbb{N}_0) \).

To conclude, we are going to prove that, for \( r = \sup\{r_1, \ldots, r_j\} \), the set

\[
\{\eta_{u,n} : u \in D, n \in \mathbb{N}_0\}
\]

has the interpolation property on \( \mathcal{A}_r(\Omega; D) \). By Theorem 1.1, as these continuous linear functionals on \( \mathcal{A}_r(\Omega; D) \) are linearly independent, we just need to prove that, for every \( s \in \mathbb{N} \), the dimension of the vector space \( L \cap \text{span} V_{\mathcal{A}_r}^{\beta_0,\ldots,\beta_N} \) is finite.

Let us fix \( s \in \mathbb{N} \) and, in order to simplify the notations, let us set \( U = V_{\mathcal{A}_r}^{\beta_0,\ldots,\beta_N} \). For every \( j \in \{1, \ldots, J\} \), the restriction of \( T_j \) to \( \mathcal{A}_r(\Omega; D) \) is surjective hence the dimension of \( L_j \cap \text{span} U^0 \) is finite: there is \( N(j) \in \mathbb{N} \) such that

\[
\left(\beta_0, \ldots, \beta_N \in \mathbb{C} : \beta_N \neq 0; \sum_{n=0}^{N} \beta_n \eta_{u_j,n} \in \text{span} U^0\right) \implies N \leq N(j).
\]

To conclude, it is then sufficient to prove that

\[
\zeta = \sum_{j=1}^{J} \sum_{n=0}^{N} \alpha_{j,n} \eta_{u_j,n} \in \text{span} U^0
\]

with \( N > \sup\{s, N(1), \ldots, N(J)\} \) implies \( \alpha_{j,N} = 0 \) for every \( j = 1, \ldots, J \).

This we do by contradiction: let us suppose the existence of such a functional \( \zeta \) with \( N > \sup\{s, N(1), \ldots, N(J)\} \) and \( \alpha_{k,N} \neq 0 \) for some \( k \in \{1, \ldots, J\} \), belonging to \( q U^0 \) for some \( q \in \mathbb{N} \).

We need some preparation. Let us denote by \( P(z) \) the polynomial

\[
\prod_{1 \leq j \leq J, j \neq k} (z - u_j)^{N+1}.
\]

It can also be written as

\[
P(z) = c_0 + c_1(z - u_k) + \ldots + c_{(N+1)(J-1)}(z - u_k)^{(N+1)(J-1)}
\]

with \( c_0 \neq 0 \). Then we choose a closed disk \( B \) in \( \mathbb{C} \), centered at the origin and containing

\[
K_s \cup \left( \bigcup_{j=1}^{J} \{ z \in \Omega : |z - u_j| \leq \frac{1}{r} \} \right).
\]

Finally we set

\[
C_1 = \sup \left\{ \left\| \frac{P(\cdot)}{(\cdot-u_j)^k} \right\|_B : h \in \{0, \ldots, s\}, j \in \{1, \ldots, J\} \setminus \{k\} \right\}
\]
\[ C_2 = \sup \left\{ \left\| c_h + \cdots + c_{N+1}(J-1)(z - u_k)^{(N+1)(J-1)-h} \right\|_B : h = 0, \ldots, s \right\} \]

and introduce the polynomial

\[
Q(z) = \frac{P(z)}{2(C_1 + C_2)(s + 1)^2}.
\]

For every \( g \in U \), the function \( Qg \) clearly belongs to \( A_\tau(\Omega; D) \). In fact, it belongs to \( U \):

a) For every \( j \in \{1, \ldots, J\} \setminus \{k\} \) and \( h \in \{0, \ldots, s\} \), we have

\[
(Qg)^[\Lambda](z, u_j) = \frac{Q(z)}{(z - u_j)^h} g(z)
\]

hence

\[
\left| (Qg)^[\Lambda](z, u_j) \right| \leq \frac{C_1}{2(C_1 + C_2)(s + 1)^2} \cdot \frac{1}{s} \leq \frac{1}{2s(s + 1)^2}
\]

for every \( z \in \Omega \) such that \( |z - u_j| \leq 1/r \).

b) For \( h \in \{0, \ldots, s\} \), we also have

\[
(Qg)^[\Lambda](z, u_k) = \sum_{l=0}^{h-1} c_l g^{[\Lambda-l]}(z, u_k) + (c_h + \cdots + c_{N+1}(J-1)(z - u_k)^{(N+1)(J-1)-h}) g^{[0]}(z, u_k)
\]

hence

\[
\left| (Qg)^[\Lambda](z, u_k) \right| \leq \frac{(s + 1)c_2}{2(C_1 + C_2)(s + 1)^2} \cdot \frac{1}{s} \leq \frac{1}{2s(s + 1)}
\]

for every \( z \in \Omega \) such that \( |z - u_k| \leq 1/r \).

c) For every \( z \in K_s \), it is clear that

\[
\left| (Qg)(z) \right| \leq \frac{C_1}{2(C_1 + C_2)(s + 1)^2} \cdot \frac{1}{s} \leq \frac{1}{2s(s + 1)^2}.
\]

At this stage, we consider the continuous linear functional

\[
\eta = \sum_{n=0}^{N} \alpha_{k,n} \sum_{j=0}^{n} Q^{[n-j]}(u_k) \eta_{u_k,j}.
\]

The coefficient of \( \eta_{u_k,N} \) is \( \alpha_{k,N} Q^{[0]}(u_k) \) and differs from 0. Moreover as \( N > N(k) \), \( \eta \) does not belong to \( \text{span} U^\circ \). Therefore there is \( f \in U \) such that \( \left| (f, \eta) \right| > q \). As \( Qf \) belongs to \( U \), we finally arrive at the following contradiction:

We have

\[
\left| (Qf, \zeta) \right| \leq q
\]
because $\zeta \in qU^\circ$, as well as

$$|\langle Qf, \zeta \rangle| = \left| \sum_{j=1}^{J} \sum_{n=0}^{N} \alpha_{j,n} \langle Qf, \eta_{u_j, n} \rangle \right|$$

$$= \left| \sum_{n=0}^{N} \alpha_{k,n} \langle Qf, \eta_{u_k, n} \rangle \right|$$

$$= \left| \sum_{n=0}^{N} \alpha_{k,n} \sum_{j=0}^{n} Q^{[n-j]}(u_k) f[j](u_k) \right|$$

$$= \left| \left\langle f, \sum_{n=0}^{N} \alpha_{k,n} \sum_{j=0}^{n} Q^{[n-j]}(u_k) \eta_{u_k, j} \right\rangle \right|$$

$$= |\langle f, \eta \rangle|$$

$$> q.$$ 

Thus our statement is proved

At this stage, we can extend Theorem 4 of [4] in the following way.

**Proposition 3.6:** The finite subset $D = \{u_1, \ldots, u_J\}$ of $\partial \Omega$ is regularly asymptotic for $\Omega$ if and only if the following condition (*) is satisfied:

(*) There is $r \in \mathbb{N}$ such that, for every compact subset $K$ of $\Omega$ and $j_0 \in \{1, \ldots, J\}$, there is an integer $p \in \mathbb{N}$ such that, for every $h > 0$, there is $f \in A_r(\Omega; D)$ verifying

$$|f(z)| \leq 1 \quad \text{for all } z \in K \cup \left( \bigcup_{j=1}^{J} \left\{ u \in \Omega : |u - u_j| \leq \frac{1}{r} \right\} \right)$$

and

$$|f^{[p]}(u_{j_0})| > h.$$

**Proof:** The condition is necessary. Indeed, by Proposition 3.3, there is an integer $r \in \mathbb{N}$ such that

$$S : A_r(\Omega; D) \to \omega(D \times \mathbb{N}_0), \quad f \mapsto \left( f^{[n]}(u) \right)_{(u,n) \in D \times \mathbb{N}_0}$$

is a continuous, surjective and linear map. Let us fix a compact subset $K$ of $\Omega$ and an integer $j_0$ in $\{1, \ldots, J\}$. There is then an integer $s \in \mathbb{N}$ such that $K \subset K_s$. As, for $n \in \mathbb{N}_0$, the continuous linear functionals $\eta_{u_{j_0}, n}$ on $A_r(\Omega; D)$ are linearly independent, Theorem 1.1 tells us that the dimension of the vector space

$$\text{span} \left\{ \eta_{u_{j_0}, n} : n \in \mathbb{N}_0 \right\} \cap \text{span} \mathcal{V}^{\circ}_{D, r, s}$$

is finite. Therefore there is an integer $p \in \mathbb{N}$ such that $\eta_{u_{j_0}, p}$ does not belong to $\text{span} \mathcal{V}^{\circ}_{D, r, s}$. Hence, for every $h > 0$, there is $f \in \mathcal{V}_{D, r, s}$ such that

$$|f^{[p]}(u_{j_0})| = |\langle f, \eta_{u_{j_0}, p} \rangle| > h.$$
Hence the conclusion.

The condition is sufficient. Indeed, by Theorem 3.5, it is enough to prove that, for every \( j \in \{1, \ldots, J\} \), the map

\[
S_j : A_r(\Omega; D) \to \omega(\mathcal{N}_0), \quad f \mapsto (f^{[n]}(u_j))_{n \in \mathcal{N}_0}
\]

is surjective. Let us fix \( j \in \{1, \ldots, J\} \). As, for \( n \in \mathcal{N}_0 \), the continuous linear functionals \( \eta_n = \eta_{u_{j,n}} \) are linearly independent, by Theorem 1.1, we just need to prove that, for every \( s \in \mathcal{N} \), the vector space

\[
\text{span}\{\eta_n : n \in \mathcal{N}_0\} \cap \text{span} V_{D,r,s}^S
\]

has finite dimension. Let us fix \( s \in \mathcal{N} \) and denote by \( p_j \) the least integer satisfying condition (*) for \( K = K_s \) and \( j_0 = j \). There is then \( k > 0 \) such that, for every \( g \in A_r(\Omega; D) \) verifying \( |g(z)| \leq 1 \) for every element \( z \) of the set

\[
\mathcal{K} = K_s \bigcup \left( \bigcup_{u \in D} \left\{ t \in \Omega : |t - u| \leq \frac{1}{r} \right\} \right),
\]

one has

\[
|g^{[n]}(u_j)| \leq k \quad \text{for all } n \in \{0, \ldots, p_j - 1\}.
\]

At this point, to conclude, it is sufficient to prove that a functional of the type

\[
d = \sum_{n=0}^{N} \alpha_n \eta_n \quad \text{with } N > p_j + s, \quad \alpha_n \in \mathcal{C} \quad \text{and } \alpha_N \neq 0
\]

never belongs to \( \text{span} V_{D,r,s}^S \). This we do by contradiction. Let us suppose that such a functional \( \delta \) belongs to \( hV_{D,r,s}^S \) for some \( h > 0 \). We need some preparation. We choose an integer \( d \) greater than the diameter of \( \mathcal{K} \) and set successively

\[
\alpha = \sup \{|\alpha_0|, \ldots, |\alpha_N|\}.
\]

\[
P(z) = (z - u_j)^{N - p_j} \prod_{1 \leq k \leq J, k \neq j} (z - u_k)^N.
\]

\[
L = \sup \left\{ |P^{[n]}(u_j)| : n = 0, \ldots, N \right\}.
\]

Now we choose \( f \in A_r(\Omega; D) \) such that \( |f(z)| \leq 1 \) for every \( z \in \mathcal{K} \) and

\[
|f^{[p_j]}(u_j)| > \frac{s(s + 2)hd^{N_J} + 2L\alpha N^2k}{|\alpha_N P^{[N - p_j]}(u_j)|}.
\]

Finally we set \( g = Pf \). Of course \( g \) belongs to \( A_r(\Omega; D) \); more precisely from

\[
p_{D,r,s}(g) = \sup_{u \in D} \left( \|Pf\|_{\mathcal{K},r} + \sup \left\{ \sum_{l=0}^{s} \frac{|P(z)f(z)|}{|z - u|^l} : z \in \Omega, |z - u| \leq \frac{1}{r} \right\} \right)
\]

\[
\leq d^{N_J} + (s + 1)d^{N_J}
\]

\[
= (s + 2)d^{N_J}
\]
we get that $g$ belongs to $s(s+2)d^{NJ}V_{D,r,s}$ and hence $g$ satisfies

$$|(g,\delta)| \leq hs(s+2)d^{NJ}.$$ 

But we also have

$$|f^{[N-t]}(u_j)| \leq k \quad \text{for every } t \in \{N-p_j+1, \ldots, N\}$$

and this leads to the following contradiction:

$$|\langle g,\delta \rangle| = \sum_{n=0}^{N} \alpha_n g^{[n]}(u_j)$$

$$= \sum_{n=0}^{N} \alpha_n (Pf)^{[n]}(u_j)$$

$$= \sum_{n=0}^{N} \alpha_n \sum_{t=0}^{n} p^{[t]}(u_j) f^{[n-t]}(u_j)$$

$$= \left| \sum_{n=N-p_j}^{N} \alpha_n \sum_{t=N-p_j}^{n} p^{[t]}(u_j) f^{[n-t]}(u_j) \right|$$

$$\geq |\alpha_N| \left| p^{[N-p_j]}(u_j) f^{[p_j]}(u_j) \right| - |\alpha_N| \sum_{t=N-p_j+1}^{N} \left| p^{[t]}(u_j) \right| \left| f^{[N-t]}(u_j) \right|$$

$$- \sum_{n=N-p_j}^{N-1} |\alpha_n| \sum_{t=N-p_j}^{n} \left| p^{[t]}(u_j) \right| \left| f^{[n-t]}(u_j) \right|$$

$$\geq s(s+2)hd^{NJ} + 2LaN^2k - \alpha NLk - N^2\alpha Lk$$

$$> s(s+2)hd^{NJ}.$$

Thus the assertion is proved \[.\]

Now by use of the previous result and of ideas of \[4\], we are going to establish a first generalization of Carleman's result.

**Notations:** For $u \in \mathbb{C}$ and $A \subset \mathbb{C}$, let us set

$$\delta_1(u, A) = \inf \{|u - z| : z \in A\} \quad \text{and} \quad \delta_2(u, A) = \sup \{|u - z| : z \in A\}.$$

**Definition:** The boundary $\partial \Omega$ is **quasi-connected** at $u \in \partial \Omega$ if, for every $\epsilon, \delta > 0$ such that $0 < \epsilon < \delta$, there is a connected subset $A$ of $\partial \Omega$ such that

$$\delta_2(u, A) < \delta \quad \text{and} \quad \delta_1(u, A) < \epsilon \delta_2(u, A).$$

Let us mention that in Section 5, we will prove that $\partial \Omega$ is quasi-connected at $u \in \partial \Omega$ if the connected component of $u$ in $\partial \Omega$ contains more than one point. So if $\Omega$ is simply connected, $\partial \Omega$ is quasi-connected at every point of $\partial \Omega$. 
Theorem 3.7: If $\partial \Omega$ is quasi-connected at every point of the finite subset $D$ of $\partial \Omega$, then $D$ is regularly asymptotic for $\Omega$.

Proof: As $D = \{u_1, \ldots, u_J\}$ is finite, there is $r \in \mathbb{N}$ such that the disks

$$\left\{ z \in \mathbb{C} : |z - u_j| \leq \frac{1}{r} \right\}$$

are pairwise disjoint for $j \in \{1, \ldots, J\}$. Let us fix $j \in \{1, \ldots, J\}$ and a compact subset $K$ of $\Omega$. We are going to prove that Proposition 3.6 applies with $p = 1$. Of course there is $C > 0$ such that $|z - \alpha| \cdot |z - \beta| < C^2$ for every

$$z \in K = K \bigcup \left( \bigcup_{u \in D} \left\{ t \in \mathbb{C} : |t - u| \leq \frac{1}{r} \right\} \right)$$

and every $\alpha, \beta \in \mathbb{C}$ such that $|\alpha - u_j| \leq 1/r$ and $|\beta - u_j| \leq 1/r$. Given $h > 0$, by hypothesis, there is a connected subset $A$ of $\partial \Omega$ such that

$$\delta_2(u_j, A) < \frac{1}{r} \quad \text{and} \quad 0 < \delta_1(u_j, A) < \inf \left\{ \frac{1}{2r}, \frac{1}{16C^2h^2} \right\} \delta_2(u_j, A).$$

Therefore there are points $\alpha$ and $\beta$ in $A$ such that

$$|u_j - \alpha| < \inf \left\{ \frac{1}{2r}, \frac{1}{16C^2h^2} \right\} |u_j - \beta|.$$

Then there is a determination $g$ of $\sqrt{(\cdot - \alpha)(\cdot - \beta)}$ which is holomorphic on $\Omega \cup V$ for some open neighbourhood $V$ of $D$. So $f = g/C$

a) belongs to $A_r(\Omega; D)$

b) verifies $|f(z)| < 1$ for every $z \in K$

c) verifies

$$|f^{[1]}(u_j)| = \frac{1}{C} \lim_{z \to u_j} \left| \frac{g(z) - g(u_j)}{z - u_j} \right| = \frac{|2u_j - (\alpha + \beta)|}{2C|u_j - \alpha|^{1/2}|u_j - \beta|^{1/2}}$$

$$\geq \frac{1 - \frac{|u_j - \alpha|}{|u_j - \beta|}}{2C \left| \frac{|u_j - \alpha|}{|u_j - \beta|} \right|^{1/2}}$$

$$> \frac{1 - 1/2}{2C \cdot \frac{1}{4Ch}} = h,$$

hence the conclusion by Proposition 3.6

Let us also mention the following result.
Proposition 3.8: Let $D$ be a finite subset of $\partial \Omega$. Let us suppose moreover the existence of a subset \{$(d_{u,n} : u \in D, n \in N_0)$\} of $(0, +\infty)$ such that, for every subset \{$(c_{u,n} : u \in D, n \in N_0)$\} of $\mathbb{C}$ such that $|c_{u,n}| \leq d_{u,n}$ for every $u \in D$ and $n \in N_0$, there is $f \in A(\Omega; D)$ such that $f(z) \approx \sum_{n=0}^{\infty} c_{u,n} (z-u)^n$ at every $u \in D$.

Then there is $\rho \in N^N$ such that, for every subset \{$(c_{u,n} : u \in D, n \in N_0)$\} as above, there is $g \in B_{D,r_1,\rho}$ such that $g(z) \approx \sum_{n=0}^{\infty} c_{u,n} (z-u)^n$ at every $u \in D$.

Proof: The set

$$ C = \{(c_{u,n} : (u,n) \in D \times N_0) : |c_{u,n}| \leq d_{u,n} \quad \text{for all} \quad (u,n) \in D \times N_0\} $$

is clearly an absolutely convex and compact subset of $\omega(D \times N_0)$. By the Proposition 12/(b) of [3], there are then $\rho \in N^N$ and a subset $M$ of $B_{D,r_1,\rho}$ such that $TM = C$.

Hence the conclusion.

4. If $D$ is infinite

Let us recall that every subset $D$ of $\partial \Omega$ which is regularly asymptotic for $\Omega$ is countable.

Notations: Let $D = \{u_j : j \in N\}$. Then:
1) For every $\rho = (r_j)_{n \in N} \in N^N$, we set

$$ A_\rho(\Omega; D) = \bigcap_{j=1}^{\infty} A_{r_j}(\Omega; \{u_j\}) $$

$$ = \left\{ f \in A(\Omega; D) : \sup_{z \in \Omega, |z-u_j| \leq 1/r_j} |f(z)| < \infty \quad \text{for all} \quad j \in N \right\}. $$

So $A_\rho(\Omega; D)$ is a vector subspace of $A(\Omega; D)$ and

$$ \bigcup_{\rho \in N^N} A_\rho(\Omega; D) = A(\Omega; D). $$

2) For every $\rho \in N^N$ and $n \in N$,

$$ p_{D,\rho,n} : A_\rho(\Omega; D) \to [0, +\infty), \quad f \mapsto \sup\{p_{u_j,r_j,n} : j \in \{1, \ldots, n\}\} $$

is a norm on $A_\rho(\Omega; D)$ and

$$ P_{D,\rho} = \{p_{D,\rho,n} : n \in N\} $$

is a countable system of norms on $A_\rho(\Omega; D)$ endowing it with a finer locally convex topology than the one induced by $H(\Omega)$. From now on, if $D = \{u_j : j \in N\}$ and if $\rho \in N^N$, the notation $A_\rho(\Omega; D)$ will refer to the locally convex space $(A_\rho(\Omega; D), P_{D,\rho})$. So

$$ V_{D,\rho,n} = \left\{ f \in A_\rho(\Omega; D) : p_{D,\rho,n}(f) \leq \frac{1}{n} \right\} \quad (n \in N). $$
is a fundamental sequence of neighbourhoods of the origin in \((A_\rho(\Omega; D), P_{D, \rho})\).

3) We fix once for all an infinite partition

\[ \{\{n_k : k \in \mathbb{N}\} : j \in \mathbb{N}\} \]

of \(\{2n : n \in \mathbb{N}\}\). Then, for every \(\rho \in \mathbb{N}^\mathbb{N}\),

\[ B_{D, \rho} = \bigcap_{j=1}^{\infty} B_{u_j, r_{nj-1}, \rho(j)} \]

where \(\rho(j)\) is the sequence \((r_{nj,k})_{k \in \mathbb{N}}\). Of course, \(B_{D, \rho}\) is an absolutely convex and compact subset of \(A(\Omega; D)\); moreover it is a bounded subset of the space \((A_{\rho'}(\Omega; D), P_{D, \rho'})\) where \(\rho'\) is the sequence \((r_{2n-1})_{n \in \mathbb{N}}\).

Let us remark that these notations depend heavily on the enumeration of the points of \(D\) but any enumeration will do.

**Proposition 4.1:** If \(D = \{u_j : j \in \mathbb{N}\}\), then, for every \(\rho \in \mathbb{N}^\mathbb{N}\), \(A_\rho(\Omega; D)\) is a Fréchet space.

**Proof:** One can go on with a direct proof as in Proposition 3.1 or use the technique of the (LB)-spaces as we do hereafter. \(\blacksquare\)

**Proposition 4.2:** If \(D = \{u_j : j \in \mathbb{N}\}\), the family \(\{B_{D, \rho} : \rho \in \mathbb{N}^\mathbb{N}\}\) is a quasi-LB representation of the space \(A(\Omega; D)\) made of absolutely convex and compact sets.

Now as in [31], once again, for every \(\rho \in \mathbb{N}^\mathbb{N}\), we may successively introduce the sets

\[ B_{D, r_1, \ldots, r_n} = \bigcup \{B_{D, \sigma} : \sigma \in \mathbb{N}^N, s_1 = r_1, \ldots, s_n = r_n\} \]

\[ F_{D, r_1, \ldots, r_n} = \text{span} B_{D, r_1, \ldots, r_n} \]

for every \(n \in \mathbb{N}\) and

\[ F_{D, \rho} = \bigcap_{n=1}^{\infty} F_{D, r_1, \ldots, r_n}. \]

One can describe directly these sets in our case. In particular, one gets \(F_{D, \rho} = A_{\rho'}(\Omega; D)\). This provides another way to establish that, for every \(\rho \in \mathbb{N}^\mathbb{N}\), \(A_\rho(\Omega; D)\) is a Fréchet space.

**Definition:** If \(D = \{u_j : j \in \mathbb{N}\}\), we have

\[ A(\Omega; D) = \bigcup_{\rho \in \mathbb{N}^\mathbb{N}} A_\rho(\Omega; D) \]

and, for every \(\rho, \sigma \in \mathbb{N}^\mathbb{N}\) such that \(\rho \leq \sigma\), the canonical injection from \(A_\rho(\Omega; D)\) into \(A_\sigma(\Omega; D)\) is continuous. Therefore we may endow \(A(\Omega; D)\) with the locally convex topology \(\tau\) of the inductive limit of these Fréchet spaces. Of course \(\tau\) is finer than the topology of \(A(\Omega; D)\) but we must insist on the fact that \((A(\Omega; D), \tau)\) is not an (LF)-space.

However some results similar to those obtained in the case when \(D\) is finite, can be established.
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Definition: If \( \rho = (r_n)_{n \in \mathbb{N}} \in \mathbb{N}^\mathbb{N} \), the subset \( D = \{ u_j : j \in \mathbb{N} \} \) of \( \partial \Omega \) is \( \rho \)-regularly asymptotic for \( \Omega \) if the restriction of the map \( T \) to \( A_\rho(\Omega; D) \) is surjective.

Theorem 4.3: Let \( D = \{ u_j : j \in \mathbb{N} \} \) be a subset of \( \partial \Omega \) having no accumulation point. The following conditions are equivalent:

(a) \( D \) is regularly asymptotic for \( \Omega \).
(b) For every \( j \in \mathbb{N} \) and every sequence \( (c_n)_{n \in \mathbb{N}} \) of complex numbers, there is \( f \in A(\Omega; D) \) such that \( f(z) \approx \sum_{n=0}^{\infty} c_n (z - u_j)^n \) at \( u_j \).
(c) There is \( \rho \in \mathbb{N}^\mathbb{N} \) such that \( D \) is \( \rho \)-regularly asymptotic for \( \Omega \).

Proof: (a) \( \Rightarrow \) (b) and (c) \( \Rightarrow \) (a) are trivial.

(b) \( \Rightarrow \) (c): Let us first introduce a sequence \( \rho \in \mathbb{N}^\mathbb{N} \). As \( D \) has no accumulation point, there is a sequence \( \tau = (t_j)_{j \in \mathbb{N}} \) such that the disks \( \{ z \in \mathbb{C} : |z - u_j| \leq 1/t_j \} \) are pairwise disjoint. We set \( r_1 = t_1 \) and obtain the other elements as follows, by induction. If \( r_1, \ldots, r_m \) are determined, we denote by \( d_m \) the distance of \( u_{m+1} \) to

\[
H_m = K_m \cup \left( \bigcup_{j=1}^{m} \left\{ z \in \mathbb{C} : |z - u_j| \leq \frac{1}{t_j} \right\} \right).
\]

As \( \{ u_{m+1} \} \) is a regularly asymptotic set for \( \Omega \), \( u_{m+1} \) is not an isolated point in \( \partial \Omega \); therefore there is \( v_m \in \partial \Omega \) such that

\[
0 < |v_m - u_{m+1}| < \inf \left\{ \frac{d_m}{4}, \frac{1}{t_{m+1}} \right\} \quad \text{and} \quad d(v_m, H_m) \geq \frac{d_m}{2}.
\]

Then we choose \( r_{m+1} \) in \( \mathbb{N} \) such that \( r_{m+1} \geq t_{m+1} \) and \( 1/r_{m+1} < |v_m - u_{m+1}| \).

As \( \{ \eta_{u,n} : u \in D, n \in \mathbb{N}_0 \} \) is a subset of the dual space \( (A_\rho(\Omega; D), P_{D,\rho})' \), the elements of which are linearly independent, by Theorem 1.1, all we need to prove is that, for every \( s \in \mathbb{N} \), the vector space

\[
\text{span}\{ \eta_{u,n} : u \in D, n \in \mathbb{N}_0 \} \cap \text{span} V^\circ_{D,\rho,s}
\]

has finite dimension. Let us fix \( s \in \mathbb{N} \) and, to simplify the notations, let us set \( q = P_{D,\rho,s} \) and \( U = V_{D,\rho,s} \). For every \( j \in \mathbb{N} \), we know already that

\[
\text{span}\{ \eta_{u_j,n} : n \in \mathbb{N}_0 \} \cap \text{span} U^\circ
\]

has finite dimension so there is an integer \( N(j) \in \mathbb{N} \) such that

\[
\left( \alpha_1, \ldots, \alpha_N \in \mathbb{C} : \alpha_N \neq 0; \; \sum_{n=0}^{N} \alpha_k \eta_{u_j,n} \in \text{span} U^\circ \right) \Rightarrow N \leq N(j).
\]

To conclude, it is then sufficient to prove that, if

\[
\zeta = \sum_{j=1}^{m+1} \sum_{n=0}^{N} \alpha_{j,n} \eta_{u_j,n} \quad \text{with} \; m \geq s
\]
belongs to span $U^o$, then a) and b) hereunder hold.

a) We have $\alpha_{m+1,n} = 0$ for every $n = 0, \ldots, N$. Indeed, if it is not the case, let $q \in \mathbb{N}$ be such that $\zeta \in qU^o$ and let $l$ be the greatest integer such that $\alpha_{m+1,l} \neq 0$. We then introduce the polynomial

$$P(z) = (z - u_1)^{m+N} \cdots (z - u_m)^{m+N}(z - u_{m+1})^l$$

and an integer $d$ larger than the diameter of $H_m$, choose an integer $t$ such that

$$|u_{m+1} - u_1|^{m+N} \cdots |u_{m+1} - u_m|^{m+N} 2^t > q(s + 2)d^{(m+N)(m+1)}$$

and finally set

$$g(z) = P(z) \cdot \left( \frac{d_m}{2(z - v_m)} \right)^t$$

for all $z \in \Omega$.

It is clear that $g$ belongs to $A_q(\Omega; D)$. Moreover from

$$\left| \frac{d_m}{2(z - v_m)} \right|^t \leq 1 \quad \text{for all } z \in H_m$$

we easily get

$$q(g) \leq \sup_{1 \leq j \leq s} p_{u_j, r_j, s}(g)$$

$$\leq d^{(m+N)(m+1)} + (s + 1)d^{(m+N)(m+1)}$$

$$= (s + 2)d^{(m+N)(m+1)}$$

hence $g$ belongs to $s(s + 2)d^{(m+N)(m+1)}U$ therefore

$$|(g, \zeta)| \leq q(s + 2)d^{(m+N)(m+1)}.$$

But we also have

$$\left| \frac{d_m}{2(u_{m+1} - v_m)} \right|^t > 2^t$$

hence

$$|(g, \zeta)| = \left| \sum_{j=1}^{m+1} \sum_{n=0}^{N} \alpha_{j,n} g^{[n]}(u_j) \right|$$

$$= \left| g^{[l]}(u_{m+1}) \right|$$

$$= |u_{m+1} - u_1|^{m+N} \cdots |u_{m+1} - u_m|^{m+N} \left| \frac{d_m}{2(u_{m+1} - v_m)} \right|^t$$

$$> |u_{m+1} - u_1|^{m+N} \cdots |u_{m+1} - u_m|^{m+N} \cdot 2^t$$

$$> q(s + 2)d^{(m+N)(m+1)}.$$

Hence a contradiction.

b) We have $\alpha_{j,N} = 0$ if $N > \sup\{s, N(1), \ldots, N(s)\}$, by a proof very similar to the one of Theorem 3.5.

The proof is now complete.
Corollary 4.4: If $D = \{u_j : j \in \mathbb{N}\}$ is regularly asymptotic for $\Omega$, then the kernel of $T$ has $2^{\aleph_0}$ as algebraic dimension.

Proof: The proof of Corollary 3.4 applies here too: one has just to replace $r \in \mathbb{N}$ by some appropriate $\rho \in \mathbb{N}^\mathbb{N}$ and the space $A_r(\Omega; D)$ by $A_\rho(\Omega; D)$.

Proposition 4.5: Let $D = \{u_j : j \in \mathbb{N}\}$ be a subset of $\partial \Omega$ having no accumulation point. This implies the existence of a sequence $\tau = (t_j)_{j \in \mathbb{N}} \in \mathbb{N}^\mathbb{N}$ such that the disks $\{z \in \mathbb{C} : |z - u_j| \leq 1/t_j\}$ are pairwise disjoint. Then the following conditions on a sequence $\rho \in \mathbb{N}^\mathbb{N}$ verifying $\rho \geq \tau$ are equivalent:

(a) $D$ is $\rho$-regularly asymptotic for $\Omega$.

(b) For every compact subset $K$ of $\Omega$ and every $j_0 \in \mathbb{N}$, there is $p \in \mathbb{N}$ such that, for every $h > 0$, there is $f \in A_\rho(\Omega; D)$ verifying

$$|f(z)| \leq 1 \quad \text{for all } z \in K \cup \left( \bigcup_{j=1}^p \left\{ u \in \Omega : |u - u_j| \leq \frac{1}{r_j} \right\} \right),$$

and

$$|f^{(p)}(u_{j_0})| > h.$$

Proof: The proof of (a) $\Rightarrow$ (b) is essentially the same as the one of the necessity of the condition in Theorem 3.6: one just has to replace $A_r(\Omega; D)$ by $A_\rho(\Omega; D)$.

Slight modifications to the proof of the sufficiency of the condition in Theorem 3.6 give (b) $\Rightarrow$ (a). One just needs to replace $A_r(\Omega; D)$ by $A_\rho(\Omega; D)$, to fix $j$ in $\mathbb{N}$, to impose moreover the condition $s > j$ on $s$, to replace $V_{\tau, r, s}$ by $V_{\rho, p, s}$, to set

$$\mathcal{K} = K \cup \left( \bigcup_{j=1}^p \left\{ z \in \Omega : |z - u_j| \leq \frac{1}{r_j} \right\} \right)$$

and

$$P(z) = (z - u_j)^{N-p_j} \prod_{1 \leq k \leq s, k \neq j} (z - u_k)^N,$$

and to replace $p_{\tau, r, s}$ by $p_{\rho, p, s}$.

Theorem 4.6: Let $D = \{u_j : j \in \mathbb{N}\}$ be a subset of $\partial \Omega$ having no accumulation point. This implies the existence of a sequence $\rho = (r_n)_{n \in \mathbb{N}} \in \mathbb{N}^\mathbb{N}$ such that the disks $\{z \in \mathbb{C} : |z - u_j| \leq 1/r_j\}$ for $j \in \mathbb{N}$ are pairwise disjoint.

If $\partial \Omega$ is quasi-connected at every point of $D$, then $D$ is $\rho$-regularly asymptotic, hence regularly asymptotic, for $\Omega$.

Proof: The proof is very similar to the one of Theorem 3.7. One fixes $j$ in $\mathbb{N}$, sets

$$\mathcal{K} = K \cup \left( \bigcup_{k=1}^j \left\{ t \in \mathbb{C} : |t - u_j| \leq \frac{1}{r_j} \right\} \right)$$

and chooses $A$ with the condition $\delta_2(u_j, A) < 1/r_j$ instead of $\delta_2(u_j, A) < 1/r$. The conclusion then follows from the preceding proposition.
5. Generalizations of Carleman’s results

We begin with the following

Theorem 5.1: If $D$ is a non-empty subset of $\partial \Omega$ having no accumulation point and if $\partial \Omega$ is quasi-connected at every point of $D$, then $D$ is regularly asymptotic for $\Omega$.

Moreover, for every family $\{c_{u,n} : u \in D, n \in \mathbb{N}\}$ of complex numbers, the set of the elements $f$ of $A(\Omega; D)$ such that $f(z) \approx \sum_{n=0}^{\infty} c_{u,n}(z - u)^n$ for every $u \in D$ is a linear variety of dimension $2^{N_0}$.

Proof: Having no accumulation point, $D$ must be countable. So if $D$ is finite, this is Theorem 3.7 and the Corollary 3.4, and if $D$ is infinite but countable, it is a trivial consequence of Theorem 4.6 and of Corollary 4.4.

The following result gives an easy way to verify that $\partial \Omega$ is quasi-connected at some point.

Proposition 5.2: If the connected component $C_u$ of $u \in \partial \Omega$ has more than one point, then $\partial \Omega$ is quasi-connected at $u$.

Proof: We are going to use twice the following property (cf. [6: (10.1)]): If $A$ is a closed, connected subset and $G$ a bounded, open subset of $\mathbb{C}$ such that $A \neq G \cap A \neq \emptyset$, then every connected component of $G \cap A$ has some point in the boundary of $G$.

Given $0 < \varepsilon < \delta < 1$, as $C_u$ contains more than one point, there is $r_1 \in (0, \delta/2)$ such that $C_1 = C_u \cap \{z \in \mathbb{C} : |z - u| \leq r_1\} \neq C_u$. Let $C$ be the connected component of $C_1$ containing $u$. Of course, $C_u$ is a closed, connected subset and $b = \{z \in \mathbb{C} : |z - u| < r_1\}$ is a bounded, open subset of $\mathbb{C}$ such that $C_u \neq b \cap C_u \neq \emptyset$. Therefore there is $z_1 \in C$ such that $|z_1 - u| = r_1$. Now we chose $r_2, r_3 > 0$ such that $r_2 < \varepsilon r_1$ and $r_1 < r_3$, and set $G = \{z \in \mathbb{C} : r_2 < |z - u| < r_3\}$. So we have $u \notin G$ and $z_1 \in G$, hence $C \neq G \cap C \neq \emptyset$ and the connected component $P$ of $C \cap G$ containing $z_1$ contains a point $z_2$ of the boundary of $G$. As $|z_2 - u| \neq r_3$, we must have $|z_2 - u| = r_2$. Therefore $P$ is a connected subset of $\partial \Omega$ such that $\delta_1(u, P) = |z_2 - u| = r_2$ and $\delta_2(u, P) = |z_1 - u| = r_1$ hence $\delta_2(u, P) = r_1 < \delta$ and $0 < \delta_1(u, P) = r_2 < \varepsilon r_1 = \varepsilon \delta_2(u, P)$.

Combining the previous two results, we get the following statement which constitutes the practical form of our result. Let us mention that it generalizes Proposition 10 of [4]:

If the connected component of $u \in \partial \Omega$ has more than one point, then $\{u\}$ is regularly asymptotic for $\Omega$

as well as Corollary 1 of [4]:

If $\Omega$ is simply connected, then every point of $\partial \Omega$ is regularly asymptotic for $\Omega$.

Theorem 5.3: If $D$ is a non-empty subset of $\partial \Omega$ having no accumulation point and if the connected component of every point of $D$ in $\partial \Omega$ has more than one point, then $D$ is regularly asymptotic for $\Omega$. 
Moreover, for every family \( \{ c_{u,n} : u \in D, n \in \mathbb{N} \} \) of complex numbers, the set of the elements \( f \) of \( A(\Omega; D) \) such that \( f(z) \approx \sum_{n=0}^{\infty} c_{u,n}(z - u)^n \) at every \( u \in D \) is a linear variety of dimension \( 2^{N_v} \).

References


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