Zeitschrift für Analysis
und ihre Anwendungen
Vol. 12 (1992), 751-758

A Class of One-Dimensional Variational Inequalities and Difference Schemes of Arbitrary Given Degree of Accuracy

I. P. GAWRILYUK

A new class of one-dimensional variational inequalities with obstruction on the boundary is investigated and a theorem of existence and uniqueness of solution is proved. The investigated properties of the exact solution give an opportunity to construct the three-point difference relations for this solution (exact difference scheme). A numerical three-point approximation of arbitrary given degree of accuracy (truncated difference scheme) is proposed.

Key words: *One-dimensional variational inequalities, exact and truncated difference schemes*
AMS subject classifications: 65 L 10, 65 K 10, 90 C 20

1. Introduction

The problems of determination of the temperature distribution in a long tunnel with air conditioning or determination of the fluid (gas) pressure in a long pipe with semipermeable endwalls imbedded into corresponding fluid (gas) surroundings are of practical interest. If u denotes the temperature (pressure), then under the assumptions of stationary state and of sufficient smoothness of the functions one can derive the following mathematical model (see [1,5]):

$$\begin{aligned} &Lu = -(k(x)u'(x))' + q(x)u(x) = f(x), \quad x \in (0,1) \\ &-u(0)u'(0) = u(1)u'(1) = 0, \quad u(0), -u'(0), u(1), u'(1) \geq 0, \end{aligned} \tag{1.1}$$

where f, k, q are given functions, $k(x) \geq k_0 = \text{const} > 0$, $q(x) \geq 0$. Let us consider the bilinear form

$$a^0(u,v) = \int_0^1 \Big(k(x)u'(x)v'(x) + q(x)u(x)v(x)\Big)dx,$$

the linear functional

$$l^0(v) = \int_0^1 f(x)v(x)\,dx$$

defined on certain space V and the closed convex set of functions

$$K = \{v \in V:\ v(0), v(1) \geq 0\}.$$

I. P. Gawrilyuk: Univ. Leipzig, Inst. Math., Augustuspl. 10-11, D - 04109 Leipzig
ISSN 0232-2064 / $ 2.50

It is well-known that problem (1.1) can be formulated as problem of minimization (see [1,5])

$$\text{Find } u \in K \text{ such that } J(u) = \inf_{v \in K} J(v), \quad J(v) = a^o(v,v) - 2\, l^o(v) \tag{1.2}$$

or as problem of solving the variational inequality

$$\text{Find } u \in K \text{ such that } a^o(u, v-u) \geq l^o(v-u) \quad \text{for all } v \in K. \tag{1.3}$$

The approximation of solutions of (1.1) - (1.3) by various numerical schemes can be done as in [5]. Our goal is to generalize the statement of the problem (1.3) and to construct three-point difference approximations which are either exact or have arbitrary given degree of accuracy on the network. We give minimal conditions (as far as we know) of smoothness of input data which guarantee the existence of a unique solution from the class $W_2^1(0,1)$ of the corresponding variational inequality and construct for this problem the exact and truncated difference schemes. Such schemes for the linear boundary value problems in classical and variational formulations have been considered in [8,9] and for the problem (1.1) in [2].

2. Formulation of the problem, existence and uniqueness of solution

Let us consider the bilinear form

$$\begin{aligned} a(u,v) &= a_{[0,1]}(u,v) \\ &= \int_0^1 \left(k(x)u'(x)v'(x) - Q(x)\big(u(x)v(x)\big)'\right)dx + Q(1)u(1)v(1) - Q(0)u(0)v(0) \end{aligned} \tag{2.1}$$

and the linear functional

$$l(v) = l_{[0,1]}(v) = \int_0^1 \big(f_0(x)v(x) - f_1(x)v'(x)\big)dx \tag{2.2}$$

defined on the Sobolev space $W_2^1(0,1)$ where f_0, f_1, k, Q are given functions satisfying the following conditions, where $\bar{K} = \{v \in W_2^1(0,1)\colon v(x) \geq 0\}$:

$$k \text{ is measurable, } 0 < k_0 \leq k(x) \leq k_1 < +\infty, \text{ where } k_0, k_1 \text{ some constants} \tag{2.3}$$

$$Q \in W_p^{\lambda}(0,1) \quad (p \geq 2,\ 1/p < \lambda \leq 1) \tag{2.4}$$

$$f_0 \in L_q(0,1) \text{ and } f_1 \in W_r^{\vartheta}(0,1) \quad (q, r \geq 2,\ 0 < \vartheta \leq 1) \tag{2.5}$$

$$-\int_0^1 Q(x)v'(x)\,dx + Q(1)v(1) - Q(0)v(0) \geq q_0 \int_0^1 v(x)\,dx \quad \forall\, v \in \bar{K},\ q_0 \geq 0 \text{ a constant.} \tag{2.6}$$

We first study some properties of the bilinear form and connected linear functionals.

In analogy with [4], for the case $q_0 > 0$ one can obtain

$$|a(u,v)| \leq c\, \|u\|_{W_2^1} \|v\|_{W_2^1}$$

$$a(v,v) \geq \alpha \|v\|^2_{W_2^1} \ \left(\alpha = \min\{k_0, q_0\}\right) \quad \text{and} \quad |l(v)| \leq \beta \|v\|_{W_2^1} \ \left(\alpha = \max\{\|f_0\|_{L_2}, \|f_1\|_{L_2}\}\right)$$

where the constant c does not depend on u and v. Hence for the functional $J(v) = a(v,v) - 2\, l(v)$

there holds

$$J(v) \geq \|v\|^2_{W_2^1} - 2\beta\|v\|_{W_2^1} \to +\infty \text{ when } \|v\|_{W_2^1} \to +\infty. \tag{2.8}$$

Let $q_0 = 0$, i.e. instead of (2.6) the inequality

$$-\int_0^1 Q(x)v'(x)\,dx + Q(1)v(1) - Q(0)v(0) \geq 0 \quad \text{for all } v \in \bar{K} \tag{2.9}$$

is true. Let us clarify the sufficient condition which implies $J(v) \to +\infty$ when $\|v\|_{W_2^1} \to +\infty$. Set $\tilde{v}(x) = v(x) - v(0)$, where $\tilde{v}(0) = 0$, $\tilde{v} \in W_2^1(0,1)$ for $v \in W_2^1(0,1)$. For such functions we have the inequality

$$\|\tilde{v}\|^2_{L_2} = \int_0^1\left(\int_0^x \tilde{v}'(s)\,ds\right)^2 dx \leq \int_0^1 x\int_0^x(\tilde{v}'(s))^2 ds\,dx \leq 0{,}5\,|\tilde{v}|^2_{W_2^1}. \tag{2.10}$$

Since $v(x) = \tilde{v}(x) + v(0)$ we can write

$$J(v) = \int_0^1 k(x)(\tilde{v}'(x))^2 dx - \int_0^1 Q(x)(v^2(x))'\,dx + Q(1)v^2(1) - Q(0)v^2(0)$$

$$-2\int_0^1\Big(f_0(x)\tilde{v}(x) - f_1(x)\tilde{v}'(x)\Big)dx - 2v(0)\int_0^1 f_0(x)\,dx.$$

By virtue of (2.3) - (2.5), (2.9), (2.7) we further obtain

$$J(v) \geq k_0|\tilde{v}|^2_{W_2^1} - 2\beta\|\tilde{v}\|_{W_2^1} - 2v(0)\int_0^1 f_0(x)\,dx. \tag{2.11}$$

In view of (2.10) we can write

$$|\tilde{v}|^2_{W_2^1} = \xi|\tilde{v}|^2_{W_2^1} + (1-\xi)|\tilde{v}|^2_{W_2^1} \geq 2\xi\|\tilde{v}\|^2_{L_2} + (1-\xi)|\tilde{v}|^2_{W_2^1}$$

$$\geq \max_{\xi\in[0,1]} \min\{2\xi, 1-\xi\}\,\|\tilde{v}\|_{W_2^1} = \tfrac{2}{3}\|\tilde{v}\|^2_{W_2^1}$$

and now from (2.11) we obtain

$$J(v) \geq \tfrac{2}{3}k_0\|\tilde{v}\|^2_{W_2^1} - 2\beta\|\tilde{v}\|_{W_2^1} - 2v(0)\int_0^1 f_0(x)\,dx.$$

Since $\tilde{v}(x) = v(x) - v(0)$, for $v(0) \geq 0$ there follows $\big|\,\|v\|_{W_2^1} - v(0)\big| \leq \|\tilde{v}\|_{W_2^1}$. It is obvious that when $\|v\|_{W_2^1} \to +\infty$ two cases are possible:

a) $\big|\,\|v\|_{W_2^1} - v(0)\big| \leq c < +\infty$ (c a constant) which yields $v(0) \to +\infty$.

b) $\big|\,\|v\|_{W_2^1} - v(0)\big| \to +\infty$, which yields $\|\tilde{v}\|_{W_2^1} \to +\infty$.

By virtue of (2.12) in both cases $J(v) \to +\infty$ if $\|v\|_{W_2^1} \to +\infty$, $v(0) \geq 0$ and the condition

$$\int_0^1 f_0(x)\,dx < 0 \tag{2.13}$$

holds. Let us state the following problem, where $K = \{v \in W_2^1(0,1): v(0), v(1) \geq 0\}$:

(P1) Find an element $u \in K$ such that $J(u) = \inf_{v \in K} J(v)$.

This problem is equivalent to the variational inequality (see [5])

(P2) Find an element $u \in K$ such that $a(u, v-u) \geq l(v-u)$ for all $v \in K$.

Now we can formulate the following assertion.

Theorem 2.1: *Let* $a(u,v)$, $l(v)$ *in* (P1), (P2) *be defined as in* (2.1), (2.2) *and the conditions* (2.3) - (2.6) *with* $q_0 > 0$ *or the conditions* (2.3) - (2.5), (2.9), (2.13) *with* $Q(0) \neq Q(1)$ *be satisfied. Then the problems* (P1), (P2) *have unique solutions.*

Proof: The existence of a solution follows from the above mentioned properties of the bilinear form $a(u,v)$, the linear functionals $l(v)$, $J(v)$ and from [5: Theorem 2.1]. If the conditions (2.3) - (2.6) with $q_0 > 0$ are satisfied, then the inequalities (2.7) yield the strict convexity of the functional $J(v)$ and due to [5: Theorem 2.2] the solution of the problems (P1) and (P2) are unique.

Let us assume that instead of (2.6) with $q_0 > 0$ the conditions (2.9), (2.13) are true which together with conditions (2.3) - (2.5) guarantee $J(v) \to +\infty$ when $\|v\|_{W_2^1} \to +\infty$ and therefore the existence of a solution. Further we give the proof of uniqueness by contradiction. Let u_1, u_2 denote two solutions of the problem (P2), namely $u_1 \neq u_2$. Then

$$a(u_1, v-u_1) \geq l(v-u_1) \quad \text{and} \quad a(u_2, v-u_2) \geq l(v-u_2) \quad \text{for all } v \in K.$$

After substitution $v = u_2$ in the first inequality and $v = u_1$ in the second one and summation we have $-a(w,w) \geq 0$ or $a(w,w) = 0$ where $w = u_1 - u_2$. As a result of the equality $a(w,w) = 0$ and (2.9) we obtain

$$\int_0^1 k(x)(w'(x))^2 dx = 0 \qquad \text{and} \qquad -\int_0^1 Q(x)(w^2(x))' dx + Q(1)w(1) - Q(0)w(0) = 0.$$

The first of these equalities yields $w(x) = c = \text{const}$. It follows from the second equality that $c(Q(1) - Q(0)) = 0$ and since $Q(0) \neq Q(1)$ we have $c = 0$. Hence $w = u_1 - u_2 = 0$ ∎

Let us consider the following problems:

(P3) Find $u_i \in K_i$ such that $J(u_i) = \inf_{v \in K_i} J(v) \quad (i = 0(1)4)$

where

$$K_0 = \{v \in W_2^1(0,1): v(0) = v(1) = 0\} \qquad K_1 = \{v \in W_2^1(0,1): v(0) = 0\}$$

$$K_2 = \{v \in W_2^1(0,1): v(1) = 0\} \qquad K_3 = W_2^1(0,1).$$

Since K_i are linear subspaces of $W_2^1(0,1)$ the problems (P3) can also be formulated in the following form:

(P4) Find Elements $u_i \in K_i$ such that $a(u_i, v) = l(v)$ for all $v \in K_i \quad (i = 0(1)4)$.

Due to the properties of the bilinear form $a(u,v)$ and the linear functionals $l(v)$, $J(v)$ each of the problems (P3) or (P4) have a unique solution.

Theorem 2.2: *Let the conditions of the Theorem* 2.1 *hold. Then the solution of the problem* (P1) *coincodes with one of the solutions of the problems* (P3) ((P4)).

Proof: *Let* $u^* \in K \subset W_2^1(0,1)$ be the unique solution of the problem (P1). We assume that $u^* \in K_1$, i.e. $u^*(0) = 0$ and $u^*(1) \geq 0$ (other possible cases are: $u^* \in K_2$ with $u^*(0) > 0$ and $u^*(1) = 0$; $u^* \in K_3$ with $u^*(0) > 0$ and $u^*(1) > 0$; $u^* \in K_0$ with $u^*(0) = 0$ and $u^*(1) = 0$). We shall prove that $J(u^*) = \min_{v \in K_1} J(v)$. Suppose (proof by contradiction) that the problem (P3) for $i = 1$ has a solution $u_1 \neq u^*$. If $u_1(1) \geq 0$, then we have $u_1 \in K$, $u^* \in K_1$ and as a result of the uniqueness theorem we obtain the contradictory inequalities

$$J(u_1) = \inf_{v \in K_1} J(v) < J(u^*) = \inf_{v \in K} J(v) < J(u_1).$$

Hence $u_1(1) < 0$. The imbedding theorem $W_2^1 \subset C$ implies the continuity of the function u_1 on $[0,1]$ therefore $u_1(x)$ vanishes at least in one point of this interval. Let $\bar{x}$ be the maximal of these points and let us put

$$l_{[\alpha,\beta]}(v) = \int_\alpha^\beta \big(f_0(x)v(x) - f_1(x)v'(x)\big)dx$$

$$a_{[\alpha,\beta]}(u,v) = \int_\alpha^\beta \Big(k(x)u'(x)v'(x) - Q(x)\big(u(x)v(x)\big)'\Big)dx + Q(\beta)u(\beta)v(\beta) - Q(\alpha)u(\alpha)v(\alpha)$$

$$J_{[\alpha,\beta]}(v) = a_{[\alpha,\beta]}(v,v) - 2\, l_{[\alpha,\beta]}(v).$$

Then

$$J(v) = J_{[0,1]}(v) = J_{[0,\bar{x}]}(v) + J_{[\bar{x},1]}(v).$$

Suppose $J_{[\bar{x},1]}(u_1) \geq 0$. In this case we consider the function $\bar{u} = u_1 \chi_{[0,\bar{x}]}$, where χ_M denotes the characteristic function of the set M. It is obvious that $\bar{u} \in K_1$ and

$$J(\bar{u}) = J_{[0,\bar{x}]}(\bar{u}) + J_{[\bar{x},1]}(\bar{u}) = J_{[0,\bar{x}]}(u_1) \leq J_{[0,\bar{x}]}(u_1) + J_{[\bar{x},1]}(u_1) = J(u_1) = \inf_{v \in K_1} J(v).$$

Since $\bar{u} \neq u_1$ we have obtained a contradiction. In the case $J_{[\bar{x},1]}(u_1) < 0$ for the function $\bar{u} = u_1 \times \chi_{[0,\bar{x}]} + c\, u_1 \chi_{[\bar{x},1]}$, $c > 1$ we arrive at the conclusions $\bar{u} \in K_1$ and

$$J(\bar{u}) = J_{[0,\bar{x}]}(u_1) + c\, J_{[\bar{x},1]}(u_1) < J_{[0,\bar{x}]}(u_1) + J_{[\bar{x},1]}(u_1) = J(u_1) = \inf_{v \in K_1} J(v)$$

and again we have a contradiction. Considering the other three possible cases in a similar way we obtain the complete proof ∎

Theorem 2.3: *Let u be the solution of problem* (P2). *Then* $u(x) = \max\{ u_i(x) : i = 1(1)3\}$ for $x \in [0,1]$, *where u_i are solutions of problem* (P4).

The **proof** of this theorem is completely similar to that of [6: Theorem 42.3] ∎

3. Exact and truncated difference schemes for variational inequality and their properties

For each problem (P4) ((P3)) analogous to [4] we can construct the following exact scheme:

$$\Lambda u_i = -\varphi(x) \text{ for } x \in \omega_h \qquad (i = 0(1)3) \tag{3.1}$$
$$l_0(\Lambda_0 u_i, u_i) = 0 \quad \text{and} \quad l_1(\Lambda_1 u_i, u_i) = 0$$

where

$$\omega_h = \{x_i = ih: i = 1(1)(N-1),\ h = 1/N\},\quad \overline{\omega}_h = \omega_h \cup \{0, 1\}$$

$$l_0(\Lambda_0 u_i, u_i) = K_0^i u_i(0) + (1 - K_0^i)\Lambda_0 u_i \quad \text{and} \quad l_1(\Lambda_1 u_i, u_i) = K_1^i u_i(1) + (1 - K_1^i)\Lambda_1 u_i$$

$$K_0^i = \begin{cases} 0 & \text{for } i = 2, 3 \\ 1 & \text{for } i = 0, 1 \end{cases} \quad \text{and} \quad K_1^i = \begin{cases} 0 & \text{for } i = 1, 3 \\ 1 & \text{for } i = 0, 2 \end{cases}$$

$$\Lambda u = (a u_{\bar{x}})_x - du,\quad \Lambda_0 u = -u_x(0) + \varkappa_0^h u(0) + \mu_0^h,\quad \Lambda_1 u = u_{\bar{x}}(1) + \varkappa_1^h u(1) + \mu_1^h$$

and the coefficients $a, d, \varkappa_i^h, \mu_i^h$ $(i = 0,1)$ are defined in [4] by means of the solutions $v_1^i(x)$, $v_2^i(x)$, $x \in e_i = (x_{i-1}, x_{i+1})$, $e_0 = (0, x_1)$, $e_N = (x_{N-1}, 1)$ of generalized Cauchy problems.

Let u be the solution of problem (P2). Due to the properties of the functions v_1^i, v_2^i we have $v_2^0, v_1^N \in K$. Since K is a cone one can obtain from (P2)

$$a(u, v) \geq l(v) \quad \text{for all } v \in K. \tag{3.2}$$

Substituting in (3.2) by turns $v = v_2^0$ and $v = v_1^N$ we find $\Lambda_0 u, \Lambda_1 u \geq 0$. Theorem 2.2 states that the function u coinsides with one of the functions u_i. For this reason (3.1) implies the following relations for the function u:

$$\Lambda u = -\varphi(x) \quad \text{for } x \in \omega_h \tag{3.3}$$
$$u(0)\Lambda_0 u = 0,\ u(1)\Lambda_1 u = 0 \ \text{ and } \ u(0), \Lambda_0 u, u(1), \Lambda_1 u \geq 0.$$

Analogously to [8] one can prove that the exact three-point scheme (3.1) for each problem (P4) is unique, hence the exact three-point difference scheme (3.3) for problem (P2) is unique as well. The question about uniqueness of the solution of exact difference schemes we shall consider later.

A constructive approach to the exact difference scheme (3.3) is the truncated difference scheme of rank m

$$\Lambda^{(m)} y^{(m)} = -\varphi^{(m)}(x) \quad \text{for } x \in \omega_h$$
$$y^{(m)}(0)\Lambda_0^{(m)} y^{(m)} = 0 \quad \text{and} \quad y^{(m)}(1)\Lambda_1^{(m)} y^{(m)} = 0 \tag{3.4}$$
$$y^{(m)}(0), \Lambda_0^{(m)} y^{(m)}, y^{(m)}(1), \Lambda_1^{(m)} y^{(m)} \geq 0$$

which one constructs as in [4]. If the solution of the exact difference scheme (3.4) exists, then it obviously coinsides with one of the solutions of the problems

$$\Lambda^{(m)} y_i^{(m)} = -\varphi^{(m)}(x) \quad \text{for } x \in \omega_h$$
$$l_0(\Lambda_0^{(m)} y_i^{(m)}, y_i^{(m)}) = 0 \quad \text{and} \quad l_1(\Lambda_1^{(m)} y_i^{(m)}, y_i^{(m)}) = 0 \quad \text{for } i = 0(1)3. \tag{3.5}$$

In analogy with [4] one can prove that under the assumptions (2.3) - (2.6) with $q_0 > 0$ the following estimates are valid:

$$\|\rho^{-1/p_0}(u_i - y_i^{(m)})\|_{\infty,\omega} \le c\, h^{2(m+1)-n_i} F(m,h) \quad \text{for } i = 0(1)3 \tag{3.6}$$

where

$\rho(x) = x(1-x)$ and $\|u\|_{\infty,\omega} = \max\{|u(x)| : x \in \overline{\omega}_h\}$,

$n_0 \equiv n_0(p_1, p_2, p, \lambda, r, \vartheta, q, m) = \max\{n_\varphi, n_d - \vartheta + 1, n_a - \vartheta + 1\}$

$n_i \equiv n_i(p_1, p_2, p, \lambda, r, \vartheta, q, m) = \max\{n_\varphi, n_d - \vartheta + 1, n_a + n_{\varphi\mu}, n_\varkappa + n_{\varphi\mu} - 1 - 1/p_2, n_\mu\}$

$p_0^{-1} + p_1^{-1} + p_2^{-1} = 1$ with $p_0, p_1, p_2 \ge 1$

$n_\varphi, n_d, n_a, n_\mu, n_\varkappa, n_{\varphi\mu}$ are defined in [4]

$F_i(m,h) = F_i(m,h,k,Q,f_0,f_1)$ are bounded or vanish when $h \to 0$

c is a constant independent of h.

Further we shall prove some properties of the exact and truncated difference schemes.

Theorem 3.1: *Let the conditions* (2.3) - (2.6) *with* $q_0 > 0$ *hold. Then a solution of the truncated difference scheme* (3.4) *exists.*

Proof: We first remark that under the assumptions made the following estimates are valid (see [4]):

$$0 < c_0^{-1} \le a(x), a^{(m)}(x) \le k_1 \quad \text{and} \quad d(x), d^{(m)}(x) \ge q_0(c_0 k_1)^{-1}$$
$$h q_0 (2k_1)^{-1} \le \varkappa_0^h, \varkappa_0^{(m)} \le c_0\left(2\|Q\|_{C[0,h]} + h^{\lambda-1/p}|Q|_{\lambda,p,e_0}\right) \tag{3.7}$$
$$h q_0 (2k_1)^{-1} \le \varkappa_1^h, \varkappa_1^{(m)} \le c_0\left(2\|Q\|_{C[1-h,1]} + h^{\lambda-1/p}|Q|_{\lambda,p,e_N}\right).$$

The theorem will be proved when among $y_i^{(m)}$ ($i = 0(1)3$) there exists $y_{i_0}^{(m)}$ such that $y_{i_0}^{(m)}(0)$, $y_{i_0}^{(m)}(1)$, $\Lambda_0^{(m)} y_{i_0}^{(m)}$, $\Lambda_1^{(m)} y_{i_0}^{(m)} \ge 0$. The existence of such $y_{i_0}^{(m)}$ can be obtained as in [2] using the inequalities (3.7) and the discrete maximum principle [7] ∎

The inequalities (3.7) and the maximum principle in analogy with [2] yield the following two theorems as well.

Theorem 3.2: *Suppose the conditions* (2.3) - (2.6) *with* $q_0 > 0$ *hold. Then* $u_i(x) \le y(x)$ and $y_i^{(m)}(x) \le y^{(m)}(x)$ *for* $x \in \overline{\omega}_h$ *and* $i = 0(1)3$, *where* y *is a solution of the exact differenece scheme* (3.3) *and* $y^{(m)}$ *is a solution of the truncated difference scheme* (3.4).

Theorem 3.3: *Suppose the conditions* (2.3) - (2.6) *with* $q_0 > 0$ *hold. Then exact difference scheme* (3.3) *and the truncated difference scheme* (3.4) *have a unique solution.*

Using these results and estimate (3.6) in complete analogy with [2] we can prove as next statement the following

Theorem 3.4: *Suppose the conditions* (2.3) - (2.6) *with* $q_0 > 0$ *hold. Then the solution of the truncated difference scheme* (3.4) *converges to the solution of the variational inequality* (2.15) *if* $h \to 0$ *and the estimate*

$$\|\rho^{-1/p_0}(u - y^{(m)})\|_{\infty,\omega} \le c\,h^{2(m+1)-n}\tilde{F}(m,h)$$

is valid, where $n = \max\{n_0, \dots, n_3\}$, *the functional* $\tilde{F}(m,h) = \tilde{F}(m,h,k,Q,f_0,f_1)$ *is bounded or tends to zero together with h, the constant c does not depend on h.*

An algorithm which is based on the statements of Theorem 3.2 and (3.5) can be constructed for computational realization of the truncated difference scheme (3.4). This algorithm can be rarried out applying the elimination method for three-point equations not more than two times (see [3]) and needs $O(N)$ arithmetical operations.

REFERENCES

[1] DUVAUT, G. and J. L. LIONS: *Inequalities in Mechanics and Physics.* Moscow: Nauka Publ. 1980

[2] GAVRILYUK, I. P.: *Exact and truncated difference schemes of arbitrary degree of accuracy for a class one-dimensional equations.* Ukraine Math. J. **38** (1986), 563 - 568.

[3] GAVRILYUK, I. P.: *An algorithm for solving a class of one-dimensional variational inequalities.* Numer. Appl. Math. (Kiev) **64** (1988), 24 - 30.

[4] GAVRILYUK, I. P.: *Exact difference schemes and difference schemes of arbitrary given degree of accuracy for generalized one-dimensional third boundary value problems.* Z. Anal. Anw. **11** (1992), 549 - 566.

[5] GLOWINSKI, R., LIONS, J.-L. and R. TRÉMOLIERES: *Numerical Analysis of Variational Inequalities.* Amsterdam: North Holland Publ. Comp. 1981.

[6] KUFNER, A. and S. FUCIK: *Nonlinear Differential Equations.* Moscow: Nauka Publ. 1988.

[7] SAMARSKI, A. A.: *Theory of Difference Schemes.* Moscow: Nauka Publ. 1983.

[8] SAMARSKI, A. A., LAZAROV, R. D. and V. L. MAKAROV: *Difference Schemes for Differential Equations with Generalized Solutions.* Moscow: Vysšaya Škola 1987.

[9] TICHONOV, A. N. and A. A. SAMARSKI: *On homogeneous difference schemes of high degree of accuracy.* Dokl. Akad. Nauk USSR **131** (1960), 514 517.

Received 28.04.1992