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On Regularity of Solutions to Inner Obstacle Problems

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We determine the class of obstacles, for which the solution of the corresponding inner obstacle problem has the same regularity as obtained when the obstacle is defined on the whole domain. We also show that this is practically the only class of obstacles having this property for arbitrary right-hand sides.

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1 Introduction

Let Ω be a bounded domain of $\mathbf{R}^n$, E a closed subset of Ω with nonempty interior and ψ a real function defined on E. In this note we discuss a regularity of the solution to the *inner obstacle problem* which can be formulated as follows: find u minimizing the Dirichlet type integral among all functions v, vanishing at $\partial\Omega$ and satisfying the inequality $v(x) \geq \psi(x)$ for $x \in E$.

Variational solvability (that is in the Sobolev space $H_0^1(\Omega)$) of such a problem follows from the general theory of *variational inequalities* [5, 6, 8]. However, whilst in the case $E = \Omega$ the variational solution can be regularized up to $H^{2,\infty}(\Omega)$ ($C^{1,1}(\overline{\Omega})$), provided data are sufficiently regular [1, 5-8], the case where E is a proper subset of Ω presents certain difficulties. In particular, the best regularity result for such a case, known to the authors, is that u is Lipschitz continuous [3, 5]. We show in Proposition 3.1 that if ψ has a regular extension outside E then this is the regularity threshold. Further investigation can be focused on determining either the class of obstacles or conditions relating the right-hand side to the geometry of the problem which yields the standard regularity results. This note deals only with the first topic and in particular we prove that the variational solution to the *inner obstacle problem* can be regularized as in the case $E = \Omega$ if (and practically only if) the obstacle is *egg-shaped* which roughly speaking means that the graph of ψ becomes vertical when one approaches the boundary of E (see assumptions $\mathbf{A}_1$ and $\mathbf{A}_2$ on page 3). This result is natural from geometrical point of view. To prove it we construct an auxiliary obstacle in such a way that the solution of the corresponding obstacle problem is regular and show that this solution coincides with the solution of the original *inner obstacle problem.*

2 Main definitions and notations

This note is focused upon the following problem. Find u satisfying

$$\langle Au, v-u\rangle = a(u, v-u) \geq \langle f, v-u\rangle \tag{2.1a}$$

for all $v \in K_\psi$, where

$$K_\psi = \{v \in H_0^1(\Omega), v \geq \psi \text{ a.e. on } E\}, \tag{2.1b}$$

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$a(\cdot,\cdot)$ is given by the formula

$$a(u,v) = \int_\Omega \left\{ \left(a^{ij} u_{,i} + b^j\right) v_{,j} + \left(c^i u_{,i} + du\right) v \right\} dx \quad (u, v \in H_0^1(\Omega)) \tag{2.2}$$

where $u_{,i}$ denotes the $i-th$ partial derivative and the summation convention is adopted. We assume that $a^{ij}, b^j \in C^{0,1}(\overline{\Omega})$ ($C^{k,\alpha}(\overline{\Omega})$ denotes the space of k-times differentiable functions with Hölder continuous derivatives of k-th order, $0 < \alpha \leq 1$), $c^i, d \in L^\infty(\Omega)$ for $i,j = 1,\ldots,n$, $a^{ij}\xi_i\xi_j \geq \nu \mid \xi \mid^2$ a.e. in Ω for some $\nu > 0$ and every $\xi \in \mathbf{R}^n$, and a is coercive over $H_0^1(\Omega)$. The last requirement is satisfied if d is sufficiently large and also for some combinations of coefficients b^j, c^i and d [8, p.95–96]. Further in (2.1) A is the operator associated with a, $f \in H^{-1}(\Omega)$, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ and ψ is continuous over E. It is known [8, Th.4.4] that if a is coercive, then (2.1a) admits exactly one solution in $H_0^1(\Omega)$. Problem (2.1) is frequently referred to as $P(f,\psi,E)$.

3 A counterexample

In this section we prove that if ψ is the restriction to E of a reasonably smooth function defined on Ω, then the variational solution to (2.1) in general does not belong even to $C^1(\overline{\Omega})$. To avoid technicalities we shall work here with slightly excessive regularity assumptions.

Proposition 3.1. *Let Ω and E be of the class C^2, $\psi \in C^2(\overline{\Omega})$ and $a(\cdot,\cdot)$ be as in Section 2. There exist $F \in C^0(\Omega)$ such that the solution to $P(F,\psi,E)$ is merely in $C^{0,1}(\Omega)$.*

Proof. It is clear that $P(0,\psi,E)$ is equivalent to $P(-A\psi, 0, E)$ and that for some $g \in C^0(\overline{\Omega})$ we have $F_1 = g - A\psi < 0$. Let [4, Th.8.12] $u_0 \in H_0^1(\Omega\backslash E) \cap C^{1,1}(\overline{\Omega\backslash E})$ satisfies

$$Au_0 = F_1 \text{ in } \Omega\backslash E \text{ and } u_0 = 0 \text{ on } \partial\Omega \cup \partial E. \tag{3.1}$$

From the maximum principle [4, Th.3.5] $u_0 < 0$, thus u, defined by

$$u = \begin{cases} 0 & \text{in } \bar{E} \\ u_0 & \text{in } \Omega\backslash E, \end{cases} \tag{3.2}$$

is the solution of $P(F_1, 0, E)$ [5, Th.6.9]. If the normal derivative $\frac{\partial u_0}{\partial n_E} \neq 0$ at some $x_0 \in \partial E$, then the proposition is proved. If not, we can modify u by adding to it a function $v \in C^2(\overline{\Omega})$, with properties $v = 0$ on $\partial E, v \leq 0$ on $\Omega\backslash E$ and $\frac{\partial v}{\partial n_E}(x_0) \neq 0$. Selecting a real N to obtain $NF_1 + Av < 0$ in Ω we can modify (3.2) by taking $\tilde{u}_N + v$ instead of u_0, where $\tilde{u}_N$ solves (3.1) with NF_1 instead of F_1. Such a function solves $P(NF_1 + Av, 0, E)$ and is not continuously differentiable at x_0 ∎

It is clear that the negative result is due to the boundedness of the derivatives of ψ close to ∂E. This leads us to the concept of the *egg-shaped obstacle* which should guarantee the regularity of the solution to (2.1). We shall make this concept precise in the next section.

4 Regularity for the egg-shaped inner obstacle problem

In this section we adopt assumptions which make available the standard regularity results for solutions to variational inequalities and associated boundary value poblems. Let $\Omega \subset \mathbf{R}^n$ be a bounded open domain with $C^{1,1}$-boundary $\partial\Omega$ and $E \subset \bar{E} \subset \Omega$ be an open set with $C^{1,\alpha}$ boundary ∂E. We can assume that $\partial E = F^{-1}(0)$ for some $F \in C^{1,\alpha}(\mathbf{R}^n)$ [4, p.143] and $\vec{n}_E$ (the inward drawn normal to E) satisfies $\vec{n}_E \doteq \text{grad}F$ on ∂E. The obstacle $\psi \in C^0(\bar{E}) \cap C^2(E)$ is assumed to satisfy the following assumptions:

(A₁) $\lim_{x\to x_0} \psi_{,i}(x)F_{,i}(x) = -\infty$ for every $x_0 \in \partial E$.

(A₂) If a vector field $\vec{\tau} \in C^{0,\alpha}(\bar{E})$ satisfies $\tau_i F_{,i} = 0$, then $\psi_{,i}\tau_i \in C^{0,\alpha}(\bar{E})$.

These assumptions mean geometrically that the graph of ψ becomes vertical close to ∂E and is sufficiently smooth in every direction which is not tranversal to it.

We start further considerations with a lemma which concerns more general situation and thus assumptions $\mathbf{A_1}$ and $\mathbf{A_2}$ are not necessary here.

Lemma 4.1. *Let $E_1 \subseteq E_2 \subseteq \Omega$ and $P(f,\psi_i,E_i)$ be uniquely solvable in $H_0^1(\Omega)$ with solutions u_i, i=1,2, respectively. If $\psi_1 \geq \psi_2$ in E_2 and $I_1 := \{x \in E_1; u_1(x) = \psi_1(x)\} \subseteq \{x \in E_1; \psi_1(x) = \psi_2(x)\}$, then $u_1 = u_2$.*

Proof. We prove that u_1 solves $P(f,\psi_2,E_2)$. Let $v \in K_{\psi_2} = \{v \in H_0^1(\Omega); v \geq \psi_2 \text{ in } E_2\}$ and let us consider

$$a(u_1, v-u_1) = a(u_1,(v-u_1)^+) + a(u_1,(v-u_1)^-),$$

where as usual $(v-u_1)^+$ (resp. $(v-u_1)^-$) denotes $\max\{v-u_1,0\}$ (resp. $\min\{v-u_1,0\}$). It is known [5, Th.A.1] that both functions belong to $H_0^1(\Omega)$ and that

$$a(u_1,(v-u_1)^+) = a(u_1,(v-u_1)^+ + u_1 - u_1) \geq \langle f,(v-u_1)^+\rangle,$$

since $(v-u_1)^+ + u_1 \in K_{\psi_1}$. However, $v \geq \psi_2 = \psi_1 = u_1$ on I_1, so $(v-u_1)^- = 0$ there, thus $(v-u_1) \in H_0^1(\Omega\backslash I_1)$ and consequently $a(u_1,(v-u_1)^-) = \langle f,(v-u_1)^-\rangle$. Combining both results we see that the inequality $a(u_1,v-u_1) \geq \langle f, v-u_1\rangle$ holds for arbitrary $v \in K_{\psi_2}$ which proves the lemma ∎

We are now able to prove the main result of this section.

Theorem 4.1. *If all introduced assumptions are satisfied and $f \in L^\infty(\Omega)$, then the solution of $P(f,\psi,E)$ belongs to $H_0^{2,p}(\Omega)$ for every $p < \infty$ (and so to $C^{1,\omega}(\bar{\Omega})$ for every $\omega < 1$).*

Proof. If F is the function which describes ∂E, then there exists $\epsilon_0 > 0$ such that the sets $F^{-1}(\epsilon)$ are, for $0 < \epsilon < \epsilon_0$, $C^{1,\alpha}$-manifolds contained in E. We denote bounded by them subdomains by E_ϵ, thus $F^{-1}(\epsilon) = \partial E_\epsilon$. By the assumption $\mathbf{A_2}$, $\psi|_{\partial E_\epsilon} \in C^{1,\alpha}(\partial E_\epsilon)$, so for every $\epsilon < \epsilon_0$ the Dirichlet problem

$$Aw_\epsilon = f \text{ in } \Omega\backslash E_\epsilon, \quad w_\epsilon = 0 \text{ on } \partial\Omega, \quad w_\epsilon = \psi \text{ on } \partial E_\epsilon \tag{4.1}$$

is uniquely solvable in $H^1(\Omega\backslash E_\epsilon) \cap C^{1,\alpha}(\overline{\Omega\backslash E_\epsilon})$ and its solution w_ϵ satisfies the inequality

$$\| w_\epsilon \|_{C^{1,\alpha}(\overline{\Omega\backslash E_\epsilon})} \leq \mathrm{M}\left(\| f \|_{L^\infty(\Omega\backslash E_\epsilon)} + \| \psi \|_{C^{1,\alpha}(\partial E_\epsilon)}\right) \tag{4.2}$$

which is the combination of the *a priori* estimates for the solution and the gradient of the solution to (4.1) [4, (8.87) and (8.39)]. The constant M in (4.2) depends on the dimension n, bounds on the coefficients of A, constant of ellipticity of A, measure $\mu(\Omega\backslash E_\epsilon)$ and ∂E_ϵ. From the proof of (4.2) it follows that the dependence on ∂E_ϵ is expressed in terms of the $C^{1,\alpha}$-norms of the diffeomorphism which flattens ∂E_ϵ; in our case they can be bounded by $C^{1,\alpha}$- norm of F in E. By assumption $\mathbf{A_2}$, the norms $\| \psi \|_{C^{1,\alpha}(\partial E_\epsilon)}$ are bounded independently of ϵ, so for some constant L, which is independent of ϵ, we have

$$\| w_\epsilon \|_{C^{1,\alpha}(\overline{\Omega\backslash E_\epsilon})} \leq \mathrm{L}, \quad \text{for } 0 < \epsilon < \epsilon_0. \tag{4.3}$$

Let us consider the family of obstacle problems $P(f,\psi_\epsilon,\Omega)$ where

$$\psi_\epsilon(x) = \begin{cases} \psi(x) & \text{for } x \in E_\epsilon \\ w_\epsilon(x) & \text{for } x \in \Omega\backslash E_\epsilon. \end{cases} \tag{4.4}$$

These are Lipschitz obstacles satisfying

$$A\psi_\epsilon = [A\psi_\epsilon]_{class} + \left(\frac{\partial\psi}{\partial n_A} - \frac{\partial w_\epsilon}{\partial n_A}\right)\delta_{\partial E_\epsilon}, \tag{4.5}$$

where $[A\psi_\epsilon]_{class} \in L^\infty(\Omega)$ denotes the classical (a.e.) value of A at ψ_ϵ, $\delta_{\partial E_\epsilon}$ is the Dirac distribution concentrated at ∂E_ϵ and $\partial/\partial n_A$ denotes the conormal derivative associated with A. By assumption $\mathbf{A}_1$ and (4.3) we obtain

$$(A\psi_\epsilon)^+ = ([A\psi_\epsilon])^+ \in L^\infty(\Omega), \tag{4.6}$$

provided $\epsilon > 0$ is sufficiently small. The inequality of Lewy and Stampacchia [1, Th.I.1, 6, 7, 8, Th.4.32] implies then that the solution u_ϵ of $P(f,\psi_\epsilon,\Omega)$ is in $H^{1,p}(\Omega)$, $p < \infty$. We shall prove that u_ϵ solves also $P(f,\psi,E)$: First we note that we have $\{x \in \Omega;\ \psi_\epsilon(x) = u_\epsilon(x)\} \subset E_\epsilon \subset E$. Indeed, it is clear from the regularity of u_ϵ that $\inf_{x\in\partial E_\epsilon}(u_\epsilon(x) - w_\epsilon(x)) > 0$ and $u_\epsilon(x) = w_\epsilon(x)$ on $\partial\Omega$, thus $u_\epsilon(x) - w_\epsilon(x) > 0$ in $\Omega\backslash E_\epsilon$. From (4.3), assumption $\mathbf{A}_2$ and the identity $\psi \equiv w_\epsilon$ on ∂E_ϵ it follows that $\psi(x) < w_\epsilon(x)$ in $E\backslash E_\epsilon$ provided $\epsilon > 0$ is sufficiently small. Hence the assumptions of Lemma 4.1 are satisfied with $\Omega = E_1$, $E = E_2$ and $u_1 = u_\epsilon$ and therefore u_ϵ is the unique solution of $P(f,\psi,E)$ ∎

Corollary 4.1. *Let A and Ω satisfy additional assumptions which ensure the $H^{2,\infty}(\Omega)$-solvability of* (2.1) *with $E = \Omega$ (e.g. $\partial\Omega$ is of class $C^{2,\alpha}, a^{ij}, b^j \in C^{1,\alpha}(\bar\Omega), c^i, d, f \in C^{0,\alpha}(\bar\Omega)$ for some $\alpha \in (0,1)$* [7]*). Then the solution of $P(f,\psi,E)$ belongs to $H^{2,\infty}(\Omega)$ (and so to $C^{1,1}(\bar\Omega)$).*

Proof. From the proof of Theorem 4.1 we infer that the coincidence set I satisfies $I \subset E_\epsilon$ and certainly $\operatorname{dist}(I,\partial E_\epsilon) = \delta > 0$ with ϵ selected there. Let η be a $C_0^\infty(\Omega)$-function, such that $\eta \equiv 1$ in I and supp $\eta \subset E_\epsilon$. Then the function $\tilde\psi := \eta\left(\tilde\psi_\epsilon + |\inf_\Omega \psi_\epsilon|\right) - |\inf_\Omega \psi_\epsilon|$ satisfies $\tilde\psi \in C^2(\bar\Omega), \tilde\psi \le 0$ on $\partial\Omega, \tilde\psi \le \psi_\epsilon$ in Ω and $\tilde\psi = \psi_\epsilon$ in I. Hence by Lemma 4.1 and Theorem 4.1 the solutions of $P(f,\psi,E)$ and $P(f,\tilde\psi,\Omega)$ coincide and the latter belongs to $H^{2,\infty}(\Omega)$ by the standard regularity result ∎

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