Eigenvalue Distribution of Invariant Linear Second Order Elliptic Differential Operators with Constant Coefficients

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Let $\mathfrak{G}$ be a properly discontinuous group of affine transformations acting on an $n$-dimensional affine space and $P$ a $\mathfrak{G}$-invariant linear elliptic differential operator with constant coefficients. In this paper the $\mathfrak{G}$-automorphic eigenvalue problem to $P$ is solved. For the number $N(x)$ of the eigenvalues which are less than or equal to the "frequency bound" $x^2$ the asymptotic estimation $N(x) = c_0 x^n + c_1 x^{n-1} + O(x^{n-2+2/(n+1)})$ is given with $c_0$ and $c_1$ being interesting geometric invariants.

Key words: Eigenvalue problem, eigenvalue distribution, invariant linear elliptic differential operator, lattice remainder, asymptotic estimation, principal vector

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0. Problem

Let $\mathcal{B}$ be an $n$-dimensional vector space or later at the same time also an affine space, $\mathcal{B}^*$ its dual, $\mathfrak{G}$ a properly discontinuous group of affine transformations acting on $\mathcal{B}$ and having a compact fundamental domain [3]. For a $\mathfrak{G}$-invariant positive definite quadratic form $\mathcal{Q}$ on $\mathcal{B}^*$ and for a fixed vector $\psi \in \mathcal{B}^*$ we consider the differential operator

$$P[\psi] = \mathcal{Q}(\frac{\partial}{\partial x} - 2\pi i \psi, \frac{\partial}{\partial x} - 2\pi i \tilde{\psi}), \psi \in \mathcal{B},$$

and the assigned polynom

$$P(\nu) = -\mathcal{Q}(\nu - 2\pi \nu, \nu - 2\pi \nu), \nu \in \mathcal{B}^*.$$  

$\mathfrak{G}$-invariant means for $P[\psi]$ that the following relation is valid:

$$P[\psi \circ S] = P[\psi] \circ S, \text{ for all } S \in \mathfrak{G}. \quad (1')$$

Now look at the $\mathfrak{G}$-automorphic eigenvalue problem

$$P[\psi] + \mu \psi = 0, \psi \in L_2(\mathfrak{G}). \quad (2)$$
$L_2(\Theta)$ is the Hilbert space over $\mathbb{C}$ of locally square-integrable $\Theta$-automorphic functions. $\text{spec}_\Theta(P)$ denotes the eigenvalue spectrum of (2). We will investigate the eigenvalue distribution $\text{dis}(\text{spec}_\Theta(P))$ over $\mathbb{R}^*$, where "dis" is defined by the distribution function

$$N(\lambda) = \# \{ \mu \in \text{spec}_\Theta(P) : \mu \leq \lambda^2 \}. \quad (3)$$

Here sometimes $\lambda$ instead of $\lambda^2$ is taken and called in Weyl's considerations "frequency bound" [25]. To establish a good asymptotic estimation of $N(\lambda)$ we will work out the following subjects:

1. Solution of the $\Theta$-automorphic eigenvalue problem (2).
2. Description of $N(\lambda)$ by a certain number of so-called "principal lattice vectors" in a convex domain $\lambda \cdot \Omega \subset \mathbb{R}^n$ (see (23)/(23')).
3. Formulation of $N(\lambda)$ as a finite sum of Weyl sums.
4. Asymptotic estimation $N(\lambda) = c_0 \lambda^n + c_1 \lambda^{n-1} + O(\lambda^{n-2+2/(n+1)})$ with explicit calculation of $c_0$ and $c_1$ as geometric invariants. Survey of influence of fixed (fixed point-free) elements of the group $\Theta$ on the asymptotic expression for $N(\lambda)$.

1. Solution of the $\Theta$-automorphic eigenvalue problem (2)

1.1 The orthonormal system of $\Theta$-automorphic functions in $L_2(\Theta)$. To introduce such a system we follow the proceeding of P. Günther in [7, § 1 and § 2].

The lattice $\Gamma \subset \mathbb{B}$: We will write the affine transformation $S: \mathbb{B} \rightarrow \mathbb{B}$ ($S \in \Theta$) of the $n$-dimensional affine space $\mathbb{B}$ as a Seitzian space group symbol $S = (\sigma, t)$ with $\xi = S(\xi) = \sigma \xi + t$ ($\xi, \sigma \in \mathbb{B}$) as transformation formula. The components $\sigma$ and $t$ are said to be fixed point and translation part of $S$, respectively. For $R = (\rho, t) \in \Theta$ and $S \in \Theta$ the composition $R \circ S = (\rho \sigma, \sigma t + t)$ is defined by $(R \circ S)(\xi) = S(R(\xi))$. The inverse to $S$ with respect to the identity element $E = (e, 0) \in \Theta$ is $S^{-1} = (\sigma^{-1}, -\sigma^{-1} \cdot t)$, where $e = \text{id}$ and $0 \in \mathbb{B}$ is the null vector. Now we consider the "point group" $\mathcal{B}$ of $\Theta$,

$$\mathcal{B} = \{ (\sigma, f) \in \Theta \text{ for some } f \in \mathbb{B} \} \quad (4)$$

and the "translation group" $\mathcal{L} \subset \Theta$ of all translations in $\Theta$,

$$\mathcal{L} = \{ (e, t) \in \Theta \}. \quad (4')$$

We know about $\mathcal{B}$ and $\mathcal{L}$ the following [1, 3, 5]: $\mathcal{L}$ is an invariant subgroup of $\Theta$. The factor group $\Theta/\mathcal{L}$ and $\mathcal{B}$ are isomorphic and $\text{ord}(\Theta/\mathcal{L})$ is finite. Therefore we can introduce

$$r := \text{ord}(\Theta/\mathcal{L}) = \text{ord} \mathcal{B}. \quad (4'')$$

$\mathcal{L}$ has $n$ generators $(e, b_1), \ldots, (e, b_n)$ with $n$ linear independent translation parts $b_k$ which are used to form the basis $\mathcal{B}$ and also to form the $\mathcal{B}$-invariant $n$-dimensional lattice

$$\Gamma := \text{orb}_{\mathcal{L}}(e) = \{ t = t^k b_k : t^k \in \mathcal{L} \} \subset \mathbb{B}. \quad (5)$$

The vector $\alpha \in \mathbb{B}$ is said to be "belonging to $\sigma \in \mathcal{B}$" if $(\sigma, \alpha) \in \Theta$. Together with $\alpha$ then also all vectors $\alpha + \tau$ and only these are belonging to $\sigma$. So modulo $\Gamma$ exactly one vector $\alpha$ is belonging to $\sigma$ and will be denoted by $\alpha = \{ \}$. In the coset decomposition of $\Theta$ relative to $\mathcal{L}$,
\[ \Theta = S_1 \circ \mathcal{I} + \ldots + S_r \circ \mathcal{I}, \quad S_\nu = (\sigma_\nu, f_\nu) \]  
(6)
the elements of one of the same coset \( S_\nu \circ \mathcal{I} \) have the same fixed point part \( \sigma_\nu \) but different cosets have different such parts. If \( (\sigma_1, f_1), (\sigma_2, f_2), (\sigma_\nu, f_\nu) \in \Theta \) it may be advantageous to think of the Frobenius congruence
\[ \sigma_2 f_2 + f_1 = f \mod \Gamma. \]  
(7)

The dual lattice \( \Gamma^* \subset \mathcal{B}^* \): A usually in crystallography here we turn to the dual situation. Let \( \mathcal{B}^* \) be the dual space of linear functionals on \( \mathcal{B} \), \( \langle \nu, \xi \rangle \) the value of \( \nu \in \mathcal{B}^* \) in \( \xi \in \mathcal{B} \). Relative to \( \Gamma \subset \mathcal{B} \), let
\[ \Gamma^* = \{ \nu = \nu_k b^k : \nu_k \in \mathbb{Z} \} \subset \mathcal{B}^*, \quad \langle b^h, b^k \rangle = \delta_h^k, \]  
(8)
be the dual lattice in \( \mathcal{B}^* \). As basis \( \mathcal{B}^* \) we use then \( \{ b^1, \ldots, b^n \} \). Instead of \( \sigma \in \mathcal{B} \) here we need the adjoint mapping \( \sigma^* \) to \( \sigma \): \( \sigma^* \) is defined by
\[ \sigma^* : \mathcal{B}^* \to \mathcal{B}^* \text{ with } \sigma^* \nu = \nu \circ \sigma. \]

The principal classes \( \mathcal{H} \subset \Gamma^* \): For a fixed lattice functional \( \nu \in \Gamma^* \) we introduce the equivalence class
\[ \mathcal{E} := \{ \mu \in \Gamma^* : \mu^* = \sigma^* \mu \text{ for all } \sigma \in \mathcal{B} \} = \{ \mu_1, \ldots, \mu_l \}. \]  
(9)
Here is \( l = \text{ord} \mathcal{E} \leq r = \text{ord} \mathcal{B} \) as we can see by help of the decomposition \( \mathcal{B} = \mathcal{K}(\mu) \cup (\mathcal{B} \setminus \mathcal{K}(\mu)) \) relative to the adjoint isotropy group to \( \mu \),
\[ \mathcal{K}(\mu) = \{ \sigma \in \mathcal{B} : \sigma^* \mu = \mu \}. \]  
(10)
So \( \Gamma^* \) is decomposed completely in a set \( \mathcal{K} \) of classes \( \mathcal{E} \). Among these classes the so-called principal classes \( \mathcal{H} \) play a leading part: For \( \mathcal{K}(\mu) \) we consider the character \( \chi(\mu, \cdot) \) with
\[ \chi(\mu, \sigma) = \exp \{ 2\pi i \langle \mu, \sigma \rangle \}, \quad (\sigma, \nu) \in \Theta. \]  
(11)
In \( (\sigma, \nu) \) the vector \( \nu \) is well-established and
\[ \varphi_\nu(\xi) = \exp \{ 2\pi i \langle \nu, \xi \rangle \} \]  
(12)
is a \( \mathcal{I} \)-automorphic function on \( \mathcal{B} \). Therefore \( \chi \) is correctly defined. If
\[ \chi(\mu, \sigma) = 1 \text{ for all } \sigma \in \mathcal{K}(\mu) \]  
(13)
so \( \chi(\mu, \cdot) \) is said to be principal character of \( \mathcal{K}(\mu) \) and \( \mu \) principal vector of \( \Gamma^* \). Now if \( \mu \in \mathcal{E} \) is a principal vector, \( \mathcal{E} \) contains only principal vectors and is called principal class \( \mathcal{H} \). Otherwise \( \mathcal{E} \) contains only non-principal vectors (\( \mathcal{E} \) is a non-principal class). Let \( \mathcal{H} \) be the set of all principal classes \( \mathcal{H} \subset \Gamma^* \).

The orthonormal system of \( \Theta \)-automorphic functions: Let \( \mathcal{F} = \{ \mu_1, \ldots, \mu_l \} \subset \mathcal{H} \) be a principal class and \( \text{rep}(\mathcal{E} / \mathcal{K}(\mu_1))_L = \{ \mu_1, \ldots, \mu_l \} \) a system of representatives of the left coset decomposition of \( \mathcal{B} \) with respect to \( \mathcal{K}(\mu_1) \). Then \( \mu_1, \ldots, \mu_l \in \mathcal{B} \) shall be vectors belonging to \( \sigma_1, \ldots, \sigma_l \), respectively, i.e. \( S_\nu = (\sigma_\nu, f_\nu) \) for \( \nu = 1, \ldots, l \).
Definition: The sum

\[ \psi_0 = \frac{1}{i} \sum_{v=1}^{L} \varphi u_1 \circ S_v \]  \hspace{1cm} (14)

is said to be a \( h \)-corresponding function on \( \mathfrak{B} \).

Remark 1: For each \( v \in \mathfrak{B}^* \) the function \( \varphi_0 \) is satisfying the relation

\[ \varphi_0 \circ S = \varphi_0(t) \varphi_0 \circ S \]  \hspace{1cm} (15)

Especially for the translations \( S = (e, t) \in \mathfrak{X} \) and lattice vectors \( v = u \in \Gamma^* \) we see that \( \varphi_0 \) is \( \mathfrak{X} \)-automorphic, even \( \varphi_0 \in L_2(\mathfrak{X}) \) (\( L_2 \)-space of \( \mathfrak{X} \)-automorphic functions).

Remark 2: If \( \sigma \) runs through \( \mathfrak{B} \), so \( \sigma^\top u_1 \) runs through \( h = \{ u_1, ..., u_l \} \) - but in general not simply \( (I \leq r) \). But if \( \sigma \) runs only through \( \text{rep}(\mathfrak{B}/\mathfrak{X}(u_1)) \), so from \( u_1 \) every vector \( u_v \in h \) arises exactly one time by \( u_v = \sigma^\top u_1 \).

The \( h \)-corresponding functions \( \psi_0 \) are elements of \( L_2(\Theta) \). As functions normed to one just the \( \psi_0 \) build a complete orthonormal system \( \{ \psi_0 : h \in \mathfrak{B} \} \) in \( L_2(\Theta) \) \([7 \S 2/(2.8)] \).

1.2 The \( \Theta \)-automorphic eigenfunctions and \( \text{spec}_{\Theta} \) of \( P \). To prove that the \( h \)-corresponding functions \( \psi_0 \) are the eigenfunctions of \( P \) we must investigate the action of \( P \) on \( \varphi_0 \circ S \).

Lemma 1: The \( \Theta \)-invariant differential operator \( P \) from (1) acts on the functions \( \varphi_0 \circ S \) from (14) or (15) according to

\[ P[\varphi_0 \circ S] = P(2\pi v) \varphi_0 \circ S \]  \hspace{1cm} (16)

Proof: The operator \( P \) can be written as

\[ P = P^h \partial_h \partial_k - 4\pi i P^h \partial_h - 4\pi^2 P^o. \]  \hspace{1cm} (17)

Here \( P^h \) are the coefficients of the quadratic form \( \Psi \) from (1'), furthermore \( P^h = P^h \partial_k \), \( P^o = P^h \partial_h \partial_k \), where \( P = P^h \partial_h \) and \( \partial_h = \partial/\partial x^h \), \( x^k = x^h b^h \) - explained altogether respectively to \( \text{bas} \mathfrak{B} \) or \( \text{bas} \mathfrak{B}^* \). Now we apply \( P \) on \( \varphi_0 \), \( v = v_j b^j \): Using (12) and (8) we obtain

\[ \partial_h \varphi_0(x) = \partial/\partial x^h (\exp 2\pi i \langle v_j b^j, x b^k \rangle) = \varphi_0(t) \cdot 2\pi i \partial/\partial x^h (v_j \cdot x^j) = 2\pi i v_h \varphi_0(t) \]

\[ \partial_h \varphi_0(x) = (2\pi i)^2 v_h v_k \varphi_0(t). \]

Now (17) and after that (1') gives

\[ P[\varphi_0] = (-P^h(2\pi v, 2\pi v_k) + 4\pi P^h(2\pi v_k) - 4\pi^2 P^o) \varphi_0 \]

\[ = -2(2\pi v - 2\pi v, 2\pi v - 2\pi v) \varphi_0 = P(2\pi v) \varphi_0. \]

So (16) follows from the \( \Theta \)-invariance of \( P \), i.e. from (1') \( \blacksquare \)

If we now take into account the \( h \)-corresponding function \( \psi_0 \) from (14), formula (16) gives
\[
P[\psi_h] = \frac{1}{l} \sum_{\nu=1}^{l} P(2\pi \nu) \varphi_{\nu} \circ S_{\nu} = P(2\pi u) \psi_h, \quad u \in h. \tag{18}
\]

**Definition:** If \( u \in \ell \), we can write
\[
P(2\pi \ell) = P(2\pi u), \tag{19}
\]
(where \( P(2\pi \ell) \) can be understood as a class norm \( \| \ell \|^2 \) of \( \ell \)).

The justification for (19) comes from the \( \mathfrak{B} \)-automorphy of \( P \) from (1'),
\[
P(\sigma \nu) = P(\nu) \quad \text{for all } \sigma \in \mathfrak{B}, \nu \in \mathfrak{B}^{*}, \tag{20}
\]
and of the fact that all \( u \in \ell = \{ u_1, \ldots, u_j \} \) arise e.g. from \( u_1 \) by means of the equivalence \( u = \sigma' u_1, \sigma \in \mathfrak{B} \).

**Remark 3:** If the class norms of \( \ell_1, \ell_2 \) are different, \( P(2\pi \ell_1) \neq P(2\pi \ell_2) \), the same is always right for the classes, \( \ell_1 + \ell_2 \). But the inverse assertion is not right; if \( \ell_1 + \ell_2 \), notwithstanding may be \( P(2\pi \ell_1) = P(2\pi \ell_2) \).

**Theorem 1:** To each principal class \( h \in \mathfrak{H} \) we can assign exactly one eigenvalue \( \mu = \mu_h \) of the \( \mathfrak{B} \)-automorphic eigenvalue problem (2), namely
\[
\mu_h = -P(2\pi h) \tag{21}
\]
with
\[
m_{\mathfrak{B}}(\mu_h) = \text{card} \{ h' \in \mathfrak{H} : P(2\pi h') = P(2\pi h) \} \tag{22}
\]
as multiplicity; thereby the \( h \)-corresponding function \( \psi_h \) belongs to \( \nu_h \) as the eigenfunction. The set \( \text{spec}_{\mathfrak{B}}(P) = \{ \mu_h : h \in \mathfrak{H} \} \) is the complete \( \mathfrak{B} \)-automorphic eigenvalue spectrum of the \( \mathfrak{B} \)-invariant differential operator \( P \) from (1).

**Proof:** The correspondence \( h \mapsto \psi_h \) from (14), and (18), prove the first part of the theorem. The completeness of \( \text{spec}_{\mathfrak{B}}(P) \) follows from the completeness of the orthonormal system \( \{ \psi_h : h \in \mathfrak{H} \} \) of \( L_2(\mathfrak{B}) \). Let \( \psi = \sum c_h \psi_h \) (summation over \( h \in \mathfrak{H} \)) be an arbitrary \( \mathfrak{B} \)-automorphic eigenfunction of \( P \) to the eigenvalue \( \mu \neq \mu_h \) for all \( h \in \mathfrak{H} \). Then from (2), (18), (19) and (21) for each \( h \in \mathfrak{H} \) there follows \( c_h (\mu_h - \mu) = 0 \). Consequently there would be \( c_h = 0 \) and therefore \( \psi = 0 \) which is a contradiction.

2. \( N(\lambda) \) as the number of principal classes \( h \) contained in a certain convex domain \( \lambda \cdot \mathfrak{D} \subset \mathfrak{D}^{*} \)

The operator \( P \) has the following geometric appearance.

**Definition:** The domains in \( \mathfrak{B}^{*} \)
\[
\mathfrak{D} = \{ \nu \in \mathfrak{B}^{*} : -P(\nu + 2\pi \nu) \leq (1/2\pi)^2 \} \tag{23}
\]
\[ \lambda \cdot \mathbb{D} = \{ \mathbf{v} \in \mathbb{B}^* : -P(\mathbf{v} + 2\pi \mathbf{p}) \leq (\lambda/2\pi)^2 \} \]  

\[ \mathbf{p} + \lambda \cdot \mathbb{D} = \{ \mathbf{v} \in \mathbb{B}^* : -P(2\pi \mathbf{v}) \leq \lambda^2 \} \]  

in this order are said to be gauge domain, homothetical expansion of \( \mathbb{D} \) with \( \lambda > 0 \) as factor, parallel translated domain by the vector \( \mathbf{p} \in \mathbb{B}^* \) (from (1)).

The \( \mathfrak{g} \)-invariance of \( P \) means for these domains

**Lemma 2:** The gauge domain \( \mathbb{D} \) and so also all its homothetical expansions \( \lambda \cdot \mathbb{D} \) are \( \mathfrak{g} \)-invariant. Therefore for an equivalence class \( \mathfrak{f} \in \mathfrak{A} \) there is valid

either \( \mathfrak{f} \subset (\mathbf{p} + \lambda \cdot \mathbb{D}) \) or \( \mathfrak{f} \cap (\mathbf{p} + \lambda \cdot \mathbb{D}) = \emptyset \).  

(24)

Now if we look at \( N(\lambda) \) from (3) and \( \mu_\mathfrak{f} \) from (21) we could ask for the geometric locus containing all \( \mathfrak{f} \) with \( \mu_\mathfrak{f} \leq \lambda^2 \). The formulas (21), (19), (1'), (23') and (24) yield

**Proposition 1:** The number of eigen values \( \mu_\mathfrak{f} \leq \lambda^2 \) is given by

\[ N(\lambda) = \text{card} \{ \mathfrak{f} \in \mathfrak{A} : \mathfrak{f} \subset (\mathbf{p} + \lambda \cdot \mathbb{D}) \} \]  

(25)

3. \( N(\lambda) \) as a finite sum of Weyl sums

3.1 A proposition of P. Günther. Let

\[ \mathbb{B}^*(\sigma) = \ker(\sigma^* - \text{id}) \quad \text{and} \quad \mathbb{D}^*(\sigma) = \mathbb{D} \cap \mathbb{B}^*(\sigma) \]  

(26)

be the eigenspace to the eigenvalue 1 of \( \sigma^* \) and the \( \mathbb{Z} \)-module of all lattice functionals of \( \mathbb{B}^*(\sigma) \), respectively (look at (8)). According to [7: Proposition 2.21], for a function \( f: \mathbb{B}^* \rightarrow \mathbb{C} \) it is valid

\[ \sum_{\mathbf{u} \in \mathfrak{D}} \frac{1}{\text{card} \mathfrak{D}} \sum_{\mathbf{u} \in \mathfrak{D}} f(\mathbf{u}) = \frac{1}{\pi} \sum_{\sigma \in \mathbb{B}} W(\sigma) \]  

(27)

so far as

\[ W(\sigma) := \sum_{\mathbf{u} \in \mathbb{D}^*(\sigma)} \chi(\mathbf{u}, \sigma) f(\mathbf{u}) \]  

(28)

is absolutely convergent for all \( \sigma \in \mathfrak{B} \).

3.2 The characteristic function \( \chi_\lambda \) of \( \lambda \cdot \mathbb{D} \). Let \( \chi \) be the characteristic function of \( \mathbb{D} \) and \( \chi_\lambda \) that of \( \lambda \cdot \mathbb{D} \). From the definition of \( \chi_\lambda \) and the \( \mathfrak{g} \)-invariance of \( \lambda \cdot \mathbb{D} \) (Lemma 2) you can easily see

**Lemma 3:** For \( \mathbf{v} \in \mathbb{B}^* \) we have

\[ \chi_\lambda(\mathbf{v}) = \chi(\frac{1}{\lambda} \cdot \mathbf{v}) \text{ for all } \lambda > 0 \]  

(29)
\[ \chi(x^o \sigma \psi) = \chi_{\lambda}(x) \quad \text{for all } x \in \Omega, \quad (29') \]

i.e. \( \chi_{\lambda} \) is \( \Omega \)-automorphic on \( \mathbb{B}' \).

Now regard \( \chi_{\lambda} \) as a partial function on \( \mathcal{P} + \Gamma^* \subset \mathbb{B}' \). Then \( \chi_{\lambda} \) is a class function depending only on the equivalence classes \( \mathcal{P} + \Gamma \) of the lattice \( \mathcal{P} + \Gamma^* \) for all \( \lambda \in \Omega \):

\[ \chi_{\lambda}(\mathcal{P} + \Gamma) = \begin{cases} 1 & \text{if } \Gamma \subset \mathcal{P} + \lambda \cdot \mathcal{D} \\ 0 & \text{if } \Gamma \not\subset \mathcal{P} + \lambda \cdot \mathcal{D} \end{cases} \quad (30) \]

(see also (24)). Now we set going proposition (27)/(28) choosing \( f(u) \) in accordance with \( f(u) := \chi_{\lambda}(-\mathcal{P} + u) = \chi_{\lambda}(-\mathcal{P} + h) \) for \( u \in h \in \Omega \). Then

\[ W(\omega) := \sum_{\mathcal{P} \in \Gamma'(\omega) \cap (\mathcal{P} + \lambda \cdot \mathcal{D})} \chi(u, \sigma) \quad (31) \]

is a finite and so an absolutely convergent series. Because of (30) and (25) the left-hand side of (27) is equal to \( N(\lambda) \) so that

\[ N(\lambda) = \frac{1}{2} \sum_{\omega \in \Omega} W(\omega). \quad (32) \]

### 3.3 Splitting of \( N(\lambda) \) into isodimensional summands

Let be

\[ n(\sigma) := \dim \mathbb{B}'(\sigma) \quad (34) \]

and

\[ \mathbb{B}_m := \{ \sigma \in \Omega : n(\sigma) = m \}, \quad m = 0, 1, \ldots, n. \quad (34) \]

For \( \sigma \in \mathbb{B}_m \) the \( \mathbb{Z} \)-module \( \Gamma^*(\sigma) \) from (26) has \( m \) linearly independent generators. Now (32) can be dissected according to

**Proposition 2:** \( N(\lambda) \) is the sum of isodimensional summands:

\[ N(\lambda) = \frac{1}{2} \sum_{m=0}^{n} \sum_{\sigma \in \mathbb{B}_m} W(\sigma) \quad (35) \]

where \( W(\sigma) \) with \( (\sigma, \mathcal{P}) \in \Omega \) are the Weyl sums (31)/(11), or for a specific purpose formulated,

\[ W(\sigma) = \sum_{\mathcal{P} \in \sigma \bmod \Gamma'(\sigma)} \exp \{ 2\pi i \langle u, \mathcal{P} \rangle \}. \quad (36) \]

The special kind of summation in (36) in comparison with that of (31) follows from (23').

**Definition:** In (35) the summand with \( m = n \) is said to be principal part and that with \( m = n - 1 \) secondary part of \( N(\lambda) \).

**Remark 4:** All the other summands of \( N(\lambda) \) with \( m \leq n - 2 \) will be proved subordinate and get into the remainder during the asymptotic estimation of \( N(\lambda) \) in Subsections 4.2/4.3 (see (49)).
4. The asymptotic estimation of $N(\lambda)$

4.1 Formulation of the Weyl sum $W(\sigma)$ in coordinates relative to $\text{bas}\Gamma^*(\sigma)$. Let be

\begin{equation}
\text{bas}\Gamma^*(\sigma) = \{c^1(\sigma), \ldots, c^m(\sigma)\}, \quad \text{bas}\Gamma^* = \{b^1, \ldots, b^n\},
\end{equation}

\begin{equation}
c^v(\sigma) = c_h^v(\sigma)b^h, \quad c_h^v(\sigma) \in \mathbb{Z} \quad (h = 1, \ldots, n; \nu = 1, \ldots, m).
\end{equation}

Because of $c^v(\sigma) \in \Gamma^*(\sigma)$ there is $(\sigma^T - \text{id})c^v(\sigma) = 0$. Therefore $c^v_\nu(\sigma)$ for each $\nu$ is a solution of the system of linear equations $(\sigma_\nu^j - \delta_\nu^j)c^v_\nu(\sigma) = 0$ ($j = 1, \ldots, n$) and naturally $\sigma^T b^j = \sigma_j^j$. Agreement: Latin indices run through $1, \ldots, n$ and Greek indices through $1, \ldots, m$ - only with the exception of $\sigma \in \mathcal{B}$.

For $u \in \Gamma^*(\sigma)$ and for $p \in \mathcal{B}^*(\sigma)$ as the invariant vector from (1'), we write

\begin{equation}
u = u_\nu c^v(\sigma) = u_\nu c_h^v(\sigma)b^h = u_h b^h \quad \text{and} \quad p = p_\nu c^v(\sigma) = p_\nu c_h^v(\sigma)b^h = p_h b^h.
\end{equation}

Then we have

\begin{equation}
\langle u, f \rangle = u_\nu s^v(\sigma) \quad \text{with} \quad s^v(\sigma) = \langle c^v(\sigma), f \rangle.
\end{equation}

Now looking at (17) we introduce the symmetric $m \times m$ - matrix $(P_{uv}(\sigma))$ with

\begin{equation}
P_{uv}(\sigma) = P_{hk} c_h^v(\sigma)c_h^u(\sigma), \quad \Delta(\sigma) := \det(P_{uv}(\sigma)).
\end{equation}

By (38) this makes possible to write $P$ in form of

\begin{equation}
- P(2\pi u) = (2\pi)^2 P_{uv}(\sigma)w_\nu w_\mu, \quad w_\nu = u_\nu - p_\nu.
\end{equation}

Therefore Proposition 2 in coordinates relative to $\text{bas}\Gamma^*(\sigma)$ can be formulated as

**Proposition 3:** $N(\lambda)$ (so as in Proposition 2) is the sum of the Weyl sums

\begin{equation}
W(\sigma) = e^{2\pi i p_\nu s_v(\sigma)} \sum_{w_\nu = -\rho_\nu \mod(1)} e^{2\pi i w_\nu s_v(\sigma)}. \quad (42)
\end{equation}

Remark 5: For $\sigma = e$ (identity in $\mathcal{B}$) we obtain

\begin{equation}
n(e) = n, \quad \Omega_n = \{e\}, \quad \mathcal{B}^*(e) = \mathcal{B}^*, \quad \Gamma^*(e) = \Gamma^*, \quad c^v(e) = b^v
\end{equation}

\begin{equation}
c_h^v(e) = \delta_h^v, \quad u_\nu = u_\nu, \quad p_\nu = p_\nu, \quad P_{uv}(e) = P_{uv}, \quad \Delta(e) = \det(P_{uv}).
\end{equation}

4.2 Landau’s estimation of lattice remainder applied to the Weyl sum $W(\sigma)$. In (42) we have the sum of the unimodular weights $\exp\{2\pi i w_\nu s_v(\sigma)\}$ which load the lattice functionals $\omega \in \Gamma^*(\sigma)$ within the $m = n(\sigma)$ - dimensional ellipsoid $(p + \lambda : \Delta) \cap \mathcal{B}^*(\sigma)$. The estimation of such a sum $W(\sigma)$ is a classical problem which was worked out above all by E. Landau ([14: Chapter 1/(7) and (10)] and [19]). As we know this leads to the result

\begin{equation}
W(\sigma) = \frac{\delta_\sigma}{2^m m^m \sqrt{\Delta(\sigma)}} \Gamma(m^2) \lambda^m + O(\lambda^{m^2 + \frac{2}{m + 1}}) \quad (44)
\end{equation}
\( \delta_\sigma = 1 \) if \( s^{\nu}(\sigma) \in \mathbb{Z} \) and \( \delta_\sigma = 0 \) otherwise. \( \tag{45} \)

**Definition:** \( \delta_\sigma \) will be called *Landau's \( \delta \)-symbol* which is assigned to \( \sigma \) (see Proposition 4).

### 4.3 \( N(\lambda) \) and the \( m \)-dimensional volumes \( \text{vol}_m(\lambda \cdot \mathbb{D} \cap \mathbb{B}^*(\sigma)) \)

Let be \( \sigma \in \mathfrak{g}_m \) and \( m = n(\sigma) \). Let \( \mathbb{B}^*(\sigma) \) be equipped with a measure \( \mu^* \) of the normalization \( \mu^*(\mathfrak{B}(I^*(\sigma))) = 1 \) (\( \mathfrak{B}(\cdot) \) - "fundamental domain of\( \cdot \)). So we can introduce the \( m \)-dimensional volume of \( \mathbb{D} \cap \mathbb{B}^*(\sigma) \),

\[
\text{vol}_m(\mathbb{D} \cap \mathbb{B}^*(\sigma)) = \int_{\mathbb{D} \cap \mathbb{B}^*(\sigma)} d\mu^*(\sigma) = \int_{\mathbb{D} \cap \mathbb{B}^*(\sigma)} d\sigma / \int_{\mathbb{B}(I^*(\sigma))} d\sigma. \tag{46}\]

**Remark 6:** In an affine space \( \mathbb{B}^* \) the affine volume \( \int_{\mathbb{D}} d\sigma \) is a relative invariant of weight \(-1\). The quotient of two such volumes, so as in (46), is an absolute invariant.

In the case that \( \mathbb{B} \) and \( \mathbb{B}^* \) are Euclidean spaces, and so especially \( \mathbb{B}^*(\sigma) \) is an Euclidean space with the metric fundamental tensor \( g^{\nu\mu}(\sigma), g(\sigma) = \det(g^{\nu\mu}(\sigma)) \), we define as usual

\[
\text{vol}_m(\mathbb{D} \cap \mathbb{B}^*(\sigma)) = \int_{\mathbb{D} \cap \mathbb{B}^*(\sigma)} \sqrt{g(\sigma)} d\mu^*(\sigma) \quad \text{with} \quad \text{vol}_m(\mathfrak{B}(I^*(\sigma))) = 1. \tag{47}\]

If \( W(\sigma) \) from (44) is belonging to a group element \( \sigma \in \mathfrak{g}_m \) with \( \delta_\sigma = 1 \), the factor before \( \lambda \) in (44) is the volume of an \( m \)-dimensional ellipsoid, namely of

\[
\lambda \cdot \mathbb{D} \cap \mathbb{B}^*(\sigma) = \left\{ \mathbf{v} = v^\nu e^\nu(\sigma); P^{\nu\mu}(\sigma)v^\nu v_\mu \leq \left( \frac{\lambda}{2\pi} \right)^2 \right\}. \tag{48}\]

Therefore \( W(\sigma) \) from (44) has the form

\[
W(\sigma) = \delta_\sigma \cdot \text{vol}_m(\mathbb{D} \cap \mathbb{B}^*(\sigma)) \lambda^m + O(\lambda^{m-2+n+1}). \tag{49}\]

Here the order of the remainder term in Proposition 2 (resp. Proposition 3) allows to carry out the summation for \( m = n \) (yielding then the principal part of \( N(\lambda) \)) and only just for \( m = n - 1 \) (producing the secondary part). Now we ascertain that \( m = n(\sigma) = n \) is true only for \( \sigma = e \) and we have \( \mathbb{D} \cap \mathbb{B}^*(e) = \mathbb{D} \) (see also Remark 5). Because the null vector \( t = \mathbf{0} \in \mathbb{B} \) is belonging to \( \sigma = e \) we get \( s^{\nu}(e) = \langle e^{\nu}(e), e \rangle = 0 \in \mathbb{Z} \) and hence \( \delta_\sigma = 1 \). We lodge all summands of \( N(\lambda) \) for \( m < n - 2 \) in (35) (Proposition 2) in \( O(\lambda^{n-2+n+1}) \). So Proposition 2 can be explained now as

**Theorem 2:** *The eigenvalue number \( N(\lambda) \) is satisfying the estimation*

\[
N(\lambda) \leq \frac{1}{\lambda^{n-2}} \text{vol}_n(\mathbb{D}) + \sum_{\delta_\sigma = 0} \text{vol}_{n-1}(\mathbb{D} \cap \mathbb{B}^*(\sigma)) \delta_\sigma \lambda^{n-1} + O(\lambda^{n-2+n+1}) \tag{50}\]

where Landau's symbol \( \delta_\sigma \) is to be taken from Proposition 4.

**Remark 7:** With regard to Remark 6 the assertion (50) of Theorem 2 can be understood also as a result of affine spectral geometry.

### 4.4 Landau's \( \delta \)-symbol and the influence of the fixed elements from \( \Theta \) on \( N(\lambda) \)

The decomposition \( \mathbb{B} = \mathbb{B}(\sigma) \oplus \mathbb{B}^*(\sigma) \) of the vector space \( \mathbb{B} \) into the subspaces
\[ B(\sigma) = \ker(\sigma - \text{id}) \quad \text{and} \quad B^1(\sigma) = \text{im}(\sigma - \text{id}) \quad (51) \]

and the sublattices

\[ \Gamma(\sigma) = \Gamma \cap B(\sigma) \quad \text{and} \quad \Gamma^1(\sigma) = \Gamma \cap B^1(\sigma) \quad (52) \]

with \( n(\sigma) = \dim B(\sigma) = \dim \Gamma(\sigma) \) makes possible to formulate the following fixed point properties.

**Lemma 4:** The affine transformation \((\sigma, f) \in \Theta\) acting on \( B \) has a fixed point \( \xi_0 \in B \) if and only if \( f \in B^1(\sigma) \).

**Proof:** From \((\sigma, f)\xi_0 = \xi_0\) there follows \((\sigma - \text{id})\xi_0 = -f\), i.e. \(-f \in B^1(\sigma)\) and so also \( f \in B^1(\sigma) \). Conversely, for \( f \in B^1(\sigma)\) there is also \(-f \in \Gamma(\sigma)\) and so by (51) there is a vector \( \xi_0 \in B \) with \(-f = (\sigma - \text{id})\xi_0\), that is \((\sigma, f)\xi_0 = \xi_0\).

**Corollary:** Assume \((\sigma, f) \in \Theta\) has a fixed point in \( B \). Then \((\sigma, f + t) \in \Theta\) has a fixed point in \( B \) if and only if \( t \in \Gamma^1(\sigma) \).

**Proof:** Let be \( f \in B^1(\sigma) \) (Lemma 4), that is \( f = \sigma\xi_0 - \xi_0, \xi_0 \in B \). a) Assume \((\sigma, f + t)\xi_1 = \xi_1, \xi_1 \in B, \) so there is true that \( \sigma(\xi_0 + \xi_1) - (\xi_0 + \xi_1) = -t \in B^1(\sigma) \) and then \( t \in \Gamma^1(\sigma) \). Because \((\sigma, f)\) and \((\sigma, f + t)\) are in \( \Theta \), by (7) there follows that \( t \in \Gamma \) and then by (52) \( t \in \Gamma^1(\sigma) \). b) Vice versa from \( t \in \Gamma^1(\sigma) \) there follows \( t \in B^1(\sigma) \), so we have \(-t, -f \in \Gamma^1(\sigma)\), i.e. \(-t = \sigma\xi_2 - \xi_2 = -f \in \Gamma^1(\sigma)\). So there is true that \( \sigma(\xi_2 + \xi_3) + t + f = (\sigma, f + t)(\xi_2 + \xi_3) = \xi_2 + \xi_3 \).

**Proposition 4:** Let be \((\sigma, f) \in \Theta\). Then \( \delta_0 = 1 \) is true if and only if there is a lattice functional \( t_0 \in \Gamma \) with the property that \((\sigma, f + t_0)\) has a fixed point \( \xi_0 \in B \), i.e. that \( f + t_0 \in B^1(\sigma) \).

**Proof:** We have to take into consideration that \( \langle u; \xi \rangle = 0 \) if \( u \in B^*(\sigma) \) and \( \xi \in B^1(\sigma) \) (see (26) and (51)); for understanding use dual bases in \( B = B(\sigma) \oplus B^1(\sigma) \) and \( B^* = B^*(\sigma) \oplus B^{*-1}(\sigma) \).

a) Assume \( \delta_0 = 1 \) for a fixed \( \sigma \in \Theta \), i.e. \( s^\sigma(\xi) = \langle c^\sigma(\xi), f \rangle \in Z \) for all \( v = 1, \ldots, m \) (see (45), (39) and (37)). Then for an arbitrary \( u = u, c^\sigma(\xi) \in \Gamma^*(\sigma) \) there is true that \( \langle u, f \rangle \in Z \). If we now decompose \( f = f_1 + f_2 \) into \( f_1 \in B(\sigma) \) and \( f_2 \in B^1(\sigma) \) we obtain \( \langle u, f_2 \rangle = 0 \) because \( u \in B^*(\sigma) \). Then we have \( \langle u, f \rangle = \langle u, f_1 \rangle \in Z \) and therefore \( f_1 \in \Gamma(\sigma) \). For each \( \xi \in \Gamma^1(\sigma) \) there is \( t_0 = -f + u \in \Gamma \) and then \( f + t_0 = f_0 + r \in B^1(\sigma) \).

b) Conversely, let there exists a \( t_0 \in \Gamma \) with \( f + t_0 \in B^1(\sigma) \); we prove that \( s^\sigma(\xi) \in Z \) for all \( v = 1, \ldots, m \), i.e. \( \delta_0 = 1 \). We write \( s^\sigma(\xi) = \langle c^\sigma(\xi), f \rangle = \langle c^\sigma(\xi), f + t_0 \rangle - \langle c^\sigma(\xi), t_0 \rangle \). Here \( \langle c^\sigma(\xi), t_0 \rangle \in Z \), because of \( t_0 \in \Gamma, c^\sigma(\xi) \in \Gamma^*(\sigma) \) and so \( c^\sigma(\xi) \in \Gamma^* \). Now using the introductory remark of the proof we find \( \langle c^\sigma(\xi), f + t_0 \rangle = 0 \) because \( f + t_0 \in B^1(\sigma) \), \( c^\sigma(\xi) \in \Gamma^*(\sigma) \) and so \( c^\sigma(\xi) \in B^*(\sigma) \). Summariting we get \( s^\sigma(\xi) \in Z \).

4.5 Survey of the influence of fixed (fixed point - free) elements of group \( \Theta \) on the asymptotic expression for \( N(\lambda) \). We ask for the intrinsic reason of the appearance of the principal term \( c_0 \lambda^n \) and the secondary term \( c_1 \lambda^{n-1} \) in \( N(\lambda) = c_0 \lambda^n + c_1 \lambda^{n-1} + O(\lambda^{n-2} + \lambda^{(n+1)}) \) we can answer (Proposition 4):
(i) For $\sigma \in \mathcal{O}_m$ the fixed elements $(\sigma, \xi + t_o) \in \mathcal{O}$ produce in (49) resp. (50) the volume terms $\text{vol}_{m}(\mathcal{O} \cap \mathcal{O}^*(\sigma)) \lambda^m$ whereas fixed point-free elements from $\mathcal{O}$ make contributions only to the remainder term $O(\lambda^{m-2+2/(m+1)})$. So we have the following knowledge:

(ii) The identity $(e, e) \in \mathcal{O}$ produces the principal part of $N(\lambda)$ (because $\delta_e = 1, e \in \mathcal{O}_n$).

(iii) The fixed elements $(\sigma, \xi + t_o) \in \mathcal{O}, \sigma \in \mathcal{O}_{n-1}$, produces the summands of the secondary part of $N(\lambda)$.

**Concluding remark:** The theory developed above can be applied e.g. for crystallographic groups, especially for the 230 space groups. For short it is recommendable to investigate an $n = 2$-dimensional group, e.g. $\mathcal{O} = \Delta^2_{p31m}$ acting on $\mathcal{B} = \mathbb{E}^2$ and having $\mathcal{P} = c(\partial_1^2 + \partial_1 \partial_2 + \partial_2^2)$ ($\partial_j = \partial/\partial x^j$) as the $\mathcal{O}$-invariant operators for all $c > 0$. The 10 possible examples for $\mathcal{O}$ in the case $n = 2$ demonstrate a considerable improvement if we turn from $N(\lambda) \sim c_0 \lambda^n$ to $N(\lambda) \sim c_0 \lambda^n + c_1 \lambda^{n-1}$ (see the Dissertation B of the author: Zur asymptotischen Verteilung der Eigenwerte $\mathcal{O}$-invarianter linearer elliptischer Differentialoperatoren mit konstanten Koeffizienten. Universität Leipzig 1989).

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