Zeitschrift für Analysis
und ihre Anwendungen
Vol. 11(1992)3, 419–424

Oscillatory Solutions of a System of *n* Nonlinear Integro-Differential Equations of Second Order with Deviating Arguments

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Some conditions are established under which solutions of a system of nonlinear integro-differential equations are oscillatory.

Key words: *Nonlinear integro-differential equations, equations of second order, oscillatory solutions*

AMS subject classification: 45J05, 45M99

1. Introduction

Consider the system of equations

$$y_i''(t) = \int_0^t f_i(t,s; y_1(s), \dots, y_n(s); y_1(g_1(s)), \dots, y_n(g_n(s)))\, ds \quad (i = 1, \dots, n) \tag{1}$$

where

$$f_i: \mathbb{R}_+^2 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \quad \text{and} \quad g_k: \mathbb{R}_+ \to \mathbb{R}_+ \ (k = 1, \dots, n)$$

are continuous functions and

$$g_k(t) \le t \text{ or } g_k(t) > t \text{ for all } t \quad \text{and} \quad \lim_{t\to\infty} g_k(t) = \infty \ (k = 1, \dots, n).$$

Sufficient conditions under which every solution of the system (1) is oscillatory will be established. Both cases of retarded and advanced arguments will be considered. A similar problem for the system of n nonlinear integro-differential equations of the first order have been considered in [1] and [2]. We shall use a method of proofs similar to that given in [4] for the system of n ordinary differential equations with delayed arguments.

2. Preliminaries

By a *solution* $y = \langle y_1, \dots, y_n \rangle$ of the system (1) we shall understand only a non-trivial solution extended to the infinity. A solution y of the system (1) is called

a) *oscillatory*, if every component y_k of y has an infinite sequence of zeros tending to infinity as the argument tends to infinity.

b) *non-oscillatory*, if every component y_k of y has a constant sign for sufficiently large values of the argument t, i.e. for $t \ge T$, *for some* $T \ge 0$.

We shall use the following **Assumptions**:

(i) g_k $(k = 1, \dots, n)$ are continuous and non-decreasing functions.

(ii) $f_i(t,s;y_1,\dots,y_n;u_1,\dots,u_n)\operatorname{sign}u_{i+1} \ge a_i(t,s)|H_i(y_{i+1})|$ $(i = 1,\dots,n-1)$
$f_n(t,s;y_1,\dots,y_n;u_1,\dots,u_n)\operatorname{sign}u_1 \le -a_n(t,s)|H_n(u_1)|$
for all $(t,s;y_1,\dots,y_n;u_1,\dots,u_n) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$, where $H_i\colon \mathbb{R} \to \mathbb{R}$ are continuous and non-decreasing functions, $H_i(u_{i+1})u_{i+1} > 0$, $u_{i+1} \in \mathbb{R}$ $(i = 1,\dots,n)$, $u_{n+1} = u_1$ and $a_i\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ are continuous functions, $a_i(t,s) > 0$ for all $(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+$ $(i = 1,\dots,n)$.

(iii) (α) $\lim\limits_{t\to\infty} \int_T^t \int_T^z a_i(z,s)\,ds\,dz = \infty$ $(i = 1,\dots,n-1)$ (β) $\lim\limits_{t\to\infty} \int_T^t \int_T^z a_n(z,s)\,ds\,dz = \infty$.

(iv) $F_i := \int_0^T f_i(\cdot,s;y_1(s),\dots,y_n(s);y_1(g_1(s)),\dots,y_n(g_n(s)))\,ds \in L_1[a,\infty)$ $(i = 1,\dots,n)$

for a constant $a > 0$, where $|F_i(t)| \le K_i$ for some constants K_i and $F_i(t)y_{i+1}(t+t_0) > 0$ $(i = 1,\dots,n-1)$, $F_n(t)y_1(t+t_0) < 0$ for some $t_0 > 0$, $t \in \mathbb{R}_+$ $(y_{n+1} = y_1)$.

In addition, we shall use the notations

$$M_i = \int_T^\infty |F_i(t)|\,dt \quad \text{and} \quad N_i = \inf_{[T_i,\infty)} |H_i(u_i)| \ \text{ for some } T_i > 0 \quad (i = 1,\dots,n).$$

3. Main results (the case $g_k(t) \le t$)

First we shall consider the system (1) with retarded arguments.

Lemma 1: *Let the assumptions* (i), (ii) *and* (iv) *hold and let* y *be a solution of the system* (1). *If one of its components is non-oscillatory, then* y *itself is non-oscillatory and monotonic.*

Proof: Let y_i be a non-oscillatory component of y. For the proof let be $y_i(t) > 0$ for $t \ge t_0 \ge 0$ and $y_i(g_i(t)) > 0$ for $t \ge T_0$, where $T_0 \ge t_0$. From the system (1) and the assumptions (ii), (iv) it follows $(y_0 = y_{n+1},\ t_0 = t_n)$

$$\begin{aligned} y_{i-1}''(t) &= \int_0^t f_{i-1}(t,s;y_1(s),\dots,y_n(s);y_1(g_1(s)),\dots,y_n(g_n(s)))\,ds \\ &= \int_0^{T_0} f_{i-1}(t,s;y_1(s),\dots,y_n(s);y_1(g_1(s)),\dots,y_n(g_n(s)))\,ds \\ &= +\int_{T_0}^t f_{i-1}(t,s;y_1(s),\dots,y_n(s);y_1(g_1(s)),\dots,y_n(g_n(s)))\operatorname{sign}y_i(g_i(s))\,ds \\ &\ge F_{i-1}(t) + \int_{T_0}^t a_{i-1}(t,s)|H_{i-1}(y_i(s))|\,ds > 0. \end{aligned}$$

Hence y_{i-1}' and y_{i-1} are monotonic functions and, for sufficiently large t (for example for $t \ge t_1$, $t_1 \ge t_0$), they have the constant sign. Now y_{i-1} may be a positive or negative function. Let be $y_{i-1}(t) < 0$ for $t \ge t_1$ and $y_{i-1}(g_{i-1}(t)) < 0$ for $t \ge T_1 \ge t_1$. Hence from the system (1) and the assumptions (ii), (iv) we get

$$y''_{i-2}(t) = \int_0^{T_1} f_{i-2}(t,s; y_1(s), \dots, y_n(s); y_1(g_1(s)), \dots, y_n(g_n(s)))\, ds$$

$$-\int_{T_1}^{t} f_{i-2}(t,s; y_1(s), \dots, y_n(s); y_1(g_1(s)), \dots, y_n(g_n(s)))\operatorname{sign} y_{i-1}(g_{i-1}(s))\, ds$$

$$\leq F_{i-2}(t) - \int_{T_1}^{t} a_{i-2}(t,s)|H_{i-2}(y_{i-1}(s))|\, ds < 0$$

or if $y_{i-1}(t) > 0$ for $t \geq t_1$ and $y_{i-1}(g_{i-1}(t)) > 0$ for $t \geq T_1$, then, from the system (1) and the assumptions (ii), (iv),

$$y''_{i-2}(t) = \int_0^{T_1} f_{i-2}(t,s; y_1(s), \dots, y_n(s); y_1(g_1(s)), \dots, y_n(g_n(s)))\, ds$$

$$= +\int_{T_1}^{t} f_{i-2}(t,s; y_1(s), \dots, y_n(s); y_1(g_1(s)), \dots, y_n(g_n(s)))\operatorname{sign} y_{i-1}(g_{i-1}(s))\, ds$$

$$\geq F_{i-2}(t) + \int_{T_1}^{t} a_{i-2}(t,s)|H_{i-2}(y_{i-1}(s))|\, ds > 0.$$

In both cases it follows that y'_{i-2} and y_{i-2} are monotonic functions and for sufficiently large t they have constant sign (for example for $t \geq t_2$). Proceeding in the same way we obtain $y_i, y_{i-1}, y_1, \dots, y_{i+1}$ for $t \geq T_i \geq t_i$ ($i = 1, \dots, n$) are monotonic and therefore non-oscillatory functions ∎

Corollary 1: *Under the assumptions of Lemma* 1, *if one of the components of the solutuion y of the system* (1) *is oscillatory, then the solution itself is oscillatory.*

Theorem 1: *If the conditions* (i) - (iv) *hold, then every bounded solution of the system* (1) *is oscillatory.*

Proof: Suppose that there exists a bounded and non-oscillatory solution $y = \langle y_1, \dots, y_n \rangle$ of the system (1). Let $y_{i+1}(t) > 0$ for $t \geq t_0 \geq 0$ and $|y_i(t)| \leq Q$ for $t \geq t_0$ ($i = 1, \dots, n$), Q being same constant and $y_{n+1} = y_1$. Then $y_{i+1}(g_{i+1}(t)) > 0$ for $t \geq T_0$, $T_0 \geq t_0$ being some constant. By system (1) and the assumptions (ii), (iv) we have

$$y''_i(t) = \int_0^{T_0} f_i(t,s; y_1(s), \dots, y_n(s); y_1(g_1(s)), \dots, y_n(g_n(s)))\, ds$$

$$+\int_{T_0}^{t} f_i(t,s; y_1(s), \dots, y_n(s); y_1(g_1(s)), \dots, y_n(g_n(s)))\operatorname{sign} y_{i+1}(g_{i+1}(s))\, ds$$

$$\geq F_i(t) + \int_{T_0}^{t} a_i(t,s)|H_i(y_{i+1}(s))|\, ds > 0.$$

Hence

$$y''_i(t) \geq F_i(t) + \int_{T_0}^{t} a_i(t,s)|H_i(y_{i+1}(s))|\, ds. \tag{2}$$

Integrating (2) from T_0 to t and using (iv) we obtain

$$y_i'(t) \geq y_i'(T_0) + \int_{T_0}^{t} |F_i(s)|\, ds + N_i \int_{T_0}^{t}\int_{T_0}^{z} a_i(z,s)\, ds\, dz > 0.$$

If $y_i''(t) > 0$ and $y_i'(t) > 0$, then $y_i(t) \to \infty$ as $t \to \infty$. This is a contradiction with the supposition that the solution of the system (1) is bounded. Therefore every bounded solution of the system (1) is oscillatory ∎

Lemma 2: *If the assumptions* (i) - (iii)/(α) *and* (iv) *are satisfied, then all components* y_i $(i = 1, \dots, n)$ *of a non-oscillatory solution* y *of the system* (1) *have the same sign for sufficiently large* t.

Proof: We shall consider the two cases a) $i = n$ and b) $i \neq n$.

a) ***The case*** $i = n$. Then y_n is a non-oscillatory component. Let for the proof $y_n(t) > 0$ for $t \geq t_{n-1}$. Then $y_n(g_n(t)) > 0$ for $t > T_{n-1} \geq t_{n-1}$ $(t_{n-1}, T_{n-1} = \text{const} \geq 0)$. We shall show that the remaining components are positive. From system (1) and the assumptions (ii), (iv) we have

$$y_{n-1}''(t) \geq F_{n-1}(t) + \int_{T_{n-1}}^{t} a_{n-1}(t,s)|H_{n-1}(y_n((s))|\, ds > 0. \tag{3}$$

Hence y_{n-1}' is an increasing function and for sufficiently large t it has constant sign. Integrating (3) from T_{n-1} to t we obtain

$$\begin{aligned} y_{n-1}'(t) &\geq y_{n-1}'(T_{n-1}) + \int_{T_{n-1}}^{t} |F_{n-1}(s)|\, ds + \int_{T_{n-1}}^{t}\int_{T_{n-1}}^{z} a_{n-1}(z,s)|H_{n-1}(y((s))|\, dz\, ds \\ &\geq c_0 + M_{n-1} + N_{n-1} \int_{T_{n-1}}^{t}\int_{T_{n-1}}^{z} a_{n-1}(z,s)\, dz\, ds, \quad c_0 = y_{n-1}'(T_{n-1}) = \text{const}. \end{aligned}$$

Hence by assumption (iii)/(α) $y_{n-1}'(t) > 0$. If $y_{n-1}''(t) > 0$ and $y_{n-1}'(t) > 0$, then $y_{n-1}(t) \to \infty$ as $t \to \infty$. In this way we can prove that, for $t \geq T_n \geq t_n \geq 0$, $y_i(t) > 0$ $(i = 1, \dots, n)$. (If we suppose that $y_n(t) < 0$, then $y_i(t) < 0$ for $i = 1, \dots, n$, $t \geq T_n \geq t_n \geq 0$.)

b) ***The case*** $i \neq n$. Then $y_i(t) > 0$ for $t \geq t_i \geq 0$ and $y_i(g_i(t)) > 0$ for $t \geq T_i \geq t_i$. We shall show that $y_{i+1}(t) > 0$ for $t \geq t_i$. Suppose conversely that $y_{i+1}(t) < 0$ for $t \geq t_{i+1}$ and $y_{i+1}(g_{i+1}(t)) < 0$ for $t \geq T_{i+1} \geq t_{i+1}$. From the i-th equation of the system (1) and the assumptions (ii), (iv) we have

$$y_i''(t) \leq F_i(t) - \int_{T_{i+1}}^{t} a_i(t,s)|H_i(y_{i+1}(s))|\, ds \leq F_i(t) - N_i \int_{T_{i+1}}^{t} a_i(t,s)\, ds < 0. \tag{4}$$

Integrating (4) from T_{i+1} to t we have

$$y_i'(t) \leq y_i'(T_{i+1}) - \int_{i+1}^{t} |F_i(s)|\, ds - N_i \int_{T_{i+1}}^{t}\int_{T_{i+1}}^{z} a_i(z,s)\, ds\, dz = C_0 - M_i - N_i \int_{T_{i+1}}^{t}\int_{T_{i+1}}^{z} a_i(z,s)\, ds\, dz.$$

Hence by (iii)/(α) $y_i'(t) < 0$. If $y_i''(t) < 0$ and $y_i'(t) < 0$, then $y_i(t) < 0$ for sufficiently large t.

This is a contradiction to the supposition $y_i(t) > 0$. Proceeding in the same way we may show that $y_i(t) > 0$ for $i = 1, \dots, n-1$ and for sufficiently large t.

Suppose that $y_n(t) < 0$ for $t \ge t_{n-1}$ and then $y_n(g_n(t)) < 0$ for $t \ge T_{n-1} \ge t_{n-1}$. From the $(n-1)$-th equation of the system (1) and the assumptions (ii), (iv) we get

$$y''_{n-1}(t) \le F_{n-1}(t) - \int_{T_{n-1}}^{t} a_{n-1}(t,s)|H_{n-1}(y_n(s))|\,ds < 0. \tag{5}$$

Integrating (5) from T_{n-1} to t we get

$$y'_{n-1}(t) \le y'_{n-1}(T_{n-1}) - \int_{T_{n-1}}^{t} |F_{n-1}(s)|\,ds - N_{n-1}\int_{T_{n-1}}^{t}\int_{T_{n-1}}^{z} a_{n-1}(z,s)\,ds\,dz\,.$$

Hence by assumption (iii)/(α) $y'_{n-1}(t) < 0$. If $y''_{n-1}(t) < 0$ and $y'_{n-1}(t) < 0$, then $y_{n-1}(t) < 0$ for sufficiently large t. This is a contradiction to $y_{n-1}(t) > 0$, which was shown early. Therefore $y_n(t) > 0$ and $y_i(t) > 0$ for $i = 1, \dots, n$ ∎

Theorem 2: *Let the assumptions* (i) - (iv) *be satisfied and*

$$\lim_{t\to\infty} \int_{T}^{t} a_n(z,s)\,ds = \infty. \tag{6}$$

Then all solutions of the system (1) *are oscillatory.*

Proof: Suppose that y_1 is a non-oscillatory component of the solution y of (1) and let $y_1(t) > 0$ for $t \ge t_0$. Then $y_1(g_1(t)) > 0$ for $t \ge T_0 \ge t_0$. By Lemma 1 all components y_i $(i = 1, \dots, n)$ are non-oscillatory and monotonic functions. Moreover by Lemma 2 it follows that all non-oscillatory components have the same sign for sufficiently large t. Suppose that all components are positive. (In the case that all components are negative - the proof is analogous.) Then by (6) from the last equation of the system (1) and the assumptions (ii), (iv) it follows that

$$y''_n(t) \le F_n(t) - \int_{T_n}^{t} a_n(t,s)|H_n(y_1(g_1(s)))|\,ds \le K_n - N_n\int_{T_n}^{t} a_n(t,s)\,ds,$$

where K_n is a constant (see (iv)). By assumption (6) $y''_n(t) < 0$. Moreover

$$y''_n(t) \le F_n(t) - N_n\int_{T_n}^{t} a_n(t,s)\,ds. \tag{7}$$

Integrating (7) from T_n to t and by assumption (iv) we have

$$y'_n(t) \le y'_n(T_n) + M_n - N_n\int_{T_n}^{t}\int_{T_n}^{z} a_n(z,s)\,ds\,dz\,.$$

By assumption (iii)/(β) $y'_n(t) < 0$. If $y''_n(t) < 0$ and $y'_n(t) < 0$, then $y_n(t) < 0$ for sufficiently large t. This contradiction proves that the component y_1 is oscillatory. By Corollary 1, if y_1 is an oscillatory component, then all components of the solution y of the system (1) are oscillatory functions ∎

4. Main results (the case $g_k(t) > t$)

In the case of advanced arguments our lemmas and theorems have the following form.

Lemma 1': *Let the assumptions* (i) *and* (ii) *be satisfied and let y be a solution of the system* (1). *If one of its components is non-oscillatory, then y itself is non-oscillatory and monotonic.*

Lemma 2': *If the assumptions* (i) - (iii)/(α) *are satisfied, then all components* y_i ($i = 1, \ldots, n$) *of a non-oscillatory solution y of the system* (1) *have the same sign for sufficiently large* t.

Theorem 1': *If the assumptions* (i) - (iii) *are satisfied, then every bounded solution of the system* (1) *is oscillatory.*

Theorem 2': *Let the assumptions* (i) - (iii) *and* (6) *be satisfied. Then all solutions of the system* (1) *are oscillatory.*

The proofs of these statements are similar as in the case $g_k(t) \leq t$.

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Received 29.05.1990; in revised version 28.05.1992

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