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On a Nonlinear Binomial Equation of Third Order

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A necessary and sufficient condition for the solution of equation $u''' + p(t)u^{\alpha} = 0$ ($\alpha > 0$ an odd integer, $p \le 0$ on (a,∞)) to be oscillatory and some sufficient conditions for the solution in the cases $p \le 0$ and $p \ge 0$ to be oscillatory or non-oscillatory are derived. For this methods and results of the theory of linear differential equations of the third order are effectively used.

Key words: *Third order nonlinear differential equations, oscillatory solutions, non-oscillatory solutions, bounded solutions*

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1. The paper investigates properties of solutions of the binomial differential equation of third order $u''' + pu^{\alpha} = 0$, (1)

where p is a continuous function on the interval (a,∞) with $a > -\infty$, and $\alpha > 1$ is an odd number. Some of our results can be generalized to the case where α is a ratio of odd integers. The problem has already been a research object of many authors, see [1, 3-6] and others. Here the methods developed in the study of linear differential equation of third order [2] are effectively used.

2. By a *solution* of equation (1) we mean a function u defined on a subinterval $\mathfrak{J} \subset (a,\infty)$, with continuous third derivative and satisfying equation (1). By an *oscillatory solution* of equation (1) we mean a solution u of (1) that has on the intervall $\mathfrak{J}$ infinitely many null points, with a limit point at the right end point of the intervall $\mathfrak{J}$. Otherwise the solution is called *non-oscillatory*. A non-extentable solution u defined on a bounded from above intervall $\mathfrak{J}$ is sometimes called *singular*.

Equation (1) can be written in the linear form

$$u''' + pu^{\alpha-1}u = 0. \qquad (1)^*$$

The *adjoint equation* to $(1)^*$ has the form

$$v''' - pu^{\alpha-1}v = 0. \qquad (2)$$

Let $t_0 \in \mathfrak{J}$ and let u be a solution of equation (1) with the property $u(t_0) = u_0$, $u'(t_0) = u_0'$, $u''(t_0) = u_0''$, where at least one of the numbers u_0, u_0', u_0'' is non-zero. Further, let v be a solution of equation (2) with the property $v(t_0) = v_0$, $v'(t_0) = v_0'$, $v''(t_0) = v_0''$, where again at least one of the numbers v_0, v_0', v_0'' is non-zero. Then for $t \in \mathfrak{J}$ we have (see [2])

$$v(t)u''(t) - v'(t)u'(t) + v''(t)u(t) = \text{const}, \qquad (3)$$

where const $= v_0u_0'' - v_0'u_0' + v_0''u_0$.

If we multiply equation (1)* by the solution u and integrate from t_0 to $t \in \mathfrak{J}$, then we obtain for all $t \in \mathfrak{J}$ the integral identity

$$u(t)u''(t) - \frac{1}{2}u'^2(t) + \int_{t_0}^{t} p(\tau)u^{\alpha-1}(\tau)u^2(\tau)\,d\tau = \text{const.} \tag{4}$$

Similarly, for equation (2) we obtain for all $t \in \mathfrak{J}$

$$v(t)v''(t) - \frac{1}{2}v'^2(t) - \int_{t_0}^{t} p(\tau)u^{\alpha-1}(\tau)v^2(\tau)\,d\tau = \text{const.} \tag{5}$$

Corollary 1: *Let $p \geq 0$ ($p \leq 0$) on (a, ∞) and $p \neq 0$ on any subinterval of (a, ∞). Further, let u be a solution of equation* (1) *defined on an interval $\mathfrak{J} \subset (a, \infty)$ and with the property $u(t_0) = u'(t_0) = 0$, $u''(t_0) \neq 0$ for some $t_0 \in \mathfrak{J}$. Then $u(t) \neq 0$, $u'(t) \neq 0$, $u''(t) \neq 0$ for all $t < t_0$ ($t > t_0$). A similar assertion holds for the solution v of the equation* (2) *with the property $v(t_0) = v'(t_0) = 0$, $v''(t_0) \neq 0$ for some $t_0 \in \mathfrak{J}$, that is $v(t) \neq 0$, $v'(t) \neq 0$, $v''(t) \neq 0$ for all $t > t_0$ ($t < t_0$).*

Proof: It follows from the identities (4) and (5) and from the equations (1) and (2), respectively ∎

Corollary 2: *Supposing p is the same as in Corollary* 1, *each solution u of equation* (1) *or* (2) *has at most one double null point.*

3. Our goal is to derive some properties of solutions of equation (1) in the case $p \leq 0$.

Theorem 1: *Let $p \leq 0$ on (a, ∞). Then any non-extendable solution u of equation* (1) *defined on an interval $\mathfrak{J} \subset (a, \infty)$ and such that $u(t_0) \geq 0$, $u'(t_0) \geq 0$, $u''(t_0) > 0$ for some $t_0 \in \mathfrak{J}$ has the property $u(t) > 0$, $u'(t) > 0$, $u''(t) > 0$, $u'''(t) \geq 0$ for all $t > t_0$ and, moreover, $u(t) \to \infty$, $u'(t) \to \infty$ as t converges to the right end point of the interval $\mathfrak{J}$.*

Proof: First of all we show that $u''(t) > 0$ for all $t > t_0$. Let us form the function $V \doteq uu'u''$. If u'' has null points to the right of t_0, let us denote by t_1 the smallest of them. Hence $u''(t_1) = 0$. Therefore $u(t) > 0$, $u'(t) > 0$ for all $t \in (t_0, t_1)$ and $V(t_0) \geq 0$, $V(t_1) = 0$. Since $p \leq 0$ there holds

$$dV(t)/dt = u''(t)u(t) + u''(t)u'^2(t) - p(t)u^{\alpha+1}(t)u'(t) > 0 \text{ for all } t \in (t_0, t_1).$$

After integration from t_0 to t_1 we obtain $0 = V(t_0) + \int_{t_0}^{t_1} V'(\tau)\,d\tau > 0$, which is a contradiction. Hence $u''(t) > 0$ *for all* $t > t_0$. From here it follows that $u(t) > 0$, $u'(t) > 0$ for all $t > t_0$. From equation (1) it also follows that $u'''(t) \geq 0$ *for all* $t > t_0$. From these inequalities we then have that $u(t) \to \infty$, $u'(t) \to \infty$ *as* t converges to the right end point of the interval $\mathfrak{J}$.

N. Parhi and S. Parhi have proved the following

Theorem A [6 : Theorem 3.1]: *Let $p \leq 0$ and $\int_{t_0}^{\infty} p(\tau)\,d\tau = -\infty$. Then every bounded solution of equation* (1) *in (t_0, ∞) is oscillatory in (t_0, ∞).*

Lemma 1: *Let the assumptions of Theorem A be fulfilled and let u be a solution of equation* (1) *with the property $u(t) > 0$ for all $t \geq t_0$, where $t_0 > a$. Then there exists such $t_1 > t_0$ that $u(t) > 0$, $u'(t) > 0$, $u''(t) > 0$ for all $t > t_1$.*

Proof: From equation (1) it follows that $u'''(t) \geq 0$ *for all* $t > t_0$. Then we have two possibilities for u'':

1. $u''(t_0) > 0$ *and hence* $u''(t) > 0$ *for all* $t > t_0$. Then after integration of equation (1) we get

$$\begin{aligned} u''(t) &= u''(t_0) - \int_{t_0}^{t} p(\tau)u^{\alpha}(\tau)\,d\tau, \\ u'(t) &= u'(t_0) + u''(t_0)(t - t_0) - \int_{t_0}^{t}(t-\tau)p(\tau)u^{\alpha}(\tau)\,d\tau, \\ u(t) &= u(t_0) + u'(t_0)(t - t_0) + u''(t_0)\frac{(t-t_0)^2}{2!} - \int_{t_0}^{t}\frac{(t-\tau)^2}{2!}p(\tau)u^{\alpha}(\tau)\,d\tau. \end{aligned} \tag{6}$$

From the second equation of (6) the existence of such $t_1 > t_0$ follows that $u'(t) > 0$ for all $t \geq t_1$. Then $u(t) > 0$, $u'(t) > 0$, $u''(t) > 0$ *for all* $t \geq t_1$.

2. $u''(t) < 0$ *for all* $t \geq t_0$. Then u' is decreasing and there are again two possibilities:

(i) $u'(t) < 0$ *for all* $t \geq t_1$ *and* u' *decreasing*. Hence $u'(t) < u'(t_1)$ from where $u(t) < u(t_1) + u'(t_1)(t - t_1)$ and this is a contradiction to the assumption that $u(t) > 0$ for all $t > t_0$.

(ii) $u'(t) > 0$ *for all* $t \geq t_0$. Then the function u is increasing for $t > t_0$ and after an integration of equation (1) we get $u''(t) = u''(t_0) - \int_{t_0}^{t} p(\tau)u^{\alpha}(\tau)\,d\tau$. From here and from the assumptions on p there follows that, for certain $t_1 > t_0$, $u''(t) > 0$ for all $t > t_1$ and this again leads to a contradiction to the assumption that $u''(t) < 0$ for all $t \geq t_0$ ∎

The following theorem answers to the question which solutions of equation (1), under the assumptions of Theorem A, can be oscillatory.

Theorem 2: *Let the assumptions of Theorem A concerning p be fulfilled. Then a necessary and sufficient condition for a solution u of equation* (1) *to be oscillatory for* $t \geq t_0$, *for some* $t_0 > a$, *is that*

$$u(t)u''(t) - u'^2(t)/2 < 0 \textit{ for all } t > t_0. \tag{7}$$

Proof: *Sufficiency.* Let (7) hold and let e.g. $u(t) > 0$ *for all* $t > t_0$. It follows from Lemma 1 that there exists such $t_1 \geq t_0$ that $u(t_1) > 0$, $u'(t_1) > 0$, $u''(t_1) > 0$ and, from Theorem 1, $u(t) \to \infty$ as $t \to \infty$. From the integral identity (4) it follows that

$$u(t)u''(t) - u'^2(t)/2 = u(t_1)u''(t_1) - u'^2(t_1)/2 - \int_{t_1}^{t} p(\tau)u^{\alpha+1}(\tau)\,d\tau \tag{8}$$

and from this and the assumptions of Theorem 2 there follows a contradiction with (7) as $t \to \infty$.

Necessity. Let the solution u of equation (1) be oscillatory in (t_0, ∞) and let t_i $(i = 1, 2, \ldots)$ be null points of u in (t_0, ∞). Then from the relation (8) it follows that the function $uu'' - u'^2/2$ is increasing in (t_1, ∞), but $u(t_i)u''(t_i) - u'^2(t_i)/2 < 0$. From this fact it follows that (7) holds for all $t > t_1$ ∎

Theorem 3: *Suppose that* $p \leq 0$ *on* (a, ∞) *and* $p \neq 0$ *on any subinterval of* (a, ∞). *Let u be a solution of equation* (1) *defined on an interval* $\mathfrak{J} \subset (a, \infty)$ *and satisfying* $k := u(t_0)u''(t_0) - u'^2(t_0)/2 \geq 0$ *for some* $t_0 \in \mathfrak{J}$. *Then u does not have a null point to the right of* t_0 *and* $|u(t)| \to \infty$, $|u'(t)| \to \infty$ *as t converges to the right end point of* $\mathfrak{J}$.

Proof: The solution u fulfils the identity (4), i.e.

$$u(t)u''(t) - u'^2(t)/2 + \int_{t_0}^{t} p(\tau)u^{\alpha+1}(\tau)d\tau = k \geq 0 \text{ for all } t \in \mathfrak{J}. \tag{9}$$

Let $u(t_1) = 0$ for some $t_1 > t_0$. Then from the identity above at the point t_1 we get a contradiction. To prove the second part of the assertion let us suppose for simplicity that $u(t) > 0$ for all $t > t_0$. Then also $u'''(t) \geq 0$ for all $t > t_0$ and from the identity (9) it follows that $u''(t) \geq 0$ for all $t > t_0$. Suppose that $\mathfrak{J}$ is a bounded interval with right end point b and let u be bounded on it. Then also u'' is bounded as follows from the first relation in (6). Note that u'' is a monotone function. From the second relation in (6) it follows that the function u' is also monotone and bounded. Hence $u(t) \to u_0$, $u'(t) \to u_0'$, $u''(t) \to u_0''$ as $t \to b$, where u_0, u_0', u_0'' are real numbers. That means u can be extended to b, which is a contradiction and therefore $u(t) \to \infty$, $u'(t) \to \infty$ as $t \to b$. In the case $b = \infty$ the proof is trivial - it follows from the monotonicity of the functions u'', u''' and from (6) ∎

Theorem 4: *Let $p(t) < -k^2$ $(k > 0)$ for all $t > t_0$. Then each oscillatory solution u of the equation* (1) *defined on (t_0, ∞) belongs to the class $\mathfrak{L}^{\alpha+1}$ on $[t_0, \infty)$, i.e. $\int_0^{\infty} u^{\alpha+1}(\tau)\, d\tau < \infty$.*

Proof: It follows again from the identity (4). Really, from Theorem 2 it follows that

$$u(t)u''(t) - u'^2(t)/2 = u(t_0)u''(t_0) - u'^2(t_0)/2 - \int_{t_0}^{t} p(\tau)u^{\alpha+1}(\tau)d\tau < 0.$$

This implies $-\int_{t_0}^{\infty} p(\tau)u^{\alpha+1}(\tau)d\tau < \infty$ ∎

4. Now our goal is to derive properties of solutions of equation (1) in the case $p \geq 0$. For this let u be a solution of the differential equation (1) defined on an interval $\mathfrak{J} \subset (a, \infty)$ and suppose that it fulfils the initial conditions $u(t_0) = u_0$, $u'(t_0) = u_0'$, $u''(t_0) = u_0''$ for some $t_0 \in \mathfrak{J}$. Notice that the relations (6) hold.

Lemma 2: *Let $p \geq 0$ on (a, ∞) and let $u \not\equiv 0$ be a non-extentable solution of equation* (1) *defined on $[t_0, b)$, for some $b \in (t_0, \infty]$. Then $b = \infty$.*

Proof: It follows from the relations (6). Indeed, suppose $b < \infty$, $u(t) > 0$ for all $t \in [t_0, b)$ and bounded from above. Then from the relations (6) it follows that u can be extended to b. If u is unbounded on $[t_0, b)$ and $\int_{t_0}^{b}(b - \tau)^2 p(\tau)u^{\alpha}(\tau)d\tau$ exists, then u and also u', u'' can be extended to b. If $\int_{t_0}^{b}(b - \tau)^2 p(\tau)u^{\alpha}(\tau)d\tau = \infty$, then from the third relation in (6) it follows that u must have a zero and this is a contradiction to $u(t) > 0$ for all $t \in [t_0, b)$ ∎

Remark 1: Lemma 2 does not hold in the case of extendability of the solution to the left of the point t_0. For example the equation $u''' + \alpha(\alpha + 1)(\alpha + 2)t^{\alpha^2 - \alpha - 3}u^{\alpha} = 0$ has a solution $u = t^{-\alpha}$ defined on $(0, \infty)$. It cannot be extended to the left of 0.

Lemma 3: *Let $p \geq 0$ on (a, ∞) and let u be a solution of equation* (1) *which for some $t_0 > a$ and $b \in (a, \infty)$ is oscillatory on $[t_0, b)$. Then u is unbounded on $[t_0, b)$.*

Proof: It again follows from the relations (6). If we suppose that u is bounded on $[t_0, b)$, then from the third relation in (6) and from the Cauchy Criterion we obtain that u can be extended to b, too ∎

The paper [5] contains a theorem of I. Ličko and M. Švec that we restate for the equation (1), $\alpha > 1$ and odd.

Theorem B: *A necessary and sufficient condition for either oscillatority or monotonic convergence to zero together with its first and second derivative of each solution of the equation* (1) *on* $[t_0,\infty)$ $(t_0 > a)$ *is that* $p(t) > 0$ *for all* $t > a$ *and* $\int_{t_0}^{\infty} t^2 p(t)dt = \infty$.

The problem is which solutions of equation (1) are oscillatory on the interval (t_0,∞) and which on the subinterval $\mathfrak{J} \subset (a,\infty)$.

Theorem 5: *Suppose that p fulfils the conditions of Theorem B. Then each solution u of equation* (1) *defined on the subinterval* $\mathfrak{J} \subset (a,\infty)$ *and such that*

$$u(t_0)u''(t_0) - u'^2(t_0)/2 = -\delta < 0 \text{ for some } t_0 \in \mathfrak{J} \tag{10}$$

is oscillatory for $t > t_0$.

Proof: Let $t_0 \in (a,\infty)$ and let u be a solution of equation (1) with the property (10) and defined on $\mathfrak{J}$. Then either $\mathfrak{J}$ is bounded from above or $\mathfrak{J} = [t_0,\infty)$. In the first case u must be oscillatory for $t > t_0$ as follows from Lemma 2. In the second case let us suppose that $u(t) > 0$ for $t \in (t_0,\infty)$. Theorem B then implies that $u'(t) < 0$, $u''(t) > 0$ for all $t > t_1$, for some $t_1 \geq t_0$, and $u(t) \to 0$, $u'(t) \to 0$ as $t \to \infty$. However from the integral identity (4) we get $u(t)u''(t) - u'^2(t)/2 + \int_{t_0}^{t} p(\tau)u^{\alpha+1}(\tau)d\tau = -\delta < 0$, which implies

$$u'^2(t)/2 = u(t)u''(t) + \int_{t_0}^{t} p(\tau)u^{\alpha+1}(\tau)d\tau + \delta \geq \delta \text{ for all } t \geq t_0,$$

but this contradicts the assumption $u'(t) \to 0$ as $t \to \infty$ ∎

Theorem 6: *Suppose that p satisfies the conditions of Theorem B. Then each solution u of equation* (1) *with double null point at* $t_0 > a$ *oscillates on the right of* t_0.

Proof: Again there are two cases. In the case when u is defined on a bounded from above interval it must, by Lemma 2, oscillate. In the second case when u is defined on $[t_0,\infty)$ and we suppose that $u(t) > 0$ for all $t > t_1$, for some $t_1 \geq t_0$, it has to converge together with its first and second derivatives to zero as $t \to \infty$ and moreover it has to satisfy $u(t) > 0$, $u'(t) < 0$, $u''(t) > 0$ for all $t > t_2$, for some $t_2 \geq t_1$. Let us substitute u into equation (2) and suppose that v is its solution with the property $v(t_0) = v'(t_0) = 0$, $v''(t_0) > 0$. From Corollary 1 we have that $v(t) > 0$, $v'(t) > 0$, $v''(t) > 0$ for all $t > t_0$. We use u and v to generate equation (3), i.e.

$$vu'' - v'u' + v''u = 0. \tag{11}$$

If u is non-oscillatory we get from equation (11) a contradiction from the fact that $v(t)u''(t) - v'(t)u'(t) + v''(t)u(t) > 0$ for all $t > t_2$ ∎

Let $p(t) > 0$ for all $t \in (a,\infty)$ and let u be a solution of equation (1) defined on $\mathfrak{J}$ and satisfying $u(t_0) = u'(t_0) \geq 0$, $u''(t_0) > 0$ *for some* $t_0 \in \mathfrak{J}$. Further let v be a solution of equation (2) defined on $\mathfrak{J}$ and satisfying $v(t_0) = v'(t_0) = 0$, $v''(t_0) > 0$. Then equation (11) holds for $t > t_0$, where $v(t) > 0$, $v'(t) > 0$ for $t > t_0$. Let us make the substitution $u = \sqrt{v}\,y$ into equation (11). It then takes the form

$$y'' + (3v''/2v - 3v'^2/4v^2)y = 0. \tag{12}$$

From the integral identity (5) for v and $t > t_0$ we get

$$3v''(t)/2v(t) - 3v'^2(t)/4v^2(t) = 3/2v^2(t)\int_{t_0}^{t} p(\tau)u^{\alpha-1}(\tau)v^2(\tau)d\tau$$

and equation (12) is transformed into

$$y''(t) + \left(3/2v^2(t)\int_{t_0}^{t} p(\tau)u^{\alpha-1}(\tau)v^2(\tau)d\tau\right)y = 0. \tag{13}$$

From the reasoning above we obtain

Theorem 7: *A necessary and sufficient condition for a solution u of equation* (1) *to be oscillatory for* $t > t_0 \in \mathfrak{J}$ *is that equation* (13) *or* (12) *is oscillatory fort* $> t_0 \in \mathfrak{J}$.

Apparently, Theorem 7 does not have any practical significance for determination of oscillatoricity or non-oscillatoricity of solutions of equation (1). However, as we shall see in the following, it has a theoretical importance.

Corollary 3: *Let* $p(t) > 0$ *for* $t \in (a, \infty)$ *and let u be a non-extendable solution of equation* (1) *on* $\langle t_0, b)$, $a < t_0 < b < \infty$, *with the property* $u(t_0) = 0$, $u'(t_0) \geq 0$, $u''(t_0) > 0$. *Then*

$\int_{t_0}^{t} p(\tau)u^{\alpha-1}(\tau)d\tau \to \infty$ *as* $t \to b$.

Proof: From Lemma 3 we have that u is oscillatory on $\langle t_0, b)$ and from equation (1)* it follows that the limit of its null points is b. Suppose that v is a solution of equation (2) which is adjoint to the solution u and has the property $v(t_0) = v'(t_0) = 0$, $v''(t_0) > 0$. From Corollary 1 we have that $v(t) > 0$, $v'(t) > 0$, $v''(t) > 0$ for all $t > t_0$. The function u is obviously a solution of equation (11) and hence by Theorem (7) equation (13) must be oscillatory on $\langle t_0, b)$, $b < \infty$. This is possible only if

$$1/v^2(t)\int_{t_0}^{t} p(\tau)u^{\alpha-1}(\tau)v^2(\tau)d\tau \to \infty \text{ as } t \to b. \tag{14}$$

However for $t > t_0$ clearly the inequality

$$1/v^2(t)\int_{t_0}^{t} p(\tau)u^{\alpha-1}(\tau)v^2(\tau)d\tau \leq \int_{t_0}^{t} p(\tau)u^{\alpha-1}(\tau)d\tau$$

holds. Hence the assertion follows ∎

Corollary 4: *Suppose that the assumptions of Corollary* 3 *hold. Then any solution v of the equation* (2) *satisfying the condition* $v(t_0) = v'(t_0) = 0$, $v''(t_0) > 0$ *has the property* $v(t) \to \infty$, $v'(t) \to \infty$, $v''(t) \to \infty$ *as* $t \to b$.

Proof: Relation (14) implies the relation

$$\int_{t_0}^{t} p(\tau)u^{\alpha-1}(\tau)v^2(\tau)d\tau \to \infty \text{ as } t \to b. \tag{15}$$

The integral identity (5) for the solution v has the form

$$v(t)v''(t) - v'^2(t)/2 - \int_{t_0}^{t} p(\tau)u^{\alpha-1}(\tau)v^2(\tau)d\tau = 0.$$

Suppose that v is bounded on $\langle t_0, b)$. Then

$$v(t)v''(t) = v'^2(t)/2 + \int_{t_0}^{t} p(\tau)u^{\alpha-1}(\tau)v^2(\tau)d\tau .$$

From this and from relation (15) it follows that $v''(t) \to \infty$ as $t \to b$ and therefore also $v'(t) \to \infty$, $v(t) \to \infty$ as $t \to \infty$. But this is in contradiction with the assumption that v is bounded ∎

Suppose we have linear differential equations of the third order

$$(\mathbf{p_1}) \quad y''' + p_1 y = 0 \qquad \text{and} \qquad (\mathbf{p_2}) \quad z''' + p_2 z = 0,$$

where p_1, p_2 are continuous functions on (a,∞), $p_1(t) > 0$ and $p_2(t) > 0$ for all $t \in (a,\infty)$.

Lemma 4: *Let $p_1 \le p_2$ on (a,∞). If equation (p_2) is non-oscillatory in (a,∞) (i.e. each of its solutions has at most a finite number of null points in (a,∞)), then the equation (p_1) is also non-oscillatory in (a,∞).*

Proof: The assertion is contained in Theorem 2.5 and Corollary 2.5 of [2], respectively ∎

Let us denote the adjoint equation $z''' - p_1 z = 0$ to equation (p_1) by $(\mathbf{\overline{p}_1})$.

Lemma 5: *Let $p_1(t) > 0$ for $t \in (a,\infty)$ and w be a solution of equation $(\overline{p}_1)$ with the property $w(t_0) = w'(t_0) = 0$, $w''(t_0) > 0$ for some $t_0 \in (a,\infty)$. Then the set of solutions y of equation (p_1) with the property $y(t_0) = 0$ (called the bundle of solutions of equation (p_1) in the point t_0) satisfies the equation* **(w)** *$wy'' - w'y' + w''y = 0$. Differentiating equation (w) term by term we obtain the equation (p_1). If equation (w) is non-oscillatory on $\langle t_0,\infty)$, then equation (p_1) is also non-oscillatory on $\langle t_0,\infty)$.*

The proof of this lemma is not included since it is the basic property of linear equations of third order [2].

Remark 2: The assertion of Lemma 5 holds for arbitrary solutions w of equation $(\overline{p}_1)$, but the interesting case is $w(t) \neq 0$ for $t > t_0$.

Theorem 8: *Suppose $p(t) > 0$ for all $t \in (a,\infty)$ and let f be a given function with continuous third derivative*, $f(t) > 0$ and $f'''(t) > 0$ *for all $t \in (a,\infty)$, such that the equation*

$$y'' + (3f''/2f - 3f'^2/4f^2)y = 0 \tag{16}$$

is non-oscillatory in (a,∞). Then each solution $\overline{u}$ of equation (1), with the property $\overline{u}(t_0) = 0$ for some $t_0 > a$ and which is defined on $\langle t_0,\infty)$ and satisfies the inequality

$$p(t)\overline{u}^{\alpha-1}(t) \le f'''(t)/f(t) \text{ for all } t \ge t_0, \tag{17}$$

is non-oscillatory on $\langle t_0,\infty)$.

Proof: Besides of the equation

$$u''' + p\overline{u}^{\alpha-1}u = 0 \tag{18}$$

we have the equation

$$v''' + (f'''/f)v = 0, \tag{19}$$

that has been obtained by differentiating the equation

$$fv'' - f'v' + f''v = 0 \qquad (20)$$

and which by the transformation $v = \sqrt{f}\,y$ can be converted into the equation (16). From the assumption that equation (16) is non-oscillatory on $\langle t_0, \infty)$ it follows that equation (20) is non-oscillatory on $\langle t_0, \infty)$ and from Lemma 5 we have that equation (19) is non-oscillatory, too. From the assumption (17) and from Lemma 4 it follows that equation (18) is non-oscillatory on $\langle t_0, \infty)$. Since $\bar{u}$ is a solution of equation (18), it is therefore non-oscillatory on $\langle t_0, \infty)$ ∎

Corollary 5: *Let $f(t) = t^n$, where $n = 1 + 2/\sqrt{3}$ and let $a \geq 0$. Then the equation* (1) *does not have an oscillatory solution $\bar{u}$ with null point in the point $t_0 > a$ on the interval $\langle t_0, \infty)$ that would satisfy the relation* (17), *i.e. the relation*

$$\bar{u}^{\alpha-1}(t) \leq 2/(3\sqrt{3}\,t^3 p(t)) \text{ for all } t \geq t_0. \qquad (21)$$

Proof: The equation (16) has the form

$$y'' + (3(n^2 - 2n)/4t^2)y = 0. \qquad (22)$$

Let $3(n^2 - 2n) = 1$. The positive root of this equation is $n = 1 + 2/\sqrt{3}$. By the well-known Kneser criterion equation (22) is non-oscillatory and hence equation (19) is non-oscillatory if $f'''(t)/f(t) = n(n-1)(n-2)/t^3 = 2/(3\sqrt{3}\,t^3)$. This and the relation (17) imply the relation (21) ∎

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