Determination of a Real Parameter in the Coefficient of a Quasilinear Elliptic Differential Equation

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We study the quasilinear elliptic differential equation
\[ \text{div} \left( (b(u - u_0)^l + c) \nabla u \right) = 0 \]
with boundary value conditions of Dirichlet type on an \( n \)-dimensional cylinder. Provided we know the numbers \( u_0, l, \) and \( c, \) the solution \( u \) on a plane which is parallel to the basic area of the cylinder, an eigenvalue and an eigenfunction of a certain eigenvalue problem, we give an explicit formula for the calculation of the real parameter \( b. \)

Key words: Inverse problems, boundary value problems, quasilinear elliptic differential equations

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1. Introduction

Let \( D \) be a bounded region of the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) with piecewise smooth boundary \( \partial D. \) Points of \( \mathbb{R}^{n-1} \) are denoted by \( x^* = (x_1, \ldots, x_{n-1}) \), those of \( \mathbb{R}^n \) by \( x = (x', x_n). \) We consider the quasilinear elliptic boundary value problem
\[
Lu(x) = \text{div}(a(u(x))\nabla u(x)) = 0 \quad \text{in } D \tag{1.1}
\]
\[
u(x) = g(x) \quad \text{on } \partial D, \tag{1.2}
\]
where \( a \) is a function of the real function \( u = u(x) \) which is strictly positive, continuously differentiable and satisfies the condition
\[
u \in C(\overline{D}) \cap C^2(D). \tag{1.3}
\]
We prove the validity of a weak maximum principle in the following sense.

**Definition 1** [3]: Let \( a(u) \) be strictly positive and continuously differentiable, \( u \in C'(D). \) Then, in a generalized sense, \( u \) is said to satisfy the relation \( Lu = 0 \) \((\geq 0, \leq 0)\) in \( D, \) in dependence on whether \( \int_D a(u(x)) \nabla u(x) \nabla \varphi(x) \, dx = 0 \) \((\leq 0, \geq 0)\) for all non-negative functions \( \varphi \in C_0^2(D). \)

**Theorem 1**: Let \( a(u) \) be strictly positive and continuously differentiable, \( u \) satisfy the condition (1.3) and suppose
\[
\int_D a(u(x)) \nabla u(x) \nabla \varphi(x) \, dx = 0 \tag{1.4}
\]
for all non-negative functions \( \varphi \in C_0^2(D). \) Then
\[
\inf_{\partial D} u \leq \inf_D u \tag{1.5}
\]
and
\[
\sup_D u \leq \sup_{\partial D} u. \tag{1.6}
\]
Proof: We prove (1.6) supposing, contrary to the assertion, that \( \sup_{D'} u > \sup_{\partial D} v = u_1 \). Then, for some constant \( c > 0 \), there is a subdomain \( D' \subset D \) in which \( v = u - u_1 - c > 0 \) and \( v > 0 \) on \( \partial D' \). The relation (1.4) remains true with \( u \) replaced by \( v \), and with \( \varphi = v \) on \( D' \), \( \varphi = 0 \) elsewhere. We have \( \varphi \in C_0^1(D) \), but (1.4) can be seen to hold by approximating \( \varphi \) with functions in \( C_0^1(D) \). It follows \( \int_{D'} a(\nabla v)^2 \, dx = 0 \) and hence, since \( a \) is a positive function, we infer \( \nabla v = 0 \) in \( D' \). On account of \( v = 0 \) on \( \partial D' \) we have \( v = 0 \) in \( D' \), which contradicts the definition of the function \( v \). Analogously the inequality (1.5) can be proved.

We set \( u_0 = \inf_{\partial D} u \) and suppose \( u_0 < u_1 \). Then we obtain from (1.5) and (1.6) \( u_0 \leq u(x) \leq u_1 \) for all \( x \in \bar{D} \). In the following the coefficient \( a(u) \) in the equation (1.1) may be given in the form

\[
a(u) = b(u - u_0)^l + c \quad \text{for all } [u_0, u_1],
\]

where \( l \) is a natural number and \( b, c \) are real constants, with \( c > 0 \) and \( b > -c(u_0 - u_1)^{-l} \). This implies \( a(u) > 0 \) for all \( u \in [u_0, u_1] \). Further we assume \( c = a(u_0) \) and \( l \) to be known. Then \( a(u) \) is available completely if the constant \( b \) is known. Finally, in addition to the assumptions stated, suppose that for every function \( a(u) \) of the form (1.7) (i.e. for every \( b > -c(u_0 - u_1)^{-l} \)) there exists a unique solution \( u = u(x,b) \) of the boundary value problem (1.1) - (1.2). For brevity we write \( u(x) = u(x,b) \).

The inverse problem to the boundary value problem (1.1) - (1.2) consists in the determination of the coefficient \( a(u) \) (i.e. of the real parameter \( b \)) under some additional information on \( u \).

In the second section we describe (under some restrictions on the domain \( D \)) a procedure for the calculation of the number \( b \), which also implies its existence and uniqueness.

The third section contains three special cases and one example.

A quasilinear parabolic equation with a coefficient of the form (1.7) is investigated by MEYER [4,5]. CANNON [2] has considered the equation (1.1) with the Neumann boundary condition \( a(u) \partial u/\partial v = g \) on \( \partial D \) and the additional condition \( u = f \) on \( C \), where \( C \) denotes a smooth curve on a portion of \( \partial D \). Some results on inverse problems in quasilinear equations are given by ANGER [1].

2. A method for the determination of the number \( b \)

We use the following notations:

- \( D_{n-1} \subset \mathbb{R}^{n-1} \) is a bounded region with sufficiently smooth boundary \( \partial D_{n-1} \),
- \( (d_1, d_2) \) is an interval on the \( x_n \)-axis,
- \( D = D_{n-1} \times (d_1, d_2), \bar{B} = D_{n-1} \times [d_1, d_2] \),
- \( \Delta_k \) is the Laplacian in the \( k \)-dimensional space.

The boundary value problem (1.1) - (1.2) is considered with a coefficient \( a(u) \) of the form (1.7) and the special boundary conditions

\[
\begin{align*}
  u(x', d_1) &= p_1(x') \quad \text{for } x' \in D_{n-1}, x_n = d_1 \\
  u(x', d_2) &= p_2(x') \quad \text{for } x' \in D_{n-1}, x_n = d_2 \\
  u(x) &= q(x) \quad \text{for } x \in \bar{B},
\end{align*}
\]

where \( p_1, p_2, q \) are given functions satisfying the conditions

\[
p_1, p_2 \in C(D_{n-1}), q \in C(\bar{B})
\]
p_i(x') = q(x', d_i) for x' \in D_{n-1} and i = 1, 2. \tag{2.5}

From (2.4) and Theorem 1 it follows

\[
\begin{align*}
\mu_0 &= \min \left\{ \min_{x' \in D_{n-1}} p_i(x') \ (i = 1, 2), \ \min_{x \in B} q(x) \right\}, \\
\mu_i &= \max \left\{ \max_{x' \in D_{n-1}} p_i(x') \ (i = 1, 2), \ \max_{x \in B} q(x) \right\}.
\end{align*}
\]

Moreover, suppose that the following additional information on the solution \( u(x', x_n) \) of the direct problem is available:

\[ u(x', x_n) = f(x') \text{ for } x' \in D_{n-1}, \tag{2.6} \]

where \( x_n^0 \) is a fixed point in the interval \((d_1, d_2)\) and

\[ f \in C(\overline{D_{n-1}}). \tag{2.7} \]

Further, we consider the eigenvalue problem

\[ \Delta_{n-1} y(x') = -\lambda y(x'), \quad x' \in D_{n-1} \quad \text{and} \quad y(x') = 0, \quad x' \in \partial D_{n-1}. \tag{2.8} \]

It is known \([6]\) that for every \( n \) and for a bounded domain with piecewise smooth boundary the set of eigenvalues of this problem is discrete and all eigenvalues are positive. We assume that we know one of the eigenvalues \( \lambda > 0 \) and the corresponding eigenfunction \( y = y(x') \) of the problem (2.8). Setting

\[ v(x) = v(x', x_n) = \int_{x_0}^{x_n} a(s) \, ds = \int_{x_0}^{x_0} (b(s - u_0)^1 + c) \, ds \tag{2.9} \]

we obtain from (1.1) and (2.1)-(2.3) the linear boundary value problem

\[ \Delta_n v = \Delta_{n-1} v + v_{x_n x_n} = 0, \quad (x \in D) \tag{2.10} \]

\[ v(x', d_i) = \int_{x_0}^{x_n} (b(s - u_0)^1 + c) \, ds = b g_i(x') + c g_2(x') \quad (x' \in D_{n-1}) \tag{2.11} \]

\[ v(x', d_2) = \int_{x_0}^{x_n} (b(s - u_0)^1 + c) \, ds = b g_3(x') + c g_4(x') \quad (x' \in D_{n-1}) \tag{2.12} \]

\[ v(x', x_n) = \int_{x_0}^{x_n} (b(s - u_0)^1 + c) \, ds = b h_1(x', x_n) + c h_2(x', x_n) \quad (x \in \overline{B}). \tag{2.13} \]

From the additional information (2.6) and (2.9) we receive

\[ r(x') = \int_{x_0}^{x_n} (b(s - u_0)^1 + c) \, ds = b r_1(x') + c r_2(x') \quad (x' \in D_{n-1}). \tag{2.14} \]

We use the following notations \((i = 1, 2; j = 1, 2, 3, 4)\):

\[ A_j(x_n) = \int_{\partial D_{n-1}} \left( \frac{\partial y(x')}{\partial n} \right) h_j(x', x_n) \, d\sigma \tag{2.15} \]

\[ B_j = \int_{D_{n-1}} g_j(x') y(x') \, dx' \tag{2.16} \]
\[ E_i(x_n) = -\nu_2^{1/2} \left( e^{-\sqrt{\lambda} x_n} \int_{a_i}^{x_n} e^{\sqrt{\lambda} t} A_i(t) \, dt - e^{\sqrt{\lambda} x_n} \int_{a_i}^{x_n} e^{-\sqrt{\lambda} t} A_i(t) \, dt \right) \]  \hspace{1cm} (2.17)

\[ F_i = \int_{D_{n-1}} r_i(x') y(x') \, dx', \]  \hspace{1cm} (2.18)

where \( y \) denotes the known eigenfunction of the problem (2.8) and \( \partial / \partial v \) the normal derivative in the \((n - 1)\)-dimensional space. Further, we set \( w = e^{\sqrt{\lambda}(d_1 - d_2)} - e^{-\sqrt{\lambda}(d_1 - d_2)} \).

**Theorem 2:** Let \( p_1, p_2, q \) and \( f \) be given functions satisfying (2.4), (2.5) and (2.7), respectively. Suppose

\[ +(B_3 e^{d_2} - B_1 e^{d_1} - E_1 (d_2)) w - E_1 (x_n^0) \neq 0, \]  \hspace{1cm} (2.19)

where \( \lambda \) denotes an eigenvalue of the problem (2.8). Then the inverse problem (1.1), (2.1) - (2.3), (2.6) of the determination of the real parameter \( b \) has a unique solution, which can be found explicitly.

**Proof:** Multiplication of the formula (2.10) by \( y(x') \) and integration with respect to \( x' \) supplies

\[ \int_{D_{n-1}} (\Delta_{n-1} + \nu(x', x_n)) v(x', x_n) \, dx' = 0. \]  \hspace{1cm} (2.20)

We put

\[ z(x_n) = \int_{D_{n-1}} v(x', x_n) y(x') \, dx'. \]  \hspace{1cm} (2.21)

Then we have

\[ z'(x_n) = \int_{D_{n-1}} v(x', x_n) y(x') \, dx'. \]  \hspace{1cm} (2.22)

Applying (2.8) and (2.13) we obtain by partial integration

\[ \int_{D_{n-1}} \Delta_{n-1} + \nu(x', x_n) y(x') \, dx' \]  \hspace{1cm} (2.23)

\[ = \int_{D_{n-1}} \Delta_{n-1} + y(x') v(x', x_n) \, dx' + \int_{\partial D_{n-1}} (\nu(x', x_n) \nu(x') - (\partial y(x') / \partial v) v(x', x_n)) \, d\sigma \]

\[ = -\lambda \int_{D_{n-1}} v(x', x_n) y(x') \, dx' - \int_{\partial D_{n-1}} (\partial y(x') / \partial v) [b h_i(x', x_n) + c h_2(x', x_n)] \, d\sigma. \]

Using (2.21)-(2.23) we receive from (2.20) an inhomogeneous linear ordinary differential equation of second order with respect to \( z(x_n) \):

\[ z''(x_n) - \lambda z(x_n) = s(x_n), \]  \hspace{1cm} (2.24)

with right-hand side

\[ s(x_n) = \int_{\partial D_{n-1}} (\partial y(x') / \partial v) [b h_i(x', x_n) + c h_2(x', x_n)] \, d\sigma = b A_i(x_n) + c A_2(x_n), \]
where \( A_i(x, n) \) \((i = 1, 2)\) given by (2.15) are known functions independent of \( b \) and \( c \). Moreover, we obtain from (2.11) and (2.12) the boundary conditions

\[
\begin{align*}
  z(d_1) &= \int_{D_{n-1}} \nu(x', d_1) y(x') dx = bB_1 + cB_2, \\
  z(d_2) &= \int_{D_{n-1}} \nu(x', d_2) y(x') dx = bB_3 + cB_4,
\end{align*}
\]  

(2.25)  
(2.26)

where \( B_j \) \((j = 1, 2, 3, 4)\) given by (2.16) are known constants independent of \( b \) and \( c \). Because \(-\lambda < 0\) we conclude that the boundary value problem (2.24)-(2.26) is always uniquely solvable [7]. The general solution of the equation (2.24) can be written in the form

\[
z(x_n) = C_1 e^{\sqrt{\lambda} x_n} + C_2 e^{-\sqrt{\lambda} x_n} + z_s(x_n),
\]  

(2.27)

where \( C_i \) \((i = 1, 2)\) are arbitrary constants and \( z_s(x_n) \) is a particular solution of the inhomogeneous equation (2.24). For brevity we write \( z_s(x_n) = bE_1(x_n) + cE_2(x_n) \), where the functions \( E_i(x_n) \) \((i = 1, 2)\) given by (2.17) are independent of \( b \) and \( c \). From (2.17) it follows \( z_s(d_1) = 0 \). Inserting (2.27) into (2.25) and (2.26) we obtain a particular solution

\[
z(x_n) = \int_{D_{n-1}} \nu(x', x_n) y(x') dx = bF_1 + cF_2,
\]  

(2.29)

where the constants \( F_i \) \((i = 1, 2)\) given by (2.18) are known and independent of \( b \) and \( c \). Replacing \( z(x_n) \) by \( z(x_n^0) \) in the solution (2.28) and using (2.29) we arrive at an algebraic equation for the determination of the real parameter \( b \):

\[
b = c \left[ \left( B_2 e^{\sqrt{\lambda} d_2} - B_4 e^{\sqrt{\lambda} d_1} + E_2(d_2) e^{\sqrt{\lambda} d_1} \right) e^{\sqrt{\lambda} x_n^0} \right. \\
+ \left. \left( B_4 e^{\sqrt{\lambda} d_1} - B_2 e^{\sqrt{\lambda} d_2} - E_2(d_2) e^{\sqrt{\lambda} d_1} \right) e^{\sqrt{\lambda} x_n^0} \right] w^{-1} + E_2(x_n^0) - F_2 \\
\times \left( F_1 - \left[ \left( B_1 e^{\sqrt{\lambda} d_2} - B_3 e^{\sqrt{\lambda} d_1} + E_1(d_2) e^{\sqrt{\lambda} d_1} \right) e^{\sqrt{\lambda} x_n^0} \right. \\
+ \left. \left( B_3 e^{\sqrt{\lambda} d_1} - B_1 e^{\sqrt{\lambda} d_2} - E_1(d_2) e^{\sqrt{\lambda} d_1} \right) e^{\sqrt{\lambda} x_n^0} \right] w^{-1} - E_1(x_n^0) \right)^{-1}.
\]

From (2.19) the uniqueness of the parameter \( b \) follows.
3. Special cases

Throughout what follows the compatibility conditions are assumed to be fulfilled.

3.1 We consider the boundary value problem

\[ \begin{align*}
\text{div}((b(u - u_0) + c) \text{grad } u(x)) &= 0 \quad \text{for } x \in D \\
u(x',d_1) &= \rho_1(x') \quad \text{for } x' \in D_{n-1}, x_n = d_1 \\
u(x',d_2) &= \rho_2(x') \quad \text{for } x' \in D_{n-1}, x_n = d_2 \\
u(x) &= 0 \quad \text{for } x \in \overline{B} \\
u(x',x_n^0) &= f(x') \quad \text{for } x' \in D_{n-1}, x_n^0 \in (d_1, d_2),
\end{align*} \]

(3.1)

where \( f \in C(\overline{D_{n-1}}) \) satisfies the condition

\[ F_i - \left[ \left( B_1 e^{-\sqrt{\lambda}x_{d_2}} - B_3 e^{\sqrt{\lambda}x_{d_1}} \right) e^{\sqrt{\lambda}x_{n_0}} + \left( B_3 e^{\sqrt{\lambda}x_{d_1}} - B_1 e^{-\sqrt{\lambda}x_{d_2}} \right) e^{-\sqrt{\lambda}x_{n_0}} \right] w^{-1} + 0. \]

(3.2)

The inverse problem (3.1) of the determination of \( b \) can be written as an inverse problem for the homogeneous ordinary differential equation of the variable \( x_n \):

\[ z''(x_n) - \lambda z(x_n) = 0, \]

(3.3)

with the inhomogeneous boundary condition

\[ z(d_1) = bB_1 + cB_2 \]

(3.4)

\[ z(d_2) = bB_3 + cB_4 \]

and the additional assumption

\[ z(x_n^0) = bF_1 + cF_2, \]

(3.5)

where the constants \( B_j \) \((j = 1, 2, 3, 4)\) and \( F_j \) \((j = 1, 2)\) are independent of \( b \) and \( c \) and are given by (2.16) and (2.18). The solution of the direct problem (3.3)-(3.4) has the form

\[ z(x_n) = b \left[ \left( B_1 e^{-\sqrt{\lambda}x_{d_2}} - B_3 e^{\sqrt{\lambda}x_{d_1}} \right) e^{\sqrt{\lambda}x_{n_0}} + \left( B_3 e^{\sqrt{\lambda}x_{d_1}} - B_1 e^{-\sqrt{\lambda}x_{d_2}} \right) e^{-\sqrt{\lambda}x_{n_0}} \right] w^{-1} \]

\[ + c \left[ \left( B_2 e^{-\sqrt{\lambda}x_{d_2}} - B_4 e^{\sqrt{\lambda}x_{d_1}} \right) e^{\sqrt{\lambda}x_{n_0}} + \left( B_4 e^{\sqrt{\lambda}x_{d_1}} - B_2 e^{-\sqrt{\lambda}x_{d_2}} \right) e^{-\sqrt{\lambda}x_{n_0}} \right] w^{-1}. \]

(3.6)

Because of (3.2) the inverse problem (3.3)-(3.5) is uniquely solvable:

\[ b = c \left\{ \left[ \left( B_2 e^{-\sqrt{\lambda}x_{d_2}} - B_4 e^{\sqrt{\lambda}x_{d_1}} \right) e^{\sqrt{\lambda}x_{n_0}} + \left( B_4 e^{\sqrt{\lambda}x_{d_1}} - B_2 e^{-\sqrt{\lambda}x_{d_2}} \right) e^{-\sqrt{\lambda}x_{n_0}} \right] w^{-1} - F_2 \right\} \]

\[ \times \left\{ \left[ \left( B_1 e^{-\sqrt{\lambda}x_{d_2}} - B_3 e^{\sqrt{\lambda}x_{d_1}} \right) e^{\sqrt{\lambda}x_{n_0}} + \left( B_3 e^{\sqrt{\lambda}x_{d_1}} - B_1 e^{-\sqrt{\lambda}x_{d_2}} \right) e^{-\sqrt{\lambda}x_{n_0}} \right] w^{-1} + F_1 \right\}. \]

From this we deduce the uniqueness of the parameter \( b \) in the problem (3.1).

3.2 We consider the boundary value problem

\[ \begin{align*}
\text{div}((b(u - u_0) + c) \text{grad } u(x)) &= 0 \quad \text{for } x \in D \\
u(x',d_1) &= 0 \quad \text{for } x' \in D_{n-1}, x_n = d_1 \\
u(x',d_2) &= 0 \quad \text{for } x' \in D_{n-1}, x_n = d_2 \\
u(x) &= q(x) \quad \text{for } x \in \overline{B} \\
u(x',x_n^0) &= f(x') \quad \text{for } x' \in D_{n-1}, x_n^0 \in (d_1, d_2),
\end{align*} \]

(3.7)
where \( f \in C(\bar{D}_{n-1}) \) satisfies the condition
\[
F_1 - F_2(\alpha)\left[ e^{\sqrt{x}(x_n^2 - d_i)} - e^{-\sqrt{x}(x_n^2 - d_i)} \right] w^{-1} - E_1(x_n^2) = 0.
\] (3.8)

The inverse problem (3.7) of the determination of \( b \) can be written as an inverse problem for the inhomogeneous ordinary differential equation of the variable \( x_n \):
\[
z''(x_n) - \lambda z(x_n) = bA_1(x_n) + cA_2(x_n),
\] (3.9)
with the homogeneous boundary conditions
\[
z(d_i) = 0 \quad (i = 1, 2)
\] (3.10)
and the additional assumption
\[
z(x_n^0) = bF_1 + cF_2, \quad (3.11)
\]
where the functions \( A_i(x_n) \) and the constants \( F_1 \) (\( i = 1, 2 \)) are independent of \( b \) and \( c \) and are given by (2.15) and (2.18). The solution of the direct problem (3.9)-(3.10) has the form
\[
z(x_n) = b\left[ E_1(d_2)\left( e^{\sqrt{x}(x_n^2 - d_2)} - e^{-\sqrt{x}(x_n^2 - d_2)} \right) w^{-1} + E_2(x_n^2) \right] + c\left[ E_2(d_2)\left( ... \right) w^{-1} + E_2(x_n^2) \right].
\]
Because of (3.8) the inverse problem (3.9)-(3.11) is uniquely solvable:
\[
b = c\left[ E_2(d_2)\left( e^{\sqrt{x}(x_n^2 - d_2)} - e^{-\sqrt{x}(x_n^2 - d_2)} \right) w^{-1} + E_2(x_n^2) \right] \left[ -E_1(d_2)\left( ... \right) w^{-1} - E_2(x_n^2) + F_1 \right]^{-1}.
\]
From this we deduce the uniqueness of the parameter \( b \) in the problem (3.7).

3. Let \( n = 1 \). Then we have the boundary value problem
\[
((b(u - u_0)^l + c)u')' = 0 \quad \text{for} \quad x \in (d_1, d_2) \quad (3.12)
\]
\[
u(d_1) = g_1 \quad \text{for} \quad i = 1, 2
\]
\[
u(x_0) = h \quad \text{for a fixed} \quad x_0 \in (d_1, d_2).
\] (3.13)

Let \( g_1 \neq g_2 \). It is easily seen that the solution \( u \) is a strictly monotone increasing function if \( g_1 < g_2 \) and a strictly monotone decreasing function if \( g_1 > g_2 \). With loss of generality we suppose \( g_1 < g_2 \). Then we have \( u_0 = \min \{ g_1, g_2 \} = g_1 \) and \( u_1 = \max \{ g_1, g_2 \} = g_2 \). Using the transformation
\[
u(x) = \int_{g_1}^{u(x)} a(s) \, ds = \int_{g_1}^{u(x)} (b(u - u_0)^l + c) \, ds
\]
we obtain from (3.12)
\[
u''(x) = 0 \quad \text{and} \quad \nu(d_1) = 0, \nu(d_2) = \int_{g_1}^{g_2} (b(s - g_1)^l + c) \, ds
\]
with the solution \( \nu(x) = (x - d_1)(d_2 - d_1)^{-1} \nu(d_2) \). We replace \( \nu \) by \( u \) to obtain
\[
u(x) = \int_{g_1}^{u(x)} a(s) \, ds = \int_{g_1}^{u(x)} (b(s - g_1)^l + c) \, ds
\]
and then receive the algebraic equation
\[
b = c(l + 1)\left[ (x - d_1)(d_2 - d_1)^{-1}(g_2 - g_1) - (u(x) - g_1) \right]
\]
\[\times \left[ -(x - d_1)(d_2 - d_1)^{-1}(g_2 - g_1)^{l+1} + (u(x) - g_1)^{l+1} \right]
\]
for the determination of the parameter \( b \). Because of (3.13), \( b \) is uniquely determined.
4. Now an example for $n=2$ and $l=1$ follows. Let $D = \{(x_1, x_2) | 0 < x_1, x_2 < 1\}$. We consider the boundary value problem
\[
\begin{align*}
\text{div}(b(u - u_0) + c) \nabla u(x) &= 0 \\
u(x_1, 0) = u(x_1, 1) = x_1(1 - x_1) &\quad \text{for } 0 < x_1 < 1 \\
u(0, x_2) = u(1, x_2) = 0 &\quad \text{for } 0 < x_2 < 1 \\
u(x_1, x_2^0) = f(x_1) &\quad \text{for } 0 < x_1 < 1, x_2^0 \in (0, 1),
\end{align*}
\]
where $f \in C[0,1]$ satisfies the condition
\[
F_1 - R(x_2^0)B_1 \neq 0
\]
with $R(x_2^0) = (e^{\pi x_2^0} - e^{-\pi x_2^0})e^{-\pi x_2^0}e^{\pi x_2^0}(e^{-\pi} - e^\pi)^{-1}$ and $B_1$ and $F_1$ given by (2.16),(2.18). The solutions of the eigenvalue problem $y''(x_1) + \lambda y(x_1) = 0, y(0) = y(1) = 0$ are known: $\lambda_k = k^2 \pi^2, y_k(x_1) = \sin(k \pi x_1), k \in \mathbb{N}$. For the following calculations we choose $\lambda_1 = \pi^2$, $y_1(x_1) = \sin(\pi x_1)$. By virtue of Subsection 3.1, we have
\[
\begin{align*}
z''(x_2) - \pi^2 z(x_2) &= 0 \\
z(0) &= z(1) = b\int_0^1 (x_1^2 - x_1^4) \sin(\pi x_1) \, dx_1 \\
&\quad + c\int_0^1 (x_1 - x_1^3) \sin(\pi x_1) \, dx_1 = bB_1 + cB_2 \\
z(x_2^0) &= \nu_2 b\int_0^1 (f(x_1))^2 \sin(\pi x_1) \, dx_1 \\
&\quad + c\int_0^1 f(x_1) \sin(\pi x_1) \, dx_1 = bF_1 + cF_2.
\end{align*}
\]
Setting in (3.6) $B_1 = B_2$ and $B_2 = B_4$ we get the solution of the direct problem (3.15) in the form $z(x_2) = (bB_1 + cB_2)R(x_2)$. Putting $x_2 = x_2^0$ and using (3.14),(3.16) it finally results that $b = 2c(R(x_2^0)B_2 - F_2)(F_1 - R(x_2^0)B_1)^{-1}$ is uniquely determined.

REFERENCES


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