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Non-Existence of Complex and Purely Imaginary Zeros of a Transcendental Equation Involving Bessel Functions

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Some results concerning the non-existence of complex and purely imaginary zeros of the transcendental equation $F(z)J_\nu(z)+G(z)J_\nu'(z)=0$ are given, where $J_\nu(z)$ is the Bessel function of first kind and order ν (in general complex), $J_\nu'(z)$ is the derivative of $J_\nu(z)$, and $F(z)$, $G(z)$ are analytic functions. The obtained results improve and generalize previously known ones.

Key words: *Complex zeros, purely imaginary zeros, Bessel functions*
AMS subject classification: 33A40, 47B05

1. Introduction

In the solution of certain boundary-value problems by means of complex integrals, Laplace transformation or separation of variables, transcendental equations of the form

$$F(z)J_\nu(z)+G(z)J_\nu'(z)=0 \tag{1.1}$$

arise, where $J_\nu(z)$ is the Bessel function of first kind and order ν, $J_\nu'(z)$ is the derivative of $J_\nu(z)$ and $F(z)$, $G(z)$ are given analytic functions of z without common zeros (these functions are usually polynomials). The number ν may be, in general, complex. In all these solutions a knowledge of the zeros of the equation (1.1) is very important and especially in some cases it is of some importance to be able to know whether complex or purely imaginary zeros of equation (1.1) can occur and under what conditions. When $G(z)$ and $F(z)$ take particular forms, the zeros of equation (1.1) occur in a large number of physical problems. So, they occur in the problem of the radiation of sound from a semi-infinite unflanged rigid duct with an internal accoustically absorbent lining and correspond to those waves which could propagate along the duct [13]. They occur in the problem of the zero-point energy of the quantum electro-magnetic field too and correspond to the frequencies for transverse electric and magnetic normal modes in a spherical cavity with conducting walls [2]. In the special case $G(z)=0$, and $\nu = in$ ($i^2=-1$, $n \in \mathbb{R}$) the purely imaginary zeros of (1.1) occur in a problem of propagating of sound in a half-space, of variable sound velocity under conditions of formation of shadow zone [11] and also in a potential problem [1]. Also the purely imaginary zeros of (1.1) when $F(z)=\beta z^2+\alpha$, α,β in general complex, $G(z)=z$, $\nu>-1$ was studied in [6] and recently for $\alpha,\beta \in \mathbb{R}$, $\nu>-1$ in [9, 10]. This particular case arised during the study of the delta function initial condition solution of the generalized Feller equation. Finally many applications, where particular forms of (1.1) occur, one can find in the interesting book of B. BUDAK, A. SAMARSKI, and A. TICHONOV [3]. In this paper we give some results concerning the non-existence of complex and the purely imaginary zeros of the equation (1.1) when the order ν is in general complex. These results improve and generalize previously known results given in [5, 7, 11, 15], and give also a partial answer (Corollary 2.5) to a question posed by M. K. KERIMOV and S. L. SKOROKHODOV in [8, p. 107] concerning the complex double zeros of the function $J_\nu'(z)$, in the case of purely imaginary order $\nu = in$, $n \in \mathbf{R}$.

2. Main results

In the following the symbol $\langle \cdot,\cdot \rangle$ means the scalar product in an abstract separable Hilbert space H with the orthonormal basis $\{e_n\}_{n\in \mathbf{N}}$ and bar means complex conjugation. Also,

$$M_\nu(z) = F(z)J_\nu(z) + G(z)J_\nu'(z),$$

where F and G are analytic functions in a neighbourhood of zero. Proceeding as in [4, 5] we can prove that $p \neq 0$ is a zero of the function M_ν, where ν is in general complex, if and only if there exists an element $u(\nu) \neq 0$ in H such that

$$(C_0 + \bar{\nu})u(\nu) - \frac{\bar{p}}{2} T_0 u(\nu) = -\overline{G(p)}\frac{\bar{p}}{2} e_1 \tag{2.1}$$

with

$$\langle u(\nu), e_1 \rangle = -\left[\overline{F(p)} + \overline{G(p)}\frac{\bar{\nu}}{\bar{p}}\right]. \tag{2.2}$$

In the above C_0 means the diagonal operator defined by $C_0 e_n = n e_n$ $(n \geq 1)$ in H, $T_0 = V + V^*$, where V is the unilateral shift defined by $Ve_n = e_{n+1}$ and V^* its adjoint.

I. Non-existence of complex zeros of $M_\nu(z)$, $\nu \in \mathbf{C}$

From the above, we obtain

Theorem 2.1: *Let* $\nu = \nu_1 + i\nu_2$, $\nu_1 \geq 0$. *If* $Im\{F(z)\overline{G(z)}\} < 0$ *(resp.* $Im\{F(z)\overline{G(z)}\} > 0$*), then the function* M_ν *cannot have complex zeros in the second or first (resp. fourth or third) quadrant if* $\nu_2 > 0$ *or* $\nu_2 < 0$. *If* $Im\{F(z)\overline{G(z)}\} = 0$, *then* M_ν *cannot have complex zeros in the second and fourth (resp. first and third) quadrant if* $\nu_2 > 0$ *(resp.* $\nu_2 < 0$.*)*

Proof: Scalar product multiplication of (2.1) with u gives

$$2\langle C_0 u, u\rangle + 2\bar{\nu}\|u\|^2 - \bar{p}\langle T_0 u, u\rangle = F(p)\overline{G(p)}\bar{p} + |G(p)|^2\frac{\nu\bar{p}}{p}$$

or

$$2p\langle C_0 u, u\rangle + 2\bar{\nu}p\|u\|^2 - |p|^2\langle T_0 u, u\rangle = |p|^2 F(p)\overline{G(p)} + |G(p)|^2 \nu\bar{p}. \tag{2.3}$$

For $\nu = \nu_1 + i\nu_2$ and $p = p_1 + ip_2$ in (2.3) we obtain

$$2p_1\langle C_0 u, u\rangle + 2ip_2\langle C_0 u, u\rangle + 2(\nu_1 p_1 + \nu_2 p_2)\|u\|^2$$
$$+ 2i(\nu_1 p_2 - \nu_2 p_1)\|u\|^2 - |p|^2\langle T_0 u, u\rangle$$

$$= |p|^2 Re\left\{F(p)\overline{G(p)}\right\} + i|p|^2 Im\left\{F(p)\overline{G(p)}\right\} + |G(p)|^2(\nu_1 p_1 + \nu_2 p_2) + i|G(p)|^2(\nu_2 p_1 - \nu_1 p_2). \tag{2.4}$$

Since $\langle T_0 u, u\rangle$ is real, equating the imaginary parts of (2.4) we obtain

$$2\langle C_0 u, u\rangle = \left(\nu_2 \frac{p_1}{p_2} - \nu_1\right)\left[2\|u\|^2 + |G(p)|^2\right] + \frac{|p|^2}{p_2}\mathrm{Im}\{F(p)\overline{G(p)}\}. \tag{2.5}$$

Since $\langle C_0 u, u\rangle > 0$, we see that for $\nu_1 \geq 0$ and $\mathrm{Im}\{F(p)\overline{G(p)}\} < 0$, $p_2 > 0$, $p_1\nu_2 < 0$ [resp. $\mathrm{Im}\{F(p)\overline{G(p)}\} > 0$, $p_2 < 0$, $p_1\nu_2 > 0$] the relation (2.5) is impossible, so the desired result follows ∎

Remark 2.1: In [12] RAWLINS derived the same results using the more complicated hypothesis $\nu_1 > 0$, $p_1 \neq 0$, $p_2 \neq 0$,

$$p_2^{-1}\left[2\, Im\left\{F(p)G(p) + \nu p^{-1}\right\} + p_1^{-1}\nu_2\left\{|F(p)G(p) + \nu p^{-1}|^2 - 1\right\}\right] \leq 0.$$

Corollary 2.1: *For $\nu > 0$ and $\mathrm{Im}\left\{F(z)\overline{G(z)}\right\} > 0$ [resp. $\mathrm{Im}\left\{F(z)\overline{G(z)}\right\} < 0$] the function M_ν cannot have complex zeros in the lower half-plane (resp. in the upper half-plane).*

Proof: It follows from (2.5), for $\nu_2 = 0$ ∎

Remark 2.2: **i)** Corollary 2.1. can be also obtained using Rawlins method [12] or using the Mittag-Leffler expansion. **ii)** For $F(z) = a \in \mathbb{R}$ and $G(z) = \beta + \gamma z$ ($\beta, \gamma \in \mathbb{R}$) we obtain Corollary 2.2 of [5]. **iii)** For $F(z) = a + \delta z$ and $G(z) = \beta + \gamma z$ ($a, \beta, \gamma, \delta \in \mathbb{R}$) we obtain Theorem 2.1. of [7]. **iv)** For $\nu = 0$, $F(z) = 1$ and $G(z) = i$ ($i^2 = -1$) it follows that the function $J_0 + iJ_0'$ has no zeros in the upper half-plane. This result was proved numerically in [15] and also in [14] using Mittag-Leffler expansion.

Corollary 2.2: *For $\nu = \nu_1 + i\nu_2$ ($\nu_1 \geq 0$, $\nu_2 \neq 0$) the function J_ν cannot have complex zeros in the second and fourth quadrant if $\nu_2 > 0$ (or in the first and third quadrant if $\nu_2 < 0$).*

Proof: It follows from (2.5) for $G(p) = 0$ ∎

Remark 2.3: Corollary 2.2 was proved in [11] with a different method, only in the case where $\nu = i\nu_2$, $\nu_2 \in \mathbb{R}$.

Corollary 2.3: *For $\nu = \nu_1 + i\nu_2$ ($\nu_1 \geq 0$, $\nu_2 \neq 0$) the function J_ν' cannot have complex zeros in the second and fourth (resp. third and first) quadrant if $\nu_2 > 0$* (resp. $\nu_2 < 0$).

Proof: It follows from (2.5), for $F(p) = 0$ ∎

Corollary 2.4: *For $\nu = \nu_1 + i\nu_2$ ($0 \leq \nu_1 \leq 1/2$, $\nu_2 \neq 0$ and $|\nu_2| > \nu_1$) the function J_ν'' cannot have complex zeros in the second and fourth (resp. first and third) quadrant of the disk $|z| < \sqrt{\nu_2^2 - \nu_1^2}$ if $\nu_2 > 0$ (resp. $\nu_2 < 0$).*

Proof: Since

$$-z^2 J_\nu''(z) = (z^2 - \nu^2)J_\nu(z) + zJ_\nu'(z) \tag{2.6}$$

for $F(z) = z^2 - \nu^2$ and $G(z) = z$ we have $Im\left\{F(z)\overline{G(z)}\right\} = p_2(|p|^2 + \nu_1^2 - \nu_2^2 - 2\nu_1\nu_2 p_1/p_2)$. So, (2.5) becomes

$$2\langle C_0 u, u\rangle = \left(\nu_2 \frac{p_1}{p_2} - \nu_1\right)\left[2\|u\|^2 + |p|^2\right] + |p|^2\left(|p|^2 + \nu_1^2 - \nu_2^2 - 2\nu_1\nu_2 \frac{p_1}{p_2}\right)$$

or

$$2\langle C_0 u, u\rangle = \nu_2 \frac{p_1}{p_2}\Big[(1-2\nu_1)|p|^2 + \|u\|^2\Big] - \nu_1\Big[\|u\|^2 + |p|^2\Big] + |p|^2\Big[|p|^2 + \nu_1^2 - \nu_2^2\Big]. \tag{2.7}$$

From the above equation, since $|p| < \sqrt{\nu_2^2 - \nu_1^2}$ and $\langle C_0 u, u\rangle > 0$, the result follows ∎

Corollary 2.5: *Let $\nu = i\nu_2$, $\nu_2 \neq 0$. Then the double complex zeros of the function J_ν' can only lie in the first and third (or second and fourth) quadrant if $\nu_2 > 0$ (or $\nu_2 < 0$).*

Proof: It follows immediately from Corollaries 2.3 and 2.4, for $\nu_1 = 0$ ∎

Remark 2.4: **i)** Corollary 2.5 gives a partial answer to a question posed by KERIMOV and SKOROKHODOV in [8, p. 107], concerning the existence of double complex zeros of J_ν', for $\nu = i\nu_2$, $\nu_2 \neq 0$. **ii)** From the differential equation (2.6) we see that the double zeros of J_ν' if they exist satisfy the relation

$$z^2 = \nu^2 . \tag{2.8}$$

For $z = p_1 + ip_2$ and $\nu = \nu_1 + i\nu_2$, $\nu_1 \neq 0$ in (2.8) we have that

$$p_1^2 - p_2^2 = \nu_1^2 - \nu_2^2, \quad p_1 p_2 = \nu_1\nu_2 . \tag{2.9}$$

From (2.9) follows that the double complex zeros of J_ν' can only lie in the first and third (resp. second and fourth) quadrant if $\nu_1\nu_2 < 0$ (resp. $\nu_1\nu_2 > 0$). It is clear that Corollary 2.5 doesn't follow from this elementary result.

II. Non-existence of purely imaginary zeros of $M_\nu(z)$, $\nu \in \mathbf{C}$.

Corollary 2.6: *Let $\nu = \nu_1 + i\nu_2$ ($\nu_1 \geq 0$, $\nu_2 \neq 0$). Then the function M_ν cannot have purely imaginary zeros in the upper (resp. lower) half-plane for* $\mathrm{Im}\left\{F(z)\overline{G(z)}\right\} < 0$ *(resp.* $\mathrm{Im}\left\{F(z)\overline{G(z)}\right\} > 0$*). If* $\mathrm{Im}\left\{F(z)\overline{G(z)}\right\} = 0$*, then M_ν cannot have purely imaginary zeros.*

Proof: It follows from (2.5), for $p_1 = 0$ ∎

Remark 2.5: This result cannot be obtained from Rawlins' theorem in [12].

Corollary 2.7: *Let $\nu = \nu_1 + i\nu_2$ ($\nu_1 \geq 0$, $\nu_2 \neq 0$). Then the function*

$$(\beta z^2 + \alpha)J_\nu(z) + zJ_\nu'(z) \tag{2.10}$$

where α, β are in general complex, cannot have purely imaginary zeros $\pm ip$ for $\mathrm{Re}\,\alpha \geq 0$, $\mathrm{Re}\,\beta \leq 0$, *or* $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,\beta > 0$, $|p| < \sqrt{\mathrm{Re}\,\alpha/\mathrm{Re}\,\beta}$.

Proof: It follows from (2.5) for $p_1 = 0$, because $\mathrm{Im}\left\{F(z)\overline{G(z)}\right\} = p\left[\mathrm{Re}\,\beta\,|p|^2 - \mathrm{Re}\,\alpha\right]$ ∎

Remark 2.6 : In [6] the authors studied the purely imaginary zeros of (2.10) for ν real, $\nu > -1$. Here we study such zeros of (2.10) in the case where $\nu = \nu_1 + i\nu_2$ ($\nu_1 \geq 0$, $\nu_2 \neq 0$).

Corollary 2.8: *For $\nu = \nu_1 + i\nu_2$ ($\nu_1 \geq 0$, $\nu_2 \neq 0$) the functions J_ν and J_ν' cannot have purely imaginary zeros.*

Proof: It follows from Corollary 2.6 because for these functions $\mathrm{Im}\left\{F(z)\overline{G(z)}\right\} = 0$ ∎

Corollary 2.9: *For $\nu = \nu_1 + i\nu_2$ ($\nu_2 \neq 0$, $\nu_1 \geq 0$, $|\nu_2| > \nu_1$) the function J_ν'' cannot have purely imaginary zeros for $|z| < \sqrt{\nu_1^2 - \nu_2^2}$.*

Proof: It follows from (2.7), for $p_1 = 0$ ∎

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