Zeitschrift für Analysis
und ihre Anwendungen
Vol. 10 (1991) 4, 551-562

Some Remarks on Trigonometric Interpolation on the n-Torus

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Let $\mathbf{T}^n$ be the n-torus and $(I_j f)_{j=0}^{\infty}$,

$$I_j f(x) = \sum_{k_1=-j}^{j} \cdots \sum_{k_n=-j}^{j} f\left(\tfrac{2\pi k}{2j+1}\right) \prod_{i=1}^{n} \frac{\sin\left(\frac{2j+1}{2}\left(x_i - \frac{2\pi k_i}{2j+1}\right)\right)}{(2j+1)\sin\frac{1}{2}\left(x_i - \frac{2\pi k_i}{2j+1}\right)} \quad \begin{pmatrix} x = (x_1, \ldots, x_n) \in \mathbf{R}^n \\ k = (k_1, \ldots, k_n) \in \mathbf{Z}^n \end{pmatrix}$$

the sequence of Lagrange interpolating polynomials. Then we give a complete characterization of the set of functions f with

$$\left(\sum_{j=1}^{\infty} [j^s \| f - I_j f \mid L_p(\mathbf{T}^n) \|]^q\right)^{1/q} < \infty \qquad \text{if } 1 < p < \infty,\ 0 < q \le \infty,\ s > n/p$$

and

$$\left\| \left(\sum_{j=1}^{\infty} [j^s | f(x) - I_j f(x) |]^q\right)^{1/q} \Big| L_p(\mathbf{T}^n) \right\| < \infty \quad \text{if } 1 < p < \infty,\ 0 < q \le \infty,\ s < n/p$$

in terms of Besov-Triebel-Lizorkin spaces on $\mathbf{T}^n$.

Key words: *Periodic spaces of Besov-Triebel-Lizorkin type, Lagrange interpolating polynomials, approximation of functions*

AMS subject classification: 41A25, 42A15, 46E15

0. Introduction

As usual, $\mathbf{R}^n$ denotes the Euclidean n-space, $\mathbf{Z}^n$ the set of all lattice points having integer components, $\mathbf{N}$ the set of all natural numbers and $\mathbf{N}_0$ the set of all non-negative integers. The aim of the paper is to show that the Besov-Triebel-Lizorkin spaces on the n-torus $\mathbf{T}^n$ can be completely characterized by the sequence $(I_j f)_{j=0}^{\infty}$ of Lagrange interpolating polynomials. Here $I_j f$ is given by

$$\sum_{k_1=-j}^{j} \cdots \sum_{k_n=-j}^{j} f\left(\tfrac{2\pi k}{2j+1}\right) \prod_{i=1}^{n} \frac{\sin\left(\frac{2j+1}{2} x_i - k_i \pi\right)}{(2j+1)\sin\frac{1}{2}\left(x_i - \frac{2\pi k_i}{2j+1}\right)}, \quad \begin{matrix} x = (x_1, \ldots, x_n) \in \mathbf{T}^n \\ k = (k_1, \ldots, k_n) \in \mathbf{Z}^n \end{matrix}. \tag{0.1}$$

It turns out that for periodic continuous functions f the following equivalences are true ($1 < p < \infty$, $s > n/p$):

$$f \in B^s_{p,q}(\mathbf{T}^n) \iff \left(\sum_{j=1}^{\infty} [j^{s-1/q} \| f - I_j f \mid L_p(\mathbf{T}^n) \|]^q\right)^{1/q} < \infty \qquad (0 < q \le \infty),$$

$$f \in F^s_{p,q}(\mathbf{T}^n) \iff \left\| \left(\sum_{j=1}^{\infty} [j^{s-1/q} | f(x) - I_j f(x) |]^q\right)^{1/q} \Big| L_p(\mathbf{T}^n) \right\| < \infty \quad (1 < q < \infty).$$

The main tools of proof used here are the characterization of the underlying function spaces via approximation and the L_p-stability of trigonometric polynomials t of degree $t \le j$ expressed by the following inequalities:

$$c_1\left(\frac{1}{(2j+1)^n}\sum_{k_1=-j}^{j}\cdots\sum_{k_n=-j}^{j}\left|t\left(\frac{2\pi k}{2j+1}\right)\right|^p\right)^{1/p} \le \|t\,|L_p(\mathbf{T}^n)\| \le c_2\sum_{k_1=-j}^{j}\cdots\sum_{k_n=-j}^{j}\left|t\left(\frac{2\pi k}{2j+1}\right)\right|^p\Big)^{1/p} \quad (0.2)$$

$(1 < p < \infty)$ for some constants $c_1, c_2 > 0$, independent of t and j (cf. A. Zygmund [19], P.I. Lizorkin and D.G. Orlovskij [4]).

The paper is organized as follows:

After collecting some necessary informations about Besov-Triebel-Lizorkin spaces on $\mathbf{T}^n$ in the first section, Section 2 deals with our main result concerning the characterization of the function spaces. Therefore we investigate the uniform boudedness of I_j in $\|\cdot|L_p(\mathbf{T}^n)\|$, $1 < p < \infty$. As a complement and more or less to show the great similarity between approximation via partial sums and approximation via Lagrange interpolating polynomials the aliasing error $f - I_j f$ is also treated in $\|\cdot|C(\mathbf{T}^n)\|$. Finally, in Section 3 we deal with approximation in stronger norms than $\|\cdot|L_p(\mathbf{T}^n)\|$, for instance in $\|\cdot|W_p^m(\mathbf{T}^n)\|$.

1. Besov-Triebel-Lizorkin spaces

1.1 Notations and definitions. The n-torus $\mathbf{T}^n$ may be represented by the set

$$\{x \in \mathbb{R}^n: -\pi \le x_j \le \pi \ (j = 1,\ldots,n)\},$$

where opposite sides are identified. D_π and D_π' denote the set of all complex-valued infinitely differentiable functions on $\mathbf{T}^n$ and its dual space, respectively. Furthermore we put

$$\hat{f}(k) = (2\pi)^{-n} f(e^{-ikx}) \quad (k \in \mathbb{Z}^n,\ f \in D_\pi').$$

Then any $f \in D_\pi'$ can be represented by its Fourier series

$$f = \sum_{k\in\mathbb{Z}^n} \hat{f}(k) e^{ikx} \quad (\text{convergence in } D_\pi')$$

(cf. H.-J. Schmeißer and H. Triebel [14]). The space of continuous functions on $\mathbf{T}^n$ is denoted by $C(\mathbf{T}^n)$, the space of p-th power integrable functions by $L_p(\mathbf{T}^n)$. If there is no confusion possible we drop $\mathbf{T}^n$ in notations.

Let ψ be an infinitely differentiable function with the properties

$$\psi(x) = 1 \text{ if } |x| \le 1 \text{ and } \psi(x) = 0 \text{ if } |x| \ge 3/2. \quad (1.1)$$

Further we put

$$\varphi_0(x) = \psi(x),\ \varphi_1(x) = \psi(x/2) - \psi(x),\ \varphi_l(x) = \varphi_1(2^{-l+1}x) \quad (l = 2,3,\ldots). \quad (1.2)$$

Hence, we have

$$\sum_{l=0}^{\infty} \varphi_l(x) = 1 \quad (x \in \mathbb{R}^n) \quad (1.3)$$

Definition: Let $0 < q \le \infty$ and $-\infty < s < \infty$.

(i) Let $1 \le p \le \infty$. Then

$$B_{p,q}^s(\mathbf{T}^n) = \left\{f \in D_\pi' : \|f\,|B_{p,q}^s\| = \left(\sum_{l=0}^{\infty} 2^{lsq}\left\|\sum_{k\in\mathbb{Z}^n} \varphi_l(k)\hat{f}(k) e^{ikx}\,|L_p\right\|^q\right)^{1/q} < \infty\right\}.$$

(ii) Let $1 < p < \infty$. Then

$$F^s_{p,q}(\mathbf{T}^n) = \left\{ f \in D'_\pi : \|f\,|F^s_{p,q}\| = \left\| \left(\sum_{l=0}^{\infty} 2^{lsq} \left| \sum_{k \in \mathbf{Z}^n} \varphi_l(k) \hat{f}(k) e^{ikx} \right|^q \right)^{1/q} \Big| L_p \right\| < \infty \right\}.$$

Remark 1: All spaces defined above are quasi-Banach spaces (Banach spaces if $q \geq 1$). They are independent of the special choice of ψ in (1.1) (equivalent quasi-norms). These periodic spaces of Besov-Triebel-Lizorkin type are extensively investigated in the book by H.-J. Schmeißer and H. Triebel [14].

Remark 2: The above definition can be understand as a uniform approach to different types of classical function spaces. In particular, we have

(i) $F^0_{p,2} = L_p$,

(ii) $F^m_{p,2} = W^m_p$ (Sobolev spaces) if $m \in \mathbf{N}$,

(iii) $B^s_{p,q} = \Lambda^s_{p,q}$ (Besov-Lipschitz classes) if $s > 0$, and

(iv) $B^s_{\infty,\infty} = \mathcal{C}^s$ (Hoelder-Zygmund classes) if $s > 0$

(cf. H.-J. Schmeißer and H. Triebel [14]).

Remark 3: Of some importance are the embedding relations

$$(B^s_{p,q} \cup F^s_{p,q}) \hookrightarrow L_p \text{ if } s > 0 \tag{1.4}$$

and

$$(B^s_{p,q} \cup F^s_{p,q}) \hookrightarrow C \text{ if } s > n/p \tag{1.5}$$

(cf. H.-J. Schmeißer and H. Triebel [14]).

1.2 Characterization via approximation. The spaces defined above are well-adapted to problems in approximation theory. To show this we recall the following facts. Let

$$T_j = \left\{ t \in D'_\pi : \hat{t}(k) = 0 \text{ for all } k \in \mathbf{Z}^n,\ |k| > j \right\} \quad (j \in \mathbf{N}_0).$$

Let X be an appropriate quasi-Banach space. Then we put

$$E_j(f, X) = \inf_{g \in T_j} \|f - g\,|X\| \quad (j \in \mathbf{N}_0).$$

Proposition 1: *Let* $0 < q \leq \infty$.

(i) *Let* $1 \leq p \leq \infty$ *and* $s > 0$. *Then*

$$B^s_{p,q} = \left\{ f \in L_p : \|f\,|L_p\| + \left(\sum_{j=0}^{\infty} [(1+j)^{s-1/q} E_j(f, L_p)]^q \right)^{1/q} < \infty \right\}$$

in the sense of equivalent quasi-norms.

(ii) *Let*

$$S_j f(x) = \sum_{|k_1| \leq j} \cdots \sum_{|k_n| \leq j} \hat{f}(k) e^{ikx} \quad (k = (k_1, \ldots, k_n),\ j \in \mathbf{N}_0). \tag{1.6}$$

We can replace $E_j(f, L_p)$ *by* $\|f - S_j f\,|L_p\|$ *in* (i) *if* $1 < p < \infty$.

(iii) *Let*

$$V_j f(x) = \sum_{k\in\mathbb{Z}^n} \psi(j^{-1}k)\hat{f}(k)\,e^{ikx} \quad (j \in \mathbb{N}), \tag{1.7}$$

where ψ *is the function from* (1.1). *Then* $E_j(f, L_p)$ *can be replaced by* $\|f - V_j f\,|L_p\|$ *in* (i).

Proposition 2: *Let* $1 < p, q < \infty$ *and* $s > 0$. *Then*

$$F^s_{p,q} = \left\{ f \in L_p: \begin{array}{l} \exists\{g_j\}_{j=1}^{\infty},\ g_j \in T_j \text{ for } j \in \mathbb{N}, \text{ such that } g_j \to f \text{ in } L_p \text{ and} \\ \|g_1|L_p\| + \left\| \left(\sum_{j=1}^{\infty} [j^{s-1/q}|f(x) - g_j(x)|]^q \right)^{1/q} |L_p \right\| < \infty \end{array} \right\}$$

in the sense of equivalent norms. Moreover, we can choose $g_j = S_j f$ $(j \in \mathbb{N})$.

Remark 4: Proofs of Propositions 1 and 2 may be found in H.-J. Schmeißer and H. Triebel [14] and W. Sickel [15] (cf. also H.-J. Schmeißer and W. Sickel [12, 13]).

Remark 5: For later use we mention also that $(0 < q < \infty,\ s \geq 0)$

$$\|f - S_j f\,|B^s_{p,q}\| \xrightarrow[j\to\infty]{} 0 \text{ if } 1 < p < \infty \quad\text{and}\quad \|f - V_j f\,|B^s_{p,q}\| \xrightarrow[j\to\infty]{} 0 \text{ if } 1 \leq p \leq \infty.$$

These are consequences of

$$\hat{f}(k) - \widehat{S_j f}(k) = 0 \text{ if } |k_i| < j \ (i = 1,\dots,n) \quad\text{and}\quad \hat{f}(k) - \widehat{V_j f}(k) = 0 \text{ if } |k| < j$$

and of

$$\sup_j \|S_j f\,|L_p\| \leq c\,\|f\,|L_p\| \ (1 < p < \infty) \quad\text{and}\quad \sup_j \|V_j f\,|L_p\| \leq c\,\|f\,|L_p\| \ (1 \leq p \leq \infty).$$

2. Trigonometric interpolation

We start with a uniform lattice on $\mathbf{T}^n$, characterized by the nodes

$$x^r = (x_1^r, \dots, x_n^r) = \left(\frac{2\pi r_1}{2j+1}, \dots, \frac{2\pi r_n}{2j+1}\right) \ (-j \leq r_i \leq j\ (i = 1,\dots,n),\ j \in \mathbb{N}_0),\ r \in \mathbb{Z}^n.$$

Let

$$Q^j_m = \left\{k \in \mathbb{Z}^n: -j - 1/2 \leq k_i - m_i(2j+1) \leq j + 1/2 \ (i = 1,\dots,n)\right\} \ (m \in \mathbb{Z}^n, j \in \mathbb{N}_0)$$

and $f \in C$. Then the function $I_j f$ defined by (0.1) is the unique solution of

$$g(x^r) = f(x^r),\ r \in Q^j_0 \text{ and } \hat{g}(k) = 0 \text{ if } k \notin Q^j_0.$$

Suppose additionally

$$\sum_{k\in\mathbb{Z}^n} |\hat{f}(k)| < \infty. \tag{2.1}$$

Then we can rewrite

$$I_j f(x) = \sum_{m\in\mathbb{Z}^n} \left(\sum_{k\in Q^j_m} \hat{f}(k)\,e^{ikx} \right) e^{-im(2j+1)x} \quad (j \in \mathbb{N}_0) \tag{2.2}$$

(cf. A. Zygmund [19]).

Remark 6: Formula (2.2) shows the great similarity between Lagrange polanomials and Whittaker's cardinal series. The latter one is defined as

$$I_j^* f(x) = \sum_{k \in \mathbf{Z}^n} f\left(\frac{2\pi k}{2j+1}\right) \prod_{i=1}^{n} \frac{\sin\left(\frac{2j+1}{2}x_i - k_i\pi\right)}{\left(\frac{2j+1}{2}x_i - k_i\pi\right)} \qquad (x \in \mathbf{R}^n).$$

We have the identity

$$I_j^* f(x) = \sum_{k \in \mathbf{Z}^n} (F^{-1}[\chi_m^j Ff])(x) e^{\,im(2j+1)x},$$

where F, F^{-1} are the Fourier transform and its inverse, respectively, and χ_m^j denotes the characteristic function of Q_m^j (cf. P.L. Butzer [3], W. Sickel [16]).

Remark 7: If we put $A_\pi = \{f \in D_\pi' : \sum_{k \in \mathbf{Z}^n} |\hat{f}(k)| < \infty\}$, then $B_{2,1}^{n/2} \hookrightarrow A_\pi \hookrightarrow B_{\infty,1}^{0}$ (cf. H. Triebel [17]).

In our investigations a crucial role is played by the following

Lemma 1: *Let* $1 < p < \infty$.

(i) *There exists a constant c such that*

$$\|I_j f \,|L_p\| \le c(1+j)^{-n/p} \|f\,|B_{p,1}^{n/p}\| \quad (j \in \mathbf{N}_0) \tag{2.3}$$

holds for all $f \in B_{p,1}^{n/p}$ *with* $\hat{f}(k) = 0$, $k \in Q_0^j$.

(ii) *There exists a constant c such that*

$$\|f - I_j f\,|L_p\| \le c(1+j)^{-n/p}\|f\,|B_{p,1}^{n/p}\| \quad (j \in \mathbf{N}_0) \tag{2.4}$$

holds for all $f \in B_{p,1}^{n/p}$.

Proof: First, note that I_j is a projection, that means $I_j f = f$ for all f with $\hat{f}(k) = 0$, $k \in Q_0^j$. Now we split

$$f - I_j f = f - S_j f + I_j(S_j f - f). \tag{2.5}$$

To prove (2.4) we can use Proposition 1 and (2.3). So, it remains to prove (2.3). Let $\{\varphi_l\}$ be the system defined in (1.2). We put $f_l(x) = \sum_{k \in \mathbf{Z}^n} \varphi_l(k)\hat{f}(k)e^{ikx}$ $(l \in \mathbf{N}_0)$. Let $2^t \le j \le 2^{t+1}$. The properties of ψ guarantee $I_j(S_j f - f) = \sum_{l=t-1}^{\infty} I_j(S_j f_l - f_l)$ in D_π'. Applying (0.2), using the interpolation property of $I_j f$, $I_j f(k) = 0$ if $k \notin Q_0^j$ and $\hat{f}_l(k) = 0$ if $|k| > (3/2)2^{l-1}$ we find

$$\begin{aligned} \|I_j(S_j f_l - f_l)|L_p\| &\le c(1+j)^{-n/p}\left(\sum_{k \in Q_0^j}\left|I_j(S_j f_l - f_l)\left(\frac{2\pi k}{2j+1}\right)\right|^p\right)^{1/p} \\ &\le c(1+j)^{-n/p}\left(\sum_{k \in Q_0^j}\left|(S_j f_l - f_l)\left(\frac{2\pi k}{2j+1}\right)\right|^p\right)^{1/p}, \end{aligned} \tag{2.6}$$

where c is independent of j, l, and f. Next we pick out a sequence of meshes $\{M_l\}$ such that

$$\left\{\left(\frac{2\pi k}{2j+1}\right): k \in Z^n\right\} \subset M_l = \left\{\left(\frac{2\pi k}{2M_{l+1}}\right): k \in Z^n\right\} \quad (l = t-1, \dots),$$

where $(3/2)2^{l-1} \le M_l \le c\,2^l$ (c independent of l and t) holds. According to M_l we apply again (0.2). This leads to

$$\Big(\sum_{k\in Q_0^j}\Big|(S_jf_l-f_l)\Big(\frac{2\pi k}{2j+1}\Big)\Big|^p\Big)^{1/p}\le c\,2^{ln/p}\|S_jf_l-f_l|L_p\|\le c\,2^{ln/p}\|f_l|L_p\|. \qquad (2.7)$$

Putting (2.7) into (2.6), summing up from $t-1$ to ∞ the desired inequality (2.3) follows ∎

Remark 8: Using Remark 5 we can sharpen (2.4) a little bit. We have

$$j^{n/p}\|f-I_jf|L_p\|\to 0 \text{ if } j\to\infty \qquad (2.8)$$

for any $f\in B^{n/p}_{p,1}$. In case $n=1$ this was observed first by K.I. Oskolkov [5].

Remark 9: In the one-dimensional case J. Prestin [7 - 10] has proved a result similar to (2.8), but with $B^{1/p}_{p,1}(\mathbf{T}^1)$ replaced by the set of functions with bounded variation.

As a consequence of Lemma 1 one obtains some estimates of the approximation error in $\|\cdot|C\|$.

Lemma 2: *Let $p<\infty$ and $s>0$.*

(i) *For any $f\in B^{n/p}_{p,1}\cup A_\pi$ there holds*

$$\|f-I_jf|C\|\to 0 \text{ as } j\to\infty. \qquad (2.9)$$

(ii) *There exists a constant c such that*

$$\sup_{j\in\mathbb{N}_0}(1+j)^s\|f-I_jf|C\|\le c\|f|B^{s+n/p}_{p,\infty}\| \text{ for all } f\in B^{s+n/p}_{p,\infty}. \qquad (2.10)$$

(iii) *There exists a constant c such that*

$$\sup_{j\in\mathbb{N}_0}(1+j)^s(\log(1+j))^{-n}\|f-I_jf|C\|\le c\|f|C^s\| \text{ for all } f\in C^s=B^s_{\infty,\infty}. \qquad (2.11)$$

Proof: (i) Let $f\in A_\pi$. Then (2.9) follows from (2.2) since

$$|f(x)-I_jf(x)|\le\sum_{|m|>0}\sum_{k\in Q^j_m}|\hat f(k)|.$$

Let $f\in B^{n/p}_{p,1}$. Then we use the decomposition

$$f-I_jf=f-V_{j/2}f+I_j(V_{j/2}f-f), \qquad (2.12)$$

with $V_{j/2}$ defined in (1.7). From the embeddings $B^{n/p}_{p,1}\hookrightarrow B^0_{\infty,1}\hookrightarrow C$ (cf. H.-J. Schmeißer and H. Triebel [14]) and Remark 5 we know that

$$\|f-V_{j/2}f|C\|\to 0 \text{ as } j\to\infty. \qquad (2.13)$$

Next we apply the Nikol'skij inequality (cf. H.-J. Schmeißer and H. Triebel [14]) and (2.3). This yields

$$\|I_j(V_{j/2}f-f)|C\|\le c(1+j)^{n/p}\|I_j(V_{j/2}f-f)|L_p\|\le c\|f-V_{j/2}f|B^{n/p}_{p,1}\|.$$

Using again Remark 5 we find

$$\|I_j(V_{j/2}f-f)|C\|\to \text{ as } j\to\infty. \qquad (2.14)$$

Now, (2.13) and (2.14) complete the proof of (2.9).

(ii) We use the splitting stated in (2.12), Proposition 1, and (2.3). This yields

$$\|f-I_jf|C\|\le\|f-V_{j/2}f|C\|+c\|f-V_{j/2}f|B^{n/p}_{p,1}\|\le c(1+j)^s\|f|B^{n/p+s}_{p,\infty}\|.$$

For the last step we have used on the one hand the embedding $B^{n/p+s}_{p,\infty} \hookrightarrow B^{s}_{\infty,\infty} = C^s$ and on the other hand Proposition 3 (see Section 3). This proves (2.10).

(iii) Since the first part of inequality (0.2) remains true if $p = 1$ (cf. A. Zygmund [19]) we obtain

$$\begin{aligned}|I_j f(x)| &= \left(\frac{1}{2j+1}\right)^n \Big| \sum_{r\in Q_0^j} \sum_{k\in Q_0^j} f(x^r) e^{ik(x-x)^r}\Big| \\ &\le \sup_{r\in Q_0^j} |f(x^r)| \left(\frac{1}{2j+1}\right)^n \sum_{r\in Q_0^j} \Big|\sum_{k\in Q_0^j} e^{ik(x-x)^r}\Big| \\ &\le c\,\|f|C\| \, \Big\| \sum_{k\in Q_0^j} e^{ikx} |L_1\Big\| \le c(\log(1+j))^n \|f|C\|.\end{aligned}$$

Using this with $f - V_{j/2}f$ instead of f, the desired inequality follows from (2.12) as in (ii) ∎

The main result of this paper is formulated in the next

Theorem 1: *Let $1 < p < \infty$ and $s > n/p$.*
(i) *Let $0 < q \le \infty$. Then*

$$B^s_{p,q} = \left\{ f\in C: |f(0)| + \left(\sum_{j=0}^{\infty} [(1+j)^{s-1/q} \|f - I_j f|L_p\|]^q\right)^{1/q} < \infty \right\}$$

in the sense of equivalent quasi-norms.
(ii) *Let $1 < q < \infty$. Then*

$$F^s_{p,q} = \left\{ f\in C: |f(0)| + \left\| \left(\sum_{j=0}^{\infty} [(1+j)^{s-1/q} |f(x)|^q\right)^{1/q} \Big| L_p \right\| < \infty \right\}$$

in the sense of equivalent norms.

Proof: (i) Comparing the above characterization of $B^s_{p,q}$ with Proposition 1 it remains to prove that

$$|f(0)| + \left(\sum_{j=0}^{\infty} [(1+j)^{s-1/q} \|f - I_j f|L_p\|]^q\right)^{1/q} \le c\,\|f|B^s_{p,q}\| \tag{2.15}$$

with c independent of f. Let $0 < q < \infty$. Again we use (2.3) and Proposition 1. This leads to

$$\begin{aligned}&\sum_{j=0}^{\infty} [(1+j)^{s-1/q} \|f - I_j f|L_p\|]^q \\ &\quad\le \sum_{j=0}^{\infty} [(1+j)^{s-1/q} (\|f - S_j f|L_p\| + \|I_j(f - S_j f)|L_p\|)]^q \\ &\quad\le c\,\|f|B^s_{p,q}\|^q + \sum_{j=0}^{\infty} [(1+j)^{s-1/q} (1+j)^{-n/p} \|f - S_j f|B^{n/p}_{p,1}\|]^q.\end{aligned} \tag{2.16}$$

We proceed with an estimate of the second term on the right-hand side of (2.16). Using $f_l(x) = \sum_{k\in \mathbf{Z}^n} \varphi_l(k)\hat{f}(k)e^{ikx}$ ($l \in \mathbf{N}_0$) we find

$$\left\{ \sum_{t=0}^{\infty} \sum_{j=2^t-1}^{2^{t+1}-2} 2^{t(s-n/p)q} 2^{-t} \left(\sum_{l=t-1}^{\infty} 2^{ln/p} \|(S_j f - f)_l|L_p\|\right)^q \right\}^{\min(1,q)/q}$$

$$\begin{aligned} &\leq c\sum_{l=0}^{\infty}\Big(\sum_{t=0}^{\infty} 2^{t(s-n/p)q}\, 2^{(l+t)qn/p}\|f_{l+t-1}|L_p\|^q\Big)^{\min(1,q)/q} \\ &\leq c\sum_{l=0}^{\infty} 2^{-l(s-n/p)\min(1,q)}\Big(\sum_{t=0}^{\infty} 2^{tsq}\|f_t|L_p\|^q\Big)^{\min(1,q)/q} \\ &\leq c\ \|f\,|B^s_{p,q}\|^{\min(1,q)}, \end{aligned} \tag{2.17}$$

since $s > n/p$ and

$$\sup_j \|(S_j f)_l|L_p\| = \sup_j \|S_j(f_l)|L_p\| \leq c\,\|f_l|L_p\|$$

(put $\varphi_{-1} = 0$). Note that $I_0 f = f(0)$. In view of this fact, Lemma 1 and (2.16), (2.17) the desired inequality (2.15) follows if $q < \infty$. In case $q = \infty$ one has to modify the above considerations in an obvious way.

(ii) Using Proposition 2 the proof is reduced to establish the inequality

$$|f(0)| + \Big\|\Big(\sum_{j=0}^{\infty}\big[(1+j)^{s-1/q}|f(x) - I_j f(x)|\big]^q\Big)^{1/q}\Big|L_p\Big\| \leq c\|f\,|F^s_{p,q}\|. \tag{2.18}$$

Step 1: In order to prove (2.18) we consider at first the case $s > n$. Because of $F^s_{p,q} \hookrightarrow A_\pi$ (cf. Remark 7) we can apply (2.2). This yields

$$f(x) - I_j f(x) = f(x) - S_j f(x) - \sum_{|m|>0}\Big(\sum_{k\in\mathbb{Z}^n}\chi^j_m(k)\hat f(k)e^{ikx}\Big)e^{-ixm(2j+1)},$$

where χ^j_m is the characteristic function of Q^j_m. With the help of Proposition 1 a proof of (2.18) is now reduced to a proof of

$$\Big\|\Big(\sum_{j=0}^{\infty}\Big[(1+j)^{s-1/q}\Big|\sum_{|m|>0}\Big(\sum_{k\in\mathbb{Z}^n}\chi^j_m(k)\hat f(k)e^{ikx}\Big)e^{-ixm(2j+1)}\Big|\Big]^q\Big)^{1/q}\Big|L_p\Big\| \leq c\|f\,|F^s_{p,q}\|. \tag{2.19}$$

In order to obtain (2.19) we make use of Lizorkin's vector-valued Fourier-multiplier theorem for cubes with sides parallel to the axis (cf. H.-J. Schmeißer and H. Triebel [14]) and of

$$Q^j_m \in K^{N_0,N_1}_{t+1} \quad \text{if } 2^t - 1 \leq j \leq 2^{t+1} - 2,\ 2^l \leq |m| < 2^{l+1},$$

where

$$K^{N_0,N_1}_0 = \{x : |x_j| \leq 2^{-N_1}\ (j = 1,\ldots,n)\},$$
$$K^{N_0,N_1}_t = \{x : |x_j| \leq 2^{t+N_0}\ (j = 1,\ldots,n)\}\setminus\{x : |x_j| \leq 2^{t-1-N_1}\ (j = 1,\ldots,n)\}\ (t \in \mathbb{N})$$

for appropriate $N_0, N_1 \in \mathbb{N}_0$. Let $\chi(K^{N_0,N_1}_t,\cdot)$ be the characteristic function of K^{N_0,N_1}_t. These yields

$$\begin{aligned} &\Big\|\Big(\sum_{t=0}^{\infty}2^{tsq}\sum_{j=2^t-1}^{2^{t+1}-2}2^{-t}\Big|\sum_{|m|>0}\Big(\sum_{k\in\mathbb{Z}^n}\chi^j_m(k)\hat f(k)e^{ikx}\Big)e^{-ixm(2j+1)}\Big|^q\Big)^{1/q}\Big|L_p\Big\| \\ &\leq \sum_{l=0}^{\infty}\sum_{2^j\leq|m|<2^{j+1}}\Big\|\Big(\sum_{t=0}^{\infty}2^{tsq}\sum_{j=2^t-1}^{2^{t+1}-2}2^{-t}\Big|\sum_{k\in\mathbb{Z}^n}\chi^j_m(k)\chi(K^{N_0,N_1}_{t+1},k)\hat f(k)e^{ikx}\Big|^q\Big)^{1/q}\Big|L_p\Big\| \end{aligned} \tag{2.20}$$

$$\leq c\sum_{l=0}^{\infty} 2^{ln}\left\|\left(\sum_{t=0}^{\infty} 2^{tsq}\left|\sum_{k\in \mathbf{Z}^n}\chi\left(K_{t+1}^{N_0,N_1},k\right)\hat{f}(k)e^{ikx}\right|^q\right)^{1/q}\Big|L_p\right\|$$

$$\leq c\sum_{l=0}^{\infty} 2^{l(n-s)}\|f\,|F_{p,q}^{s}\| \leq c\,\|f\,|F_{p,q}^{s}\|,$$

according to a so-called Lizorkin-type representation of $F_{p,q}^s$ (cf. H.-J. Schmeißer and H. Triebel [14]).

Step 2: We remove the restriction $s > n$. Note, that $B_{p,p}^s = F_{p,p}^s$. Furthermore, we have

$$\begin{aligned}
&\left[F_{p_0,q_0}^{s_1}, F_{p_1,q_1}^{s_1}\right]_\vartheta = F_{p,q}^{s} && s = (1-\vartheta)s_0 + \vartheta s_1\\
&\left[L_{p_0}(A), L_p(B)\right]_\vartheta = L_p([A,B]_\vartheta) \qquad \text{with} && \frac{1}{p} = \frac{1-\vartheta}{p_0} + \frac{\vartheta}{p_1}\\
&\left[l_{q_0}(A_j), l_{q_1}(B_j)\right]_\vartheta = l_q([A_j,B_j]_\vartheta) && \frac{1}{q} = \frac{1-\vartheta}{q_0} + \frac{\vartheta}{q_1}
\end{aligned} \tag{2.21}$$

(cf. Triebel [18]). We shall use (2.21) with $A_j = j^{s_0}\mathbb{C}$, $B_j = j^{s_1}\mathbb{C}$, and $A = l_{q_0}(A_j)$, $B = l_{q_1}(B_j)$. Here $\mathbb{C}$ is the complex plane. Considering the linear operator $R\colon F_{p,q}^s \to L_p(l_q(j^{s-1/q}\mathbb{C}))$, $Rf = \{f - I_j f\}_{j=0}^{\infty}$ we know from the proof of (i) and from Step 2 that R is bounded if $s > n/p$ and $p = q$ or $s > n$ and $1 < p,q < \infty$. Hence, R is bounded as a mapping with respect to the intermediate spaces $R\colon F_{p,q}^s \to L_p(l_q(j^{s-1/q}\mathbb{C}))$ $(1 < p,q < \infty;\ s > n/p)$. That means, (2.18) is true also under these restrictions ∎

Remark 10: The restriction $s > n/p$ in Theorem 1 seems to be natural. If $s < n/p$, then unbounded functions are contained in $B_{p,q}^s$ and hence, $I_j f$ makes no sense in general.

Remark 11: Parts of the assertions of Theorem 1 and of the Lemmas 1 and 2 are known if $n = 1$. We refer to J. Prestin [7,10] and K.I. Oskolkov [5]. Corresponding results in case of Whittaker's cardinal series are obtained in Sickel [16].

We are also interested in a characterization of function spaces if $p = \infty$. To this end we can employ an inequality due to Leindler [3]. Let $0 < \mu < \infty$. Then

$$\left\|\left(2^{-l}\sum_{j=2^l}^{2^{l+1}-1}|f(x) - S_j f(x)|^\mu\right)^{1/\mu}\Big|L_\infty(\mathbf{T}^1)\right\| \leq c\,E_{2^l}(f, C(\mathbf{T}^1)), \tag{2.22}$$

where c is independent of f and $l \in \mathbf{N}_0$. This implies

$$\sup_{l\in\mathbf{N}_0} 2^{(1+1/\mu)l}\left\|\left(2^{-l}\sum_{j=2^l}^{2^{l+1}-1}|f(x) - I_j f(x)|^\mu\right)^{1/\mu}\Big|C(\mathbf{T}^1)\right\| \leq c\,\|f\,|B_{\infty,1}^{1+1/\mu}(\mathbf{T}^1)\| \tag{2.23}$$

and

$$\sup_{l\in\mathbf{N}_0} 2^{sl}\left\|\left(2^{-l}\sum_{j=2^l}^{2^{l+1}-1}|f(x) - I_j f(x)|^\mu\right)^{1/\mu}\Big|C(\mathbf{T}^1)\right\| \leq c\,\|f\,|C^s(\mathbf{T}^1)\| \tag{2.24}$$

if $1 \leq \mu < \infty$ and $s > 1 + 1/\mu$. Extending (2.24) to l_q-norms one obtains a characterization of $B_{\infty,q}^s(\mathbf{T}^1)$.

Theorem 2: *Let* $1 \leq \mu < \infty$, $0 < q \leq \infty$ *and* $s > 1/\min(1,q) + 1/\mu$. *Then*

$$B_{\infty,q}^s(\mathbf{T}^1) = \Big\{f \in C(\mathbf{T}^1)\colon \|f\,|C(\mathbf{T}^1)\|$$

$$+\left(\sum_{l=0}^{\infty}2^{lsq}\left\|\left(2^{-l}\sum_{j=2^l}^{2^{l+1}-1}|f(x)-I_jf(x)|^{\mu}\right)^{1/\mu}\Big|C(\mathbf{T}^1)\right\|^q\right)^{1/q}<\infty\Bigg\}$$

in the sense of equivalent quasi-norms.

Remark 12: Assertions of this type with I_jf replaced by S_jf may be found in H.-J. Schmeißer and W. Sickel [13].

3. Approximation in Besov and Sobolev norms

In several papers the approximation order of $f - I_jf$ is studied in stronger norms than $\|\cdot|L_p\|$ (cf. R. Haverkamp [2], J. Prestin [7 - 10], S. Prößdorf and B. Silbermann [11]). The results derived in the preceding section can be generalized in a convenient way. The first step in doing this is the following characterization of Besov spaces (cf. A. Pietsch [6]).

Proposition 3: *Let* $1 \le p \le \infty$, $0 < q_0, q_1 \le \infty$, *and* $t, s > 0$.

(i) *We have*

$$B^{s+t}_{p,q_0}=\left\{f\in B^t_{p,q_1}:\ \|f|B^t_{p,q_1}\|+\left(\sum_{j=1}^{\infty}[j^{s-1/q_0}E_j(f,B^t_{p,q_1})]^{q_0}\right)^{1/q_0}<\infty\right\}$$

in the sense of equivalent quasi-norms.

(ii) *If* $1 < p < \infty$, *then* $E_j(f, B^t_{p,q_1})$ *can be replaced by* $\|f - S_jf\,|B^t_{p,q_1}\|$ *in* (i).

As a consequence of this proposition and Theorem 1 we obtain the following

Theorem 3: *Let* $1 < p < \infty$, $0 < q_0, q_1 \le \infty$, $t \ge 0$ *and* $s > 0$. *Let additionally* $s + t > n/p$. *Then we have*

$$B^{s+t}_{p,q_0}=\left\{f\in B^t_{p,q_1}:\ \|f|B^t_{p,q_1}\|+\left(\sum_{j=0}^{\infty}[(1+j)^{s-1/q_0}\|f-I_jf|B^t_{p,q_1}\|]^{q_0}\right)^{1/q_0}<\infty\right\}$$

in the sense of equivalent quasi-norms.

Proof: By Proposition 3 it is sufficient to prove

$$\|f|B^t_{p,q_1}\|+\left(\sum_{j=0}^{\infty}[(1+j)^{s-1/q_0}\|f-I_jf|B^t_{p,q_1}\|]^{q_0}\right)^{1/q_0}\le c\,\|f|B^{s+t}_{p,q_0}\|$$

for some constant c, independent of f. Therefore, we use the splitting from (2.5). Again by applying Proposition 3 it suffices to consider the term $I_j(S_jf - f)$. Let $2^{\nu} \le j < 2^{\nu+1}$. Then (2.3) implies

$$\|I_j(S_jf-f)|B^t_{p,q_1}\|\le\left(\sum_{l=0}^{\nu+1}2^{ltq_1}\left\|\sum_{k\in\mathbf{Z}^n}\varphi_l(k)\,I_j(S_jf-f)(k)\,e^{ikx}\Big|L_p\right\|^{q_1}\right)^{1/q_1}$$

$$\le c(1+j)^t\,\|I_j(S_jf-f)|L_p\|\le c(1+j)^t\,(\|f-S_jf|L_p\|+\|f-I_jf|L_p\|).$$

This leads to

$$\left(\sum_{j=0}^{\infty}[(1+j)^{s-1/q_0}\|f-I_jf|B^t_{p,q_1}\|]^{q_0}\right)^{1/q_0}$$

$$\leq c\left(\|f \mid B^{s+t}_{p,q_0}\| + \left(\sum_{j=0}^{\infty}[(1+j)^{s+t}(\|f - S_j f \mid L_p\| + \|f - I_j f \mid L_p\|)]^{q_0}\right)^{1/q_0}\right) \leq c\,\|f \mid B^{s+t}_{p,q_0}\|$$

since $s + t > n/p$ ensures that Theorem 1 can be applied ∎

Remark 13: As a consequence of embeddings for Besov-Triebel-Lizorkin spaces on the n-torus one obtains characterizations of $B^s_{p,q}$ via approximation by Lagrange interpolating polynomials in certain norms. For instance, by $B^0_{p,1} \hookrightarrow L_p \hookrightarrow B^0_{p,\infty}$ one obtains Theorem 1 as an application of Theorem 3. Furthermore, by $B^t_{p,1} \hookrightarrow W^t_p \hookrightarrow B^t_{p,\infty}$ $(t \in \mathbb{N})$ one can replace B^t_{p,q_1} in (3.1) by the Sobolev spaces W^t_p. This improves some results of S. Prößdorf and B. Silbermann [11] and J. Prestin [7, 10].

REFERENCES

[1] BUTZER, P.L.: *A survey of the Whittaker-Shannon sampling theorem and some of its extensions.* J. Math. Res. Exp. **3** (1983), 185 -212.

[2] HAVERKAMP, R: *Approximationsgüte der Ableitungen bei trigonometrischer Interpolation.* Math. Z. **179** (1982), 59 - 67.

[3] LEINDLER, L.: *On summability of Fourier series.* Acta Sci. Math. (Szeged) **29** (1968), 147 -162.

[4] LIZORKIN, P.I., and D.G. ORLOVSKIJ.: *Some interpolation formulas for trigonometric and exponential polynomials and estimates connected with them.* (Russian.) Trudy Sem. Sobolev Math. Inst. Akad. Nauk SSSR, Sib. Otd. **1** (1976), 60 - 71.

[5] OSKOLKOV, K.I.: *Inequalities of the "large sieve" type and applications to trigonometric approximation.* Anal. Math. **12** (1986), 143 - 166.

[6] PIETSCH, A.: *Approximation spaces.* J. Approx. Theory **32** (1981), 115 - 134.

[7] PRESTIN, J.: *Approximation in periodischen Lipschitz-Räumen.* Thesis. Rostock: Wilhelm-Pieck-Universität 1985.

[8] PRESTIN, J.: *Trigonometrische Approximation von Funktionen beschränkter Variation.* Math. Nachr. **122** (1985), 29 - 43.

[9] PRESTIN, J.: *Trigonometric interpolation of functions of bounded variation.* In: Constructive Theory of Functions. Proc. Int. Conf. Varna 1984. Sofia: Publ. House Bulg. Acad. Sci. 1984, 699 - 703.

[10] PRESTIN, J.: *On the approximation by de la Vallee' Poussin means and interpolatory polynomials in Lipschitz norms.* Anal. Math. **13** (1987), 251 - 259.

[11] PRÖSSDORF, S., and B. SILBERMANN.: *Projektionsverfahren und die näherungsweise Lösung singulärer Gleichungen* (Teubner-Texte zur Mathematik: Vol. 12). Leipzig: B.G. Teubner Verlagsges. 1977.

[12] SCHMEISSER, H.-J., and W. SICKEL: *On strong summability of multiple Fourier series and smoothness properties of functions.* Anal. Math. **8** (1982), 57 - 70.

[13] SCHMEISSER, H.-J., and W. SICKEL: *On strong summability of multiple Fourier series and approximation of periodic functions.* Math. Nachr. **133** (1987), 211 - 236.

[14] SCHMEISSER, H.-J., and H. TRIEBEL.: *Topics in Fourier Analysis and Function Spaces.* Leipzig: Akad. Verlagsges. Geest & Portig 1986, and Chichester: Wiley 1986.

[15] SICKEL, W.: *Periodische Räume: Maximalungleichungen, Fouriersche Multiplikatoren und Beziehungen zur starken Summierbarkeit mehrdimensionaler trigonometrischer Reihen.* Thesis. Jena: Friedrich-Schiller-Universität 1984.

[16] SICKEL, W.: *Characterization of Besov-Triebel-Lizorkin spaces via approximation by Whittaker's cardinal series.* Preprint. Jena: Friedrich-Schiller-Univ. 1990, 1 - 20.

[17] H.TRIEBEL: *Fourier Analysis and Function Spaces* (Teubner-Texte zur Mathematik: Vol. 7). Leipzig: B.G. Teubner Verlagsges. 1977.

[18] H.TRIEBEL: *Interpolation Theory, Function Spaces, Differential Operators.* Am-

sterdam-New York- Oxford: North-Holland Publ. Comp. 1978, and Berlin: Dt. Verlag Wiss. 1978.
[19] ZYGMUND, A.: *Trigonometric Series*, Vol. I and II. Sec. ed. Cambridge: Cambr. Univ. Press 1977.

Received 13.02.1990; in revised form 30.01.1991

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