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A Dynamic Problem of Thermoelasticity

J. NEČAS and M. RŮŽIČKA

A space-periodic problem of nonlinear thermoelasticity is considered. For an elastic, linear, isotropic, homogeneous, nonviscous body in small geometry, we obtain a nonlinear system of equations. For small coefficient of the heat extension a we find a time-global weak solution of the initial-value problem. The smallness of a is independent of the length of the time interval and of the datas. The space periodicity of the solution is related to the absence of reflected waves. A mixed problem for a bounded domain, even with a smooth boundary, seems to be an open problem. Our work is closely related to that by J. Nečas [5] and by J. Nečas, A. Novotný and V. Sverák [6].

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0. Introduction

Let $\Omega = (0,1)^3 \subset \mathbf{R}^3$ be the body considered. We denote:

$u = (u_1, u_2, u_3)$	- the displacement vector,
$\sigma = (\sigma_{ij})$	- the stress tensor,
$e = (e_{ij})$	- the small strain tensor,
T	- the temperature,
S	- the entropy,
$c = (c_1, c_2, c_3)$	- the heat flow,
$f = (f_1, f_2, f_2)$	- the body force,
W	- the internal energy,
F	- the free energy,
k	- the coefficient of heat conductivity,
d	- the coefficient of specific heat,
a	- the coefficient of heat extension,
μ, λ	- the Lamé's coefficients,
ρ	- the density.

For the linear isotropic, homogeneous body, where we also take care of heat effects, we have the following equations:

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} - (3\lambda + 2\mu) a \delta_{ij} (T - T_0) \qquad (\textit{Hooke's law}), \tag{0.1}$$

$$\rho \ddot{u}_i = \partial \sigma_{ij} / \partial x_j + f_i \qquad (\textit{equation of motion}), \tag{0.2}$$

$$\dot{W} = \sigma_{ij} \dot{e}_{ij} - \partial c_i / \partial x_i \qquad (\textit{energy equation}), \tag{0.3}$$

where T_0 is a constant and δ_{ij} denotes Kronecker's delta. For simplicity we suppose $\rho \equiv 1$. We use the summation convention over repeated indices. Let us assume that the free energy is only a function of the temperature and of the deformation, i.e. $F(e,T) = W(e,S) - TS$. From this, by some calculations, we get

$$\dot{W} = \sigma_{ij}\dot{e}_{ij} + d\dot{T} - (3\lambda + 2\mu)a\dot{e}_{kk}T. \tag{0.4}$$

If we use the equations (0.3) and (0.4) and the relation

$$c_i(T,\nabla T) = -k(T,\nabla T)\partial T/\partial x_i \tag{0.5}$$

we obtain

$$\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) = d\dot{T} + (3\lambda + 2\mu)a\dot{e}_{kk}T. \tag{0.6}$$

Relation (0.5) is motivated by Fourier's law. The heat flow is subject to the principle of material frame indifference and so $k(T,\nabla T)$ has to be an isotropic function (see [9]). Further we require that k is even with respect to $(T - T_0)$. The most simple function satisfying these properties is

$$k(T,\nabla T) = a_0 + a_1(T - T_0)^2 + |\nabla T|^2, \text{ with } a_0, a_1 = \text{const}. \tag{0.7}$$

For simplicity we set $a_0 = a_1 = 1$. If we substitute the definition of the small strain tensor into (0.1) and the result into (0.2) we obtain

$$\lambda\frac{\partial^2 u_j}{\partial x_i\partial x_j} + \mu\left(\frac{\partial^2 u_i}{\partial x_j\partial x_j} + \frac{\partial^2 u_j}{\partial x_i\partial x_j}\right) - (3\lambda + 2\mu)a\frac{\partial T}{\partial x_i} + f_i = \ddot{u}_i. \tag{0.8}$$

The system (0.6), (0.8) is a thermoelastic description of a linear, isotropic, homogeneous and thermoextensible body without viscosity. So we seek for the displacement vector u and the temperature T, both defined on $Q_I = \Omega \times I$, with $I = (0,t_0)$ and satisfying (0.6),(0.8).

By $(L^p(\Omega), \|\cdot\|_{p,\Omega})$ and $(W_p^m(\Omega), \|\cdot\|_{m,p,\Omega})$ we denote the Lebesgue and Sobolev spaces of space-periodic functions with period 1. We put

$$L^p(\Omega,\mathbf{R}^3) = L^p(\Omega)\times L^p(\Omega)\times L^p(\Omega),\ W_p^m(\Omega,\mathbf{R}^3) = W_p^m(\Omega)\times W_p^m(\Omega)\times W_p^m(\Omega)$$

with norms defined by the same symbols $\|\cdot\|_{p,\Omega}$ and $\|\cdot\|_{m,p,\Omega}$. Let V be a Banach space. Then by $L^p(I,V)$ we denote the space of Bochner-measurable functions with values in V, for which $\int_I \|u(s)\|_V^p ds$ is finite. With the norm $\|u\|_{p,V} = \left(\int_I \|u(s)\|_V^p\right)^{1/p}$, $L^p(I,V)$ is a Banach space. For details see [1]. Further we denote $\dot{u} = \partial u/\partial t$, $\nabla u = (\partial u/\partial x_1, \partial u/\partial x_2, \partial u/\partial x_3)$, $\nabla^2 u = (\partial^2 u/\partial x_1^2, \partial^2 u/\partial x_1\partial x_2, \dots, \partial^2 u/\partial x_3^2)$, weak convergence of a sequence $\{x_n\}$ to x by $x_n \rightharpoonup x$, $(u,v) = \int_\Omega u(x)v(x)dx$, $\langle g,h\rangle = \int g(t)h(t)dt$, $Q_t = \Omega \times (0,t)$, $t \in I$. At last, we will also denote by c constants in various estimations.

1. Existence of a solution

From the above, we have to find functions $u: Q_I \to \mathbf{R}^3$ and $T: Q_I \to \mathbf{R}$ solving the problem

$$(\lambda + \mu)\frac{\partial e_{kk}}{\partial x_i} + \mu\Delta u_i - \alpha\frac{\partial T}{\partial x_i} + f_i = \ddot{u}_i, \tag{1.1}$$

$$\beta\dot{T} + \alpha\dot{e}_{kk}T - \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) = 0, \tag{1.2}$$

where $\beta > 0$, $\alpha \in (0,\alpha_0)$, α_0 = const and, according to (0.7),

$$k = 1 + (T - T_0)^2 + |\nabla T|^2 , \tag{1.3}$$

with initial conditions

$$u(x,0) = u_0(x) \text{ and } \dot{u}(x,0) = u_1(x) , \quad x \in \Omega, \tag{1.4}$$

$$T(x,0) = T_0 = \text{const.} > 0 , \quad x \in \Omega . \tag{1.5}$$

Further we will assume that

$$u_0 \in W_2^2(\Omega,\mathbf{R}^3) , \quad u_0 \in W_2^1(\Omega,\mathbf{R}^3) , \; f \in L^2(I, W_2^1(\Omega,\mathbf{R}^3)). \tag{1.6}$$

Definition 1: A solution of the given problem (1.1) - (1.6) is a pair of functions u: $Q_I \to \mathbf{R}^3$ and T: $Q_I \to \mathbf{R}$ satisfying the following conditions:

(i) $u \in L^2(I, W_2^1(\Omega,\mathbf{R}^3))$, $\dot{u} \in L^\infty(I, L_2(\Omega,\mathbf{R}^3))$, $\ddot{u} \in L^2(I,(W_2^1(\Omega,\mathbf{R}^3))')$,
$T \in L^4(I, W_4^1(\Omega)) \cap L^2(I, W_2^2(\Omega))$, $\dot{T} \in L^{4/3}(I, W_4^1(\Omega))$.

(ii) $(\lambda + \mu)\dfrac{\partial e_{kk}}{\partial x_i}(x,t) + \mu \Delta u_i(x,t) - \alpha \dfrac{\partial T}{\partial x_i}(x,t) + f_i(x,e) = \ddot{u}(x,t)$ a.e. in Q_I.

(iii) $$\int_\Omega \Big(\beta \dot{T}(x,t) v(x) + \alpha \dot{e}_{kk}(x,t) T(x,t) v(x) + k \frac{\partial T}{\partial x_i}(x,t) \frac{\partial v}{\partial x_i}(x) \Big) dx = 0 \quad \forall\, v \in W_4^1(\Omega), \text{ a.e. } t \in I.$$

First, let us consider the following problem:

(P1) Let $T_1 \in L^4(I, W_4^1(\Omega))$. We seek for a weak solution u of the problem (1.1), (1.4) for fixed $T \equiv T_1$, i.e. we look for such function u: $Q_I \to \mathbf{R}^3$ that

(i) $u \in L^\infty(I, W_2^1(\Omega,\mathbf{R}^3))$, $\dot{u} \in L^\infty(I, L_2(\Omega,\mathbf{R}^3))$, $\ddot{u} \in L^2(I,(W_2^1(\Omega,\mathbf{R}^3))')$;

(ii) $$\int_\Omega \Big((\lambda + \mu) \frac{\partial u_i}{\partial x_i}(x,t) \frac{\partial v_j}{\partial x_j}(x) + \mu \frac{\partial u_i}{\partial x_j}(x,t) \frac{\partial v_i}{\partial x_j}(x) + \ddot{u}_i(x,t) v_i(x) \Big) dx$$
$$= \int_\Omega \Big(f_i(x,t) v_i(x) - \alpha \frac{\partial T_1}{\partial x_i}(x,t) v_i(x) \Big) dx \quad \forall\, v \in W_2^1(\Omega,\mathbf{R}^3), \text{ a.e. } t \in I.$$

Lemma 1: *There exists a unique weak solution u of Problem* (P1), *for which the following estimate holds :*

$$\int \big(|\dot{u}(x,t)|^2 + |\nabla u(x,t)|^2 + |u(x,t)|^2 \big) dx \le c \big(\|u_0\|^2_{1,2,\Omega} + \|u_1\|^2_{0,2,\Omega} + \|f\|^2_{2,Q_I} + \alpha \|\nabla T_1\|^2_{2,Q_I} \big). \tag{1.7}$$

Proof: The classical Galerkin method gives the existence of the weak solution of this linear problem (se, e.g., [8]) ∎

Theorem 1: *Let* $T_1 \in L^2(I, W_2^2(\Omega)) \cap L^4(I, W_4^1(\Omega))$. *Then for the weak solution u of the Problem* (P1) *we obtain*

$$u \in L^{\infty}(I, W_2^2(\Omega, \mathbf{R}^3)),\ \dot u \in L^{\infty}(I, W_2^1(\Omega, \mathbf{R}^3)),\ \ddot u \in L^2(I, L^2(\Omega, \mathbf{R}^3))$$

and

$$\int \left(|\nabla \dot u(x,t)|^2 + |\nabla^2 u(x,t)|^2\right) dx$$
$$\le c\left(\|u_0\|^2_{2,2,\Omega} + \|u_1\|^2_{1,2,\Omega} + \|\nabla f\|^2_{2,Q_I} + \alpha \|\nabla^2 T_1\|^{2'}_{2,Q_I}\right) \text{ for a.e. } t \in I.$$

Proof: We again use the Galerkin method. Let $\{g^K\} = \{(g_1^K, g_2^K, g_3^K)\}$ be a base of the space $W_2^2(\Omega, \mathbf{R}^3)$ satisfying $\Delta g^K = -\lambda_K g^K$ for all $K \in \mathbf{N}$. We define

$$u^N(x,t) = \sum_{K=1}^N C_{NK}(t) g^K(x), \text{ where } C_{NK}(t) = \left(C^1_{NK}(t), C^2_{NK}(t), C^3_{NK}(t)\right)$$

are defined by

$$\begin{aligned} &\int_\Omega \left((\lambda + \mu)\frac{\partial^2 u_j^N}{\partial x_j \partial x_i}(x,t)\Delta g_i^K(x) + \mu \Delta u_i^N(x,t)\Delta g_i^K(x) - \ddot u_i^N(x,t)\Delta g_i^K(x)\right) dx \\ &\quad = \int_\Omega \left(-f_i(x,t)\Delta g_i^K(x) + \alpha \frac{\partial T_1}{\partial x_i}(x,t)\Delta g_i^K(x)\right) dx \quad \forall\ t \in I, \\ &C_{NK}(0) = \left(u_0, g^K\right),\quad \dot C_{NK}(0) = \left(u_1, g^K\right). \end{aligned} \tag{1.8}$$

Multiplying the K-equation of (1.8) by $\dot C_{NK}(t)$, summing up over K and integrating over $(0,t)$, we obtain

$$\int_\Omega \left(|\nabla \dot u^N(x,t)|^2 + |\nabla^2 u^N(x,t)|^2\right) dx$$
$$\le c\left(\|u_0(0)\|^2_{2,2,\Omega} + \|u_1(0)\|^2_{1,2,\Omega} + \|\nabla f\|^2_{2,Q_I} + \alpha \|\nabla^2 T_1\|^2_{2,Q_I}\right). \tag{1.9}$$

From equations (1.7) and (1.9) we get that the sequence $\{u^N\}$ is bounded in $W_2^2(\Omega, \mathbf{R}^3)$ and that the sequence $\{\dot u^N\}$ is bounded in $W_2^1(\Omega, \mathbf{R}^3)$. Now, by a well-known technique we obtain our assertion ∎

Let us for fixed $T_1 \in L^4(I, W_4^1(\Omega))$ and every $t \in I$ define the operator $A(t): W_4^1(\Omega) \to (W_4^1(\Omega))'$ by

$$(A(t)v, w) = \int_\Omega \left(\frac{1}{\beta}(1 + (T_1 - T_0)^2(x,t) + |\nabla v(x)|^2)\frac{\partial v}{\partial x_i}(x)\frac{\partial w}{\partial x_i}(x)\right) dx \quad \forall v, w \in W_4^1(\Omega).$$

Lemma 2: *$A(t)$ is a monotonous, continuous and bounded operator (i.e. $A(t)$ maps bounded sets into bounded sets), which satisfies*

$$\left(A(t)v - A(t)w, v - w\right) \ge c\int_\Omega \left(|\nabla(v - w)(x)|^2 + |\nabla(v - w)(x)|^4\right) dx \quad \forall\ v, w \in W_4^1(\Omega).$$

Proof: Put $k_i(t,x,y) = \beta^{-1}(1 + (T_1 - T_0)^2(x,t) + |y|^2)y_i$. It is obvious that k_i satisfies the Carathéodory condition and that the estimate $|k_i(t,x,y)| \le c\left(g(x,t) + \sum_{i=1}^3 |y_i|^3\right)$ holds for some function $g(\cdot,t) \in L^{4/3}(\Omega)$. Now it is obvious (see [1: Lemma 1.6]) that $A(t)$ is a continuous and bounded operator. Further we have

$$\langle A(s)v - A(s)w, v - w\rangle$$
$$\geq \frac{1}{\beta}\int_\Omega \left((1+|\nabla v|^2)\frac{\partial v}{\partial x_i} - (1+|\nabla w|^2)\frac{\partial w}{\partial x_i}\right)\frac{\partial(v-w)}{\partial x_i}\,dx$$
$$= \frac{1}{\beta}\int_\Omega\int_0^1 (1+3|\nabla w + t\nabla(v-w)|^2)(\nabla(v-w),\nabla(v-w))\,dt\,dx$$
$$\geq c\left(\int_\Omega\int_0^1 |\nabla(v-w)|^2|\nabla w + t\nabla(v-w)|^2\,dt\,dx + \int_\Omega |\nabla(v-w)|^2\,dx\right)$$
$$\geq c\int_\Omega\left(|\nabla(v-w)|^2 + |\nabla(v-w)|^4\right)dx \blacksquare$$

Put $X = L^4(I, W_4^1(\Omega))$. Then $X' = L^{4/3}(I,(W_4^1(\Omega))')$. Let us define the operator $A: X \to X'$ by

$$(Av)(t) = A(t)v(t) \quad \forall\, v \in X$$

and the element $b \in X'$ by

$$(b(t),v) = \frac{\alpha}{\beta}\int_\Omega \dot{u}_i(x,t)\left(\frac{\partial T_1}{\partial x_i}(x,t)v(x) + T_1(x,t)\frac{\partial v}{\partial x_i}(x)\right)dx \quad \forall\, v \in W_4^1(\Omega),$$

where u is the corresponding solution of problem (P1) to $T_1 \in X$. If we use this notation we can for (1.2), (1.3), (1.5) formulate the following problem:

(P2) Find $T_2 \in X$ satisfying

$$\begin{aligned}&(\dot{T}_2(t),v) + (A(t)T_2(t),v) = (b(t),v) \quad \forall\, v \in W_4^1(\Omega), \text{ a.e. } t \in I,\\ &T_2(x,0) = T_0.\end{aligned} \tag{1.10}$$

Definition 2: The function $T_2 \in X$ is called a *weak solution of the Problem* (P2) if T_2 satisfies (1.10) and if $\dot{T}_2 \in X'$.

Theorem 2: *Let $T_2 \in X$ and u be the corresponding weak solution of the Problem* (P1). *Then there exists a weak solution T_2 of the Problem* (P2).

Proof: Let us again use the Galerkin method. Let $\{g_K\}_{K=1}^{\infty}$ be a base of $W_4^1(\Omega)$. We define

$$T_2^N(x,t) = \sum_{K=1}^N C_{NK}(t)g_K(x) \quad \forall\, (x,t) \in Q_I,$$

where C_{NK} are determined by

$$\begin{aligned}&(\dot{T}_2^N(t),g_K) + (A(t)T_2^N(t),g_K) = (b(t),g_K) \quad \forall\, t \in I,\\ &C_{NK}(0) = (T_0,g_K).\end{aligned} \tag{1.11}$$

From this we obtain

$$\int_0^s\left((\dot{T}_2^N(t),T_2^N(t)) + (A(t)T_2^N(t),T_2^N(t))\right)dt = \int_0^s (b(t),T_2^N(t))\,dt \quad \forall\, s \in I. \tag{1.12}$$

For the right-hand side we have in view of (1.7)

$$\left|\int_0^s\int_\Omega \dot u_i(x,t)\frac{\partial T_1}{\partial x_i}(x,t)\,T_2^N(x,t)\,dx\,dt\right|$$

$$\le \int_0^{t_0}\Big(\int_\Omega |\dot u_i(x,t)|^2dx\Big)^{1/2}\Big(\int_\Omega|\nabla T_1(x,t)|^4dx\Big)^{1/4}\Big(\int_\Omega|T_2^N(x,t)|^4dx\Big)^{1/4}dt$$

$$\le c\big(c_1+\alpha\|\nabla T_1\|_{2,Q_I}\big)\|\nabla T_1\|_{4,Q_I}\Big(\int_0^{t_0}\Big(\int_\Omega|T_2^N(x,t)|^4dx\Big)^{1/3}dt\Big)^{3/4} \tag{1.13}$$

$$\le c_0\big(c_1+\alpha\|\nabla T_1\|_{2,Q_I}\big)\|\nabla T_1\|_{4,Q_I}\big(\|\nabla T_2^N\|_{2,Q_I}+\|T_2^N\|_{2,Q_I}\big),$$

where we have used in the last line the imbedding of $W_2^1(\Omega)$ into $L^6(\Omega)$. In the same way we get

$$\left|\int_0^s\int_\Omega \dot u_i(x,t)\frac{\partial T_2^N}{\partial x_i}(x,t)\,T_1(x,t)\,dx\,dt\right|$$
$$\le c_0\big(c_1+\alpha\|\nabla T_1\|_{2,Q_I}\big)\|\nabla T_2^N\|_{4,Q_I}\big(\|\nabla T_1\|_{2,Q_I}+\|T_1\|_{2,Q_I}\big). \tag{1.14}$$

From (1.12) we obtain, in view of (1.13), (1.14) and the monotonicity of $A(t)$,

$$\int_\Omega|T_2^N(x,s)|^2dx+\int_{Q_I}\big(|\nabla T_2^N(x,t)|^2+|\nabla T_2^N(x,t)|^4\big)dx\,dt$$
$$\le c\big(\|T_2^N(x,0)\|_{2,\Omega}^2+(1+\|\nabla T_1\|_{2,Q_I}^2)(\|\nabla T_1\|_{4,Q_I}^2+\|T_1\|_{2,Q_I}^2)\big).$$

But

$$\Big(\int_0^{t_0}\Big(\int_\Omega T^2(x,t)\,dx\Big)^2+\int_\Omega|\nabla T(x,t)|^4dx\,dt\Big)^{1/4} \tag{1.15}$$

is an equivalent norm on the space X. And so there exist a weak convergent subsequence $\{T_2^N\}\subset X$ and elements $w\in X'$, $z\in L^2(\Omega)$ such that

$$T_2^N\rightharpoonup T_2 \text{ in } X,\quad AT_2^N\rightharpoonup w \text{ in } X',$$
$$T_2^N(t_0)\rightharpoonup z \text{ in } L^2(\Omega),\quad T_2^N(0)\rightharpoonup T_0 \text{ in } W_4^1(\Omega),$$

where we used the boundedness of A and the imbedding of the set $\{T\colon T\in X,\ \dot T\in X'\}$ into $C(I,L^2(\Omega))$. By some calculation we obtain from (1.11)

$$\dot T_2\in X',\quad \dot T_2+w=b,\quad T_2(0)=T_0,\ T_2(t_0)=z.$$

But

$$\varlimsup_{N\to\infty}\langle AT_2^N,T_2^N\rangle=\varlimsup_{N\to\infty}2^{-1}\big(\|T_2^N(0)\|_{2,\Omega}^2-\|T_2^N(t_0)\|_{2,\Omega}^2\big)+\langle b,T_2^N\rangle$$
$$\le 2^{-1}\big(\|T_2(0)\|_{2,\Omega}^2-\|T_2(t_0)\|_{2,\Omega}^2\big)+\langle b,T_2\rangle=\langle w,T_2\rangle.$$

This, (1.15), the weak convergence of T_2^N to T_2 in X and the monotonicity of A imply $AT_2=w$, i.e. T_2 is a weak solution of the Problem (P2) ∎

Theorem 3: *Let $T_1 \in X \cap L^2(I, W_2^2(\Omega))$. Then the weak solution T_2 of the Problem* (P2) *belongs to the space $L^2(I, W_2^2(\Omega))$ and satisfies the estimate*

$$\|\nabla^2 T_2\|^2_{2,Q_I} \le c_2\left(c_3 + \alpha(c_5 + \alpha\|\nabla^2 T_1\|^2_{2,Q_I}) + \|\nabla T_1\|^2_{4,Q_I}\|\nabla T_2\|^2_{4,Q_I}\right) \tag{1.16}$$

Proof: **(i)** First we prove that $T_2 \in L^2(I, W_2^2(\Omega))$. For $v \in X$ and almost every $t \in I$ we have

$$\int_\Omega\left(\dot{T}_2(x,t)v(x,t) + \frac{1}{\beta}\left(1 + (T_1 + T_0)^2(x,t) + |\nabla T_2(x,t)|^2\right)\frac{\partial T_2}{\partial x_i}(x,t)\frac{\partial v}{\partial x_i}(x,t)\right)dx$$
$$= -\int_\Omega \frac{\alpha}{\beta} T_1(x,t)\frac{\partial \dot{u}_i}{\partial x_j}(x,t)v(x,t)\,dx. \tag{1.17}$$

We denote $\Delta_\tau x = \tau e^I$, $|\tau| < h$, $h > 0$ fixed, where e^I $(I = 1,2,3)$ is a unit vector and $\{e^1, e^2, e^3\}$ is a base of $\mathbb{R}^3$. Further we denote $\Delta_\tau w(x,t) = \tau^{-1}(w(x + \Delta_\tau x, t) - w(x,t))$. From (1.17) we obtain

$$\int_\Omega\Big\{(\dot{T}_2(x + \Delta_\tau x, t) - \dot{T}_2(x,t))v(x,t)$$
$$+ \frac{1}{\beta}\Big[\left(1 + (T_1 - T_0)^2(x + \Delta_\tau x, t) + |\nabla T_2(x + \Delta_\tau x, t)|^2\right)\frac{\partial T_2}{\partial x_i}(x + \Delta_\tau x, t)$$
$$- \left(1 + (T_1 - T_0)^2(x,t) + |\nabla T_2(x,t)|^2\right)\frac{\partial T_2}{\partial x_i}(x,t)\Big]\frac{\partial v}{\partial x_i}(x,t)\Big\}dx$$
$$= -\int_\Omega \frac{\alpha}{\beta}\left[\frac{\partial \dot{u}_i}{\partial x_j}(x + \Delta_\tau x, t)T_1(x + \Delta_\tau x, t) - \frac{\partial \dot{u}_i}{\partial x_i}(x,t)T_1(x,t)\right]v(x,t)\,dx.$$

In particular for $v(x,t) = \Delta_\tau T_2(x,t)$ we get

$$\frac{1}{2}\int_\Omega\left((\Delta_\tau T_2)^2(x,t) - (\Delta_\tau T_2)^2(x,0)\right)dx + \frac{1}{\beta}\int_{Q_I}\Big\{\frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t)\frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t)$$
$$+ \frac{1}{2}\left((T_1 - T_0)^2(x + \Delta_\tau x, t) - (T_1 - T_0)^2(x,t)\right)\frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t)\frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t)$$
$$+ \frac{1}{2}(\Delta_\tau(T_1 - T_0))^2(x,t)\frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t)\left(\frac{\partial T_2}{\partial x_i}(x + \Delta_\tau x, t) + \frac{\partial T_2}{\partial x_i}(x,t)\right)$$
$$+ \frac{1}{2}\left(|\nabla T_2(x + \Delta_\tau x, t)|^2 + |\nabla T_2(x,t)|^2\right)\frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t)\frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t)$$
$$+ \frac{1}{2}\Delta_\tau|\nabla T_2(x,t)|^2\left(\frac{\partial T_2}{\partial x_i}(x + \Delta_\tau x, t) + \frac{\partial T_2}{\partial x_i}(x,t)\right)\frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t)\Big\}dx\,dt$$
$$\le \int_{Q_I}\frac{\alpha}{\beta}\Delta_\tau\left(\frac{\partial \dot{u}_i}{\partial x_j}(x,t)T_1(x,t)\right)\Delta_\tau T_2(x,t)\,dx\,dt$$
$$\le c\left((1 + \|\nabla^2 T_1\|_{2,Q_I})\|T_1\|_{4,Q_I}\|\Delta_\tau \Delta_\tau T_2\|_{4,Q_I}\right)$$
$$\le c\left((1 + \|\nabla^2 T_1\|_{2,Q_I})\frac{1}{\varepsilon}\|T_1\|^2_{4,Q_I} + \varepsilon\|\partial \Delta T_2/\partial x_{e^I}\|^2_{4,Q_I}\right),$$

where we used

$$\int_{Q_t} \Delta_\tau u(x,t)\, \Delta_\tau T(x,t)\, dx\, dt = -\int_{Q_t} u(x,t)\, \Delta_{-\tau} \Delta_\tau T(x,t)\, dx\, dt$$

$$\|\Delta_{-\tau} \Delta_\tau T\|_{4,\Omega} \le \|\partial \Delta_\tau T/\partial x_{el}\|_{4,Q_I} \qquad \big(T \in W_4^1(\Omega)\big).$$

From this we get

$$\begin{aligned}
\int_{Q_t} \Big\{ & \frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t) \frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t) \\
& + \big((T_1 - T_0)^2(x + \Delta_\tau x, t) - (T_1 - T_0)^2(x,t)\big) \frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t) \frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t) \\
& + \Delta_\tau (T_1 - T_0)(x,t) \frac{\partial \Delta_\tau T_2}{\partial x_i} \left(\frac{\partial T_2}{\partial x_i}(x + \Delta_\tau x, t) + \frac{\partial T_2}{\partial x_i}(x,t)\right) \\
& \times \big((T_1 - T_0)(x + \Delta_\tau x, t) + (T_1 - T_0)(x,t)\big) \\
& + \left(\frac{\partial T_2}{\partial x_i}(x + \Delta_\tau x, t) + \frac{\partial T_2}{\partial x_i}(x,t)\right)^2 \frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t) \frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t) \Big\}\, dx\, dt \\
\le\ & c\big(1 + \|\nabla^2 T_1\|_{2,Q_I}^2 \|T_1\|_{4,Q_I}^2\big).
\end{aligned} \tag{1.18}$$

The third term on the left-hand side we carry to the right-hand side, where we use the estimate

$$\begin{aligned}
\int_{Q_t} & \Delta_\tau (T_1 - T_0)(x,t) \left(\frac{\partial T_2}{\partial x_i}(x + \Delta_\tau x, t) + \frac{\partial T_2}{\partial x_i}(x,t)\right) \\
& \times \frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t) \big((T_1 - T_0)(x + \Delta_\tau x, t) + (T_1 - T_0)(x,t)\big) dx\, dt \\
\le\ & c \frac{1}{\varepsilon} \|\Delta_\tau T_1\|_{4,Q_I}^2 \|\nabla T_2\|_{4,Q_I}^2 + c\varepsilon \int_{Q_I} \frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t) \frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t) \\
& \times \big((T_1 - T_0)(x + \Delta_\tau x, t) + (T_1 - T_0)(x,t)\big) dx\, dt
\end{aligned}$$

for an ε such that $c\varepsilon < 1$. So from (1.18) we can obtain

$$\begin{aligned}
\int_{Q_I} \frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t) \frac{\partial \Delta_\tau T_2}{\partial x_i}(x,t) & \le c\big(1 + \|\nabla^2 T_1\|_{2,Q_I}^2 \|T_1\|_{4,Q_I}^2\big) \\
& \le c\big(1 + \|\nabla^2 T_1\|_{2,Q_I}^2 \|T_1\|_{4,Q_I}^2 + \|\nabla T_1\|_{4,Q_I}^2 \|\nabla T_2\|_{4,Q_I}^2\big).
\end{aligned} \tag{1.19}$$

We will use the following fact:

Let $u \in L^p(\Omega)$, $1 < p < \infty$, let $\Delta_\tau u \in L^p(\Omega_h)$ for all $h > 0$, $|\tau| \le h$, and let $\|\Delta_\tau u\|_{p,\Omega_h} \le c_1 < \infty$. Then, in sence of distributions, $\|\partial u/\partial x_{el}\|_{p,\Omega_h} \le c_1$.

Then we get from (1.19) that $T_2 \in L^2(I, W_2^2(\Omega))$.

(ii) Now we want to prove the estimate (1.16). We multiply (1.2) by T_2'', where T_2'' is some second spatial derivative, and integrate over Q_I. Using partial integration we get

$$\int_\Omega \frac{1}{2}(T_2'(x,t_0))^2\,dx - \int_\Omega \frac{1}{2}(T_2'(x,0))^2\,dx$$

$$+\int_{Q_I}\left(1+(T_1-T_0)^2(x,t)+|\nabla T_2(x,t)|^2\right)\frac{\partial T_2'}{\partial x_i}(x,t)\,\frac{\partial T_2'}{\partial x_i}(x,t)\,dx\,dt$$

$$=\int_{Q_I}\Big(\alpha T_1(x,t)\frac{\partial \dot u_i}{\partial x_i}(x,t)\,T_2''(x,t)$$

$$-\left(1+(T_1-T_0)^2(x,t)+|\nabla T_2(x,t)|^2\right)'\frac{\partial T_2}{\partial x_i}(x,t)\,\frac{\partial T_2'}{\partial x_i}(x,t)\Big)\,dx\,dt$$

$$=\int_{Q_I}\Big(\alpha T_1(x,t)\frac{\partial \dot u_i}{\partial x_i}(x,t)T_2''(x,t)-2(T_1-T_0)(x,t)T_1'(x,t)\frac{\partial T_2}{\partial x_i}(x,t)\frac{\partial T_2'}{\partial x_i}(x,t)$$

$$-2\frac{\partial T_2}{\partial x_i}(x,t)\,\frac{\partial T_2'}{\partial x_i}(x,t)\,\frac{\partial T_2}{\partial x_t}(x,t)\,\frac{\partial T_2'}{\partial x_t}(x,t)\Big)\,dx\,dt.$$

The last term on the right-hand side we carry to the left-hand side and so we get

$$\int_{Q_I}\left(|\nabla T_2'(x,t)|^2+(T_1-T_0)^2(x,t)|\nabla T_2'(x,t)|^2+|\nabla T_2(x,t)|^2\,|\nabla T_2'(x,t)|^2\right)dx\,dt$$

$$\le \int_\Omega|\nabla T_2(x,0)|^2\,dx+\alpha\left|\int_{Q_I}T_1(x,t)T_2''(x,t)\,\frac{\partial \dot u_i}{\partial x_i}(x,t)\,dx\,dt\right|$$

$$+2\left|\int_{Q_I}(T_1-T_0)(x,t)\frac{\partial T_2'}{\partial x_i}(x,t)T_1'(x,t)\,\frac{\partial T_2}{\partial x_i}(x,t)\,dx\,dt\right|$$

$$\le c_4\left(c_5+\alpha\|T_1T_2''\|_{2,Q_I}\,\|\nabla\dot u\|_{2,Q_I}+\|(T_1-T_0)\nabla T_2'\|_{2,Q_I}\,\|T_1'\,\nabla T_2\|_{2,Q_I}\right)$$

$$\le c_6\Big(c_5+\alpha\varepsilon\,\|T_1T_2''\|_{2,Q_I}+\alpha\varepsilon^{-1}\left(c_7+\alpha\|\nabla^2T_1\|_{2,Q_I}\right)$$

$$+\varepsilon\,\|(T_1-T_0)\nabla T_2'\|^2_{2,Q_I}+\varepsilon^{-1}\,\|T_1'\,\nabla T_2\|^2_{2,Q_I}\Big)$$

for all $\varepsilon>0$. From this we obtain for some ε

$$\int_{Q_I}|\nabla T_2'(x,t)|^2\,dx\,dt\le c_8\left(c_5+\alpha(c_7+\alpha\|\nabla^2T_1\|^2_{2,Q_I})+\|\nabla T_1\|^2_{4,Q_I}\,\|\nabla T_2\|^2_{4,Q_I}\right)\ \blacksquare$$

Lemma 3: *Put, for some* $R_1>0$,

$$K_1=\left\{T,\ T\in X,\ \int_{Q_I}|\nabla T(x,t)|^4\,dx\,dt\le R_1^4,\ \int_{Q_I}T^4(x,t)\,dx\,dt\le R_1^4\right\}.$$

Let T_1, T_2 *denote the same functions as in Theorem 2. Then* B: $T_1\to T_2$ *is for* R_1 *large enough a mapping from* K_1 *into* K_1.

Proof: Using the proof of Theorem 2, especially (1.13) - (1.15) we have, where all the norms are L^4-norms,

$$\int_{Q_I}\left(T_2^4(x,t) + |\nabla T_2(x,t)|^4\right) dx\,dt$$

$$\le c\int_0^{t_0}\left\{\left(\int_\Omega T_2^2(x,t)\,dx\right)^2 + \int_\Omega |\nabla T_2(x,t)|^4 dx\right\} dt$$

$$\le c_9\Big(c_{10} + \alpha(c_1 + \alpha\|\nabla T_1\|)(\|\nabla T_1\|\|\nabla T_2\| + \|\nabla T_1\|\|\nabla T_2\| + \|\nabla T_2\|\|\nabla T_1\|)\Big)$$

$$\le c_{11}\Big(c_{10} + \alpha c_1\left(\varepsilon^{-1}\|\nabla T_1\|^2 + \varepsilon\|\nabla T_2\|^2 + \varepsilon\|T_2\|^2 + \varepsilon^{-1}\|T_1\|^2\right)$$

$$+ \alpha^2\left(\varepsilon^{-1}\|\nabla T_1\|^3 + \varepsilon\|\nabla T_2\|^2 + \|\nabla T_1\|^2 + \varepsilon\|\nabla T_2\|^4 + \varepsilon\|T_2\|^4 + \|T_1\|^3\right)\Big)$$

for all $\varepsilon > 0$. For some ε we carry all the terms containing T_2 or ∇T_2 on the left-hand side and so for $T_1 \in K_1$ we have

$$\int_{Q_I}\left(T_2^4(x,t) + |\nabla T_2(x,t)|^4\right) dx\,dt$$

$$\le c_{12}\left(c_{10} + \|\nabla T_1\|^2 + \|T_1\|^2 + \|\nabla T_1\|^3 + \|T_1\|^3\right) \le c_{12}\left(c_{10} + 2R^2 + 2R^3\right),$$

and the last term is smaller than R_1^4 for R_1 large enough ■

Lemma 4: *Put, for some $R_2 > 0$,*

$$K_2 = \left\{T,\ T \in K_1,\ \|\nabla^2 T\|_{2,Q_I} \le R_2,\ \dot T \in X'\right\}.$$

Let T_1, T_2 denote the same functions as in Theorem 3. Then $B\colon T_1 \to T_2$, for R_2 large enough and α small enough but α independent of R_1 and R_2, is a mapping from K_2 into K_2.

Proof: For $T_1 \in K_2$ we have (see (1.16))

$$\|\nabla T_2\|_{2,Q_I} \le c_2\left(c_3 + \alpha(c_5 + \alpha\|\nabla^2 T_1\|_{2,Q_I}) + \|\nabla T_1\|_{4,Q_I}\|\nabla T_2\|_{4,Q_I}\right)$$

$$\le c_2\left(c_3 + \alpha c_5 + \alpha^2\|\nabla^2 T_1\|^2_{2,Q_I} + R_1^4\right) \le c_2\left(c_{13} + \alpha^2 R_2^2\right).$$

Now we take an α such that $c_2\alpha^2 < 1$, and so for R_2 large enough, T_2 is an element of K_2 ■

Theorem 4: *Put*

$$K = \{T,\ T \in K_2,\ \|\dot T\|_{X'} \le R_3\}.$$

Then the mapping $B\colon T_1 \to T_2$ has in K a fixed point T, i.e. T and the corresponding u are a solution of (1.1) - (1.6).

Proof: We take an R_3 such that $\|\dot T\|_{X'} \le R_3$ for all $T_1 \in K_2$. K is a convex, closed and bounded subset of X. So we have to show that B is a weakly continuous operator, i.e. $x_n \rightharpoonup x$ in X implies $Bx_n \rightharpoonup Bx$ in X'. Let $\{T_1^n\} \subset K$ weakly converge to T_1. We denote $T_2^n = B(T_1^n)$. From Lemma 4 we get that $\{T_2^n\} \subset K$. So there exists a $T_3 \in K$ such that $T_2^n \rightharpoonup$

T_3 in X, $\dot{T}_2^n \rightharpoonup \dot{T}_3$ in $L^6(Q_I)$. From [2: Theorem 5.1] we get the strong convergence $T_2^n \to T_3$ and $T_1^n \to T_1$ in $L^4(Q_t)$. Further we have

$$\int_{Q_I}\big(\dot{T}_2^n(x,t) - \dot{T}_3(x,t)\big)\big(T_2^n(x,t) - T_3(x,t)\big)dx\,dt$$

$$+\int_{Q_I}\frac{1}{\beta}\Big[\big(1 + (T_1^n - T_0)^2(x,t) + |\nabla T_2^n(x,t)|^2\big)\frac{\partial T_2}{\partial x_i}(x,t)$$

$$-\big(1 + (T_1^n - T_0)^2(x,t) + |\nabla T_3(x,t)|^2\big)\frac{\partial T_3}{\partial x_i}(x,t)\Big]\frac{\partial T_2^n - T_3}{\partial x_i}(x,t)\,dx\,dt$$

$$= -\frac{\alpha}{\beta}\int_{Q_I}\frac{\partial \dot{u}_i^n}{\partial x_i}(x,t)T_1^n(x,t)\big(T_2^n(x,t) - T_3(x,t)\big)dx\,dt$$

$$-\int_{Q_I}\dot{T}_3(x,t)\big(T_2^n(x,t) - T_3(x,t)\big)dx\,dt$$

$$-\frac{1}{\beta}\int_{Q_I}\big(1 + T_1^n(x,t) + |\nabla T_3(x,t)|^2\big)\frac{\partial T_3}{\partial x_i}(x,t)\Big]\frac{\partial T_2^n - T_3}{\partial x_i}(x,t)\,dx\,dt.$$

The first term on the right-hand side we can estimate by

$$c\,\|\nabla \dot{u}^n\|_{2,Q_I}\,\|T_1^n\|_{4,Q_I}\,\|T_2^n - T_3\|_{4,Q_I}$$

which converges to zero in view of the boundedness of $\{T_1^n\}$ and $\{\nabla \dot{u}^n\}$. The second term on the right-hand side also converges to zero, because T_3 is an element of X'. Also the third term on the right-hand side converges to zero, because $\partial T_2^n/\partial x_i \rightharpoonup \partial T_3/\partial x_i$ in $L^4(Q_I)$ and $\big(1 + (T_1^n - T_0)^2(x,t) + |\nabla T(x,t)|^2\big)\partial T_3/\partial x_i(x,t)$ converges strongly in $L^{4/3}(Q_I)$. The left-hand side we can estimate from below by

$$\int_\Omega\big(T_2^n(x,t_0) - T_3(x,t_0)\big)^2dx - \int_\Omega\big(T_2^n(x,0) - T_3(x,0)\big)^2dx$$

$$+\|\nabla(T_2^n - T_3)\|_{2,Q_I}^2 + \|\nabla(T_2^n - T_3)\|_{4,Q_I}^4.$$

We carry the second term on the other side, but also this term converges to zero. Thus we get that $\|T_2^n - T_3\|_X \to 0$. In view of the strong convergence of T_2^n in X we can tend with $n \to \infty$ in the equation

$$\int_{Q_I}T_2^n(x,t)\varphi(x,t)\,dx\,dt$$

$$+\frac{1}{\beta}\int_{Q_I}\big(1 + T_1^n(x,t)^2 + |\nabla T_2^n(x,t)|^2\big)\frac{\partial T_2^n}{\partial x_i}(x,t)\frac{\partial \varphi}{\partial x_i}(x,t)\,dx\,dt$$

$$= -\frac{\alpha}{\beta}\int_{Q_I}\frac{\partial \dot{u}_i^n}{\partial x_i}(x,t)T_1^4(x,t)\varphi(x,t)\,dx\,dt \quad \forall\,\varphi \in X.$$

So we get $B(T_1) = T_3$. Altogether we have the weak convergence of T_2^n to $B(T_1)$ in K, i.e. weak continuity of B. By [7: Corollary 9.3] we get our assertion ■

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Authors' addresses:

Prof. Dr. Jindřich Nečas
Faculty of Mathematics and Physics
Charles University
Sokolovská 83
ČSSR - 18600 Praha 8

Dr. Michael Růžička
Sections of Mathematics
Karl - Marx - University
Karl - Marx - Platz
D(Ost) - 7010 Leipzig