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On the Structure of the Solutions of the Jabotinsky Equations in Banach Spaces

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Dedicated in gratitude to Professor János Aczél on the occasion of his 65^{th} birthday

The differential-functional equations of Jabotinsky are closely related to the translation equation. They are of importance in the iteration theory. In this paper we will give an explicit representation of the solutions of each of these equations in Banach spaces.

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The so-called Jabotinsky equations are the three differential equations

$$\frac{\partial F(x,t)}{\partial t} = \frac{\partial F(x,t)}{\partial x} \circ G(x), \tag{1}$$

$$\frac{\partial F(x,t)}{\partial t} = G(F(x,t)), \tag{2}$$

$$\frac{\partial F(x,t)}{\partial x} \circ G(x) = G(F(x,t)) \tag{3}$$

connected with the differential-initial condition

$$(\partial F(x,t)/\partial t)|_{t=0} = G(x) \tag{4}$$

which allows us to call them a system of differential-functional equations (see Targonski [16]). They were derived, in the one-dimensional analytic case, by Jabotinsky [9,10] from the translation equation

$$F(F(x,t),s) = F(x,t+s) \tag{5}$$

and the initial condition

$$F(x,0) = x. \tag{6}$$

This can be generalized to solutions of (5) in more abstract spaces, e.g. in real or complex Banach spaces (see Aczél and Gronau [2]). Equation (1) can be derived for every differentiable solution of (5) by differentiating both sides of (5) with respect to the variable t, putting $t = 0$ afterwards and using (6). Equation (2) is derived from (5) by differentiating (5) with respect to the variable s and then putting $s = 0$ (without using (6)). Equation (3) is a combination of (1) and (2). The $\circ$ in (1) and (3) means composition of $\partial F/\partial x$ with G.

The Jabotinsky equations can be used to determine the set S of numbers t such that a given function

$$F(x) = a_1 x + a_2 x^2 + \ldots \quad (a_i \in \mathbb{R} \text{ or } a_i \in \mathbb{C},\ a_1 \neq 0)$$

has a t-th iterate, that is, there exists a family of functions $\{F_t\}_{t \in S}$ such that $F_t|_{t=1} = F$ and $F(x,t) = F_t(x)$ satisfies equation (5) whenever t, s and $t+s$ are in S. This was done in the one-dimensional analytic case by Erdös and Jabotinsky [8]. In view of this, the Jabotinsky equations may also be of advantage for the embedding problem in the higher dimensional case (see, e.g.,Targonski [15] or Reich [12]).

In Aczél and Gronau [2] the following problem was treated: Do the Jabotinsky equations imply the translation equation? We there gave explicit representations of the general local solutions of each of the three Jabotinsky equations, without and with the initial conditions (6) and/or (4), in the real one-dimensional case (see also the condensed version Aczél and Gronau [3]).

It seems to be desirable to have also similar representation theorems for the solutions of equations (1)-(3) in the higher dimensional case, say $\mathbf{R}^n$ or $\mathbf{C}^n$ or in abstract Banach spaces. This will be given in the present paper in the case where F and x take values in a real or complex Banach space X, and t is a real or complex variable. It should be emphasized that we can give existence, uniqueness and explicit representation theorems on all of the three Jabotinsky equations, without using standard existence theorems, except the one well-known theorem of the existence of a local flow, that is, the existence of a solution of the autonomous equation $dy/dt = G(y)$, with the initial condition $y(0) = x$. The solution $y = y(t,x)$ depending on the initial value x exists, and is of class C^1 if the function G is of class C^1 (see Cartan [6] or Dieudonné [7]). The presented results include of course the case that X is one of the spaces $\mathbf{R}^n$ or $\mathbf{C}^n$ $(n \geq 1)$. In this case $\partial F(x,t)/\partial x$ is the Jacobian matrix $(\partial F_i(x,t)/\partial x_j)_{i,j=1,\dots,n}$ of the function $F(x,t) = {}^t(F_1(x,t),\dots,F_n(x,t))$ written as a column vector, and the symbol $\circ$ in (1) and (3) is the matrix multiplication with the column vector function $G(x) = {}^t(G_1(x),\dots,G_n(x))$, where $x = {}^t(x_1,\dots,x_n)$ is a column vector variable in $\mathbf{R}^n$ or $\mathbf{C}^n$. The real one-dimensional case is included herein in an obvious way.

Firstly, we will state three lemmas, which are important for the sequel. They are standard results from the theory of first order differential equations. But the first lemma is crucial for what follows. We will use an idea of L. Berg who gave an explicit representation of the solutions of an autonomous system of ordinary differential equations in $\mathbf{R}^n$, using the method of rectification. As a matter of fact, equation (2) is such an autonomous system. For so, we can apply the result of Berg [5] (see also Aczél [1]) to this equation.

The contents and the proofs of the following lemmas are standard (see, e.g., Lang [11] or Arnold [4]). The first lemma is the lemma of straihtening, the others give a characterization of functions with prescribed constant directional derivative. But, we will give the proofs here for the sake of completeness. We use some standard facts of functional analysis, for example the one, that every finite-dimensional subspace V of a Banach space X has a closed complementary space $\tilde{X}$. This means that there is a closed subspace $\tilde{X}$ of X, such that $X = V \oplus \tilde{X}$.

In what follows, with X will always be denoted a real or complex Banach space and by $\mathbf{K}$ the field of real or complex numbers, according as X is real or complex.

Lemma 1: *Let $G: U \subseteq X \to X$ be continuously differentiable (U an open neighbourhood of a fixed point $x_0 \in X$) and suppose $G(x_0) \neq \mathbf{0}$ ($\mathbf{0}$ is the zero in X). Further let be given $\mathbf{a} \in X$, $\mathbf{a} \neq \mathbf{0}$. Then there exists a C^1-diffeomorphism f, from a (possibly smaller) neighbourhood of x_0 onto a neighbourhood of $\mathbf{0}$, such that $f'(x) \circ G(x) = \mathbf{a}$, this means*

$$G(x) = (f'(x))^{-1}(\mathbf{a}). \tag{7}$$

Proof: Set $\mathbf{g} = G(x_0)$ and $\langle \mathbf{a} \rangle = \{t\mathbf{a} \mid t \in \mathrm{K}\}$. Then there exists a closed complementary subspace $\widetilde{X}$ of $\langle \mathbf{a} \rangle$ in X with $X = \langle \mathbf{a} \rangle \oplus \widetilde{X}$. This subspace can be chosen in such a kind that $\mathbf{g} \notin \widetilde{X}$. Therefore we also have $X = \langle \mathbf{g} \rangle \oplus \widetilde{X}$. Let now $\varphi = \varphi(t, \tilde{x})$ be a solution of the differential equation $d\varphi/dt = G(\varphi)$ with the initial condition $\varphi(0, \tilde{x}) = x_0 + \tilde{x}$, $\tilde{x} \in \widetilde{X}$. The function φ is defined on a suitable neighbourhood $I \times U \subseteq \mathrm{K} \times X$ of $(0, \mathbf{0})$. For $x = t\mathbf{a} + \tilde{x} \in X$ we may define $\psi(x) = \varphi(t, \tilde{x})$, supposed that $(t, \tilde{x}) \in I \times \widetilde{U}$. We have $\psi(\mathbf{0}) = x_0$ and we will prove that ψ is a C^1-diffeomorphism and $f = \psi^{-1}$ is the claimed diffeomorphism.

a) ψ is of class C^1 (see, e.g., Cartan [6]).

b) $\psi'(0)|_{\widetilde{X}} = \mathrm{id}_{\widetilde{X}}$, since $\psi(\tilde{x}) = \varphi(0, \tilde{x}) = \tilde{x}$ for $\tilde{x} \in \widetilde{X}$ and $\psi'(0)|_{\langle \mathbf{a} \rangle}(h_1 \mathbf{a}) = G(x_0)h_1 = \mathbf{g}h_1$. Therefore, $\psi'(\mathbf{0})\colon \langle \mathbf{a} \rangle \oplus \widetilde{X} \to \langle \mathbf{g} \rangle \oplus \widetilde{X}$, being the direct sum of two isomorphisms, is itself an isomorphism. By the inverse mapping theorem, ψ is invertible in a suitable neighbourhood $U_1 \subseteq X$ of $\mathbf{0}$. The inverse map $f = \psi^{-1}$ is defined in a neighbourhood $U_0 = \psi(U_1)$, where $\psi(\mathbf{0}) = x_0 \in U_0$.

c) For $h = h_1\mathbf{a} + h_2$, $h_2 \in \widetilde{X}$ we have

$$\psi'(x)(h) = \frac{\partial \varphi(t, \tilde{x})}{\partial t} h_1 + \frac{\partial \varphi(t, \tilde{x})}{\partial \tilde{x}}(h_2) = G(\varphi(t, \tilde{x}))h_1 + \frac{\partial \varphi(t, \tilde{x})}{\partial \tilde{x}}(h_2).$$

Hence for $h = \mathbf{a} = 1 \cdot \mathbf{a} + 0$ the following calculations will hold:

$$\psi'(x)(\mathbf{a}) = G(\psi(x))\psi'(f(x))(\mathbf{a}) = G(x)(f^{-1})'(f(x))(\mathbf{a}) = G(x)f'(x)^{-1}(\mathbf{a}) = G(x).$$

Therefore f is the claimed diffeomorphism ∎

Remark 1: If $X = \mathbf{R}^n$, then $f'(x)$ is the Jacobian matrix of f. That is, if $f_i(x)$ are the components of $f(x)$ and $x = {}^t(x_1, \dots, x_n)$, then $f'(x) = (\partial f_i(x)/\partial x_j)_{i,j=1,\dots,n}$. If $\mathbf{a} = {}^t(a_1, \dots, a_n) \in \mathbf{R}^n$, then (7) means $f'(x)G(x) = \mathbf{a}$ and there exist n independent real-valued functions f_i, which satisfy the inhomogeneous partial differential equations $\sum_{j=1}^{n} \partial f_i(x)/\partial x_j \cdot G_j(x) = a_i$ for $i = 1, \dots, n$.

Lemma 2: A differentiable function $H\colon U \times I \subseteq X \times \mathrm{K} \to X$ *($U \times I$ open in $X \times \mathrm{K}$) is a differentiable solution of*

$$\frac{\partial H(y,t)}{\partial y}(\mathbf{a}) = \frac{\partial H(y,t)}{\partial t}, \tag{8}$$

where $\mathbf{a}$ *is a fixed element of* X, *if and only if there exists a differentiable function* $\varphi\colon U + I \cdot \mathbf{a} \to X$ *with*

$$H(y,t) = \varphi(y + t\mathbf{a}). \tag{9}$$

Proof: Every function of the form (9) satisfies equation (8). Contrary, suppose that $H(y,t)$ is a solution of equation (8), and put $\widetilde{H}(y,t) = H(y - t\mathbf{a}, t)$. Then

$$\frac{\partial \widetilde{H}(y,t)}{\partial t} = \frac{\partial H(y - t\mathbf{a}, t)}{\partial y}(-\mathbf{a}) + \frac{\partial H(y - t\mathbf{a}, t)}{\partial t} = \mathbf{0}.$$

Hence $\widetilde{H}(y,t)$ is independent of t, say $\widetilde{H}(y,t) = H(y - t\mathbf{a}, t) = \varphi(y)$, that is $H(y,t) = \widetilde{H}(y + t\mathbf{a}, t) = \varphi(y + t\mathbf{a})$ ∎

Lemma 3: *Let $U \subseteq X$ be open in X. Further, let $\mathbf{a} \in X$, $\mathbf{a} \neq \mathbf{0}$, and $\widetilde{X}$ a closed complementary subspace of $\langle \mathbf{a} \rangle$ in X, that is $X = \langle \mathbf{a} \rangle \oplus \widetilde{X}$. Denote by $\pi_{\widetilde{X}}$ the (continuous) linear projection from X onto $\widetilde{X}$. Then a function $H\colon U \subseteq X \to X$, being differentiable in U, satisfies*

$$H'(x)(\mathbf{a}) = \mathbf{0}, \tag{10}$$

if and only if there exists a differentiable function $\chi : \pi_{\tilde{X}}(U) \to X$ *such that*

$$H'(x) = \chi \circ \pi_{\tilde{X}}(x), \text{ for } x \in U. \tag{11}$$

Proof: Every function H of the form (11) is a solution of equation (10):

$$H'(x)(\mathbf{a}) = \chi'(\pi_{\tilde{X}}(x)) \circ \pi'_{\tilde{X}}(x)(\mathbf{a}) = \chi'(\pi_{\tilde{X}}(x)) \circ \pi_{\tilde{X}}(x)(\mathbf{a}) = 0,$$

because $\pi_{\tilde{X}}(\mathbf{a}) = \mathbf{0}$. Contrary, in view of $X = \langle \mathbf{a} \rangle \dotplus \tilde{X}$, we have the following unique representations for x, $H(x)$ and $h \in X$:

$$x = x_1 + x_2 \qquad (x_1 \in \langle \mathbf{a} \rangle,\ x_2 \in \tilde{X})$$
$$H(x) = H_1(x_1, x_2) + H_2(x_1, x_2) \qquad (H_1(x_1,x_2) \in \langle \mathbf{a} \rangle, H_2(x_1,x_2) \in \tilde{X})$$
$$h = h_1 + h_2 \qquad (h_1 \in \langle \mathbf{a} \rangle,\ h_2 \in \tilde{X}).$$

$H'(x)$ is therefore of the form

$$H'(x)(h) = \frac{\partial H_1(x_1,x_2)}{\partial x_1}(h_1) + \frac{\partial H_1(x_1,x_2)}{\partial x_2}(h_2) + \frac{\partial H_2(x_1,x_2)}{\partial x_1}(h_1) + \frac{\partial H_2(x_1,x_2)}{\partial x_2}(h_2)$$

For $h = \mathbf{a}$, that is, $h_1 = \mathbf{a}$ and $h_2 = \mathbf{0}$ this yields

$$H'(x)(\mathbf{a}) = \frac{\partial H_1(x_1,x_2)}{\partial x_1}(\mathbf{a}) + \frac{\partial H_2(x_1,x_2)}{\partial x_1}(\mathbf{a}) = 0.$$

This has as a consequence that $\partial H_1(x_1,x_2)/\partial x_1 = 0$ and $\partial H_2(x_1,x_2)/\partial x_1 = \mathbf{0}$ as a scalar or vector, respectively. Hence $H_1(x_1,x_2)$ and $H_2(x_1,x_2)$ are independent of x_1, therefore $H(x)$ is a function depending on $x_2 = \pi_{\tilde{X}}(x)$ only, this reads as $H(x) = \chi(\pi_{\tilde{X}}(x))$ ■

Remark 2: In the case $X = \mathbf{R}^n$ we have $\mathbf{a} = {}^t(a_1,\dots,a_n) \in \mathbf{R}^n$ and the vector-valued function $H(x) = {}^t(H_1(x),\dots,H_n(x))$ depends on $x = {}^t(x_1,\dots,x_n)$. In this way, equation (10) is of the form $\sum_{j=1}^n a_j\, \partial H(x)/\partial x_j = 0$. The condition $\mathbf{a} \neq \mathbf{0}$ implies that one of the a_i's is non-zero say, for the sake of simplicity, $a_n = 1$. Therefore we get by formula (11) as solution of (10)

$$H(x) = \chi(x_1 - a_1 x_n, \dots, x_{n-1} - a_{n-1} x_n)$$

Accordingly, in the case $n = 1$ we get the solution $H(x) = \text{const}$.

Now we can state the main results. In what follows we will assume the following suppositions:

The function $G = G(x)$ has to satisfy the conditions of Lemma 1, i.e. G is continuously differentiable and non-zero at a fixed point $x_0 \in X$. Furthermore, let us denote in the sequel by $\mathbf{a}$ a fixed non-zero element of X and, by f the invertible function, given by Lemma 1 such that $G(x) = (f'(x))^{-1}(\mathbf{a})$ holds. Finally, let $\tilde{X}$ be a closed complementary subspace of $\langle \mathbf{a} \rangle$ in X such that $X = \langle \mathbf{a} \rangle \oplus \tilde{X}$, and let $\pi_{\tilde{X}} : X \to \tilde{X}$ be the linear continuous projection from X onto $\tilde{X}$.

Theorem 1: *The following statements hold.*

a) *A function f is a solution of the Jabotinsky equation* (1),

$$\frac{\partial F(x,t)}{\partial t} = \frac{\partial F(x,t)}{\partial x} \circ G(x),$$

a neighbourhood of $(x_0, 0) \in X \times \mathbf{K}$ *if and only if there exists a differentiable function* $\varphi : V \subseteq X \to X$, *where V is a neighbourhood of $f(x_0)$ such that*

$$F(x,t) = \varphi(f(x) + t\mathbf{a}). \tag{12}$$

b) *This function F satisfies the initial condition* (6) *if and only if* $\varphi = f^{-1}$, *i.e., if*

$$F(x,t) = f^{-1}(f(x) + t\mathbf{a}). \tag{13}$$

In this case F satisfies also the differential-initial condition (4) *and equation* (5).

c) *The solution* (12) *of equation* (1) *satisfies the condition* (4) *if and only if there exists a function* $\chi : \pi_{\mathfrak{X}}(f(U)) \to X$ *such that*

$$F(x,t) = f^{-1}(f(x) + t\mathbf{a}) + \chi \circ \pi_{\mathfrak{X}} \circ f(x). \tag{14}$$

In general, the function (14) *is not a solution of equation* (5).

Proof: a) An easy computation, using (7), shows that every function F, defined by (12), is a solution of equation (1). We will prove that every solution F of equation (1) is of the form (12). For this, one could use existence and uniqueness theorems in the sense of Cauchy. But these are hard to find in the standard literature. Thus we may proceed as follows. We have

$$\frac{\partial F(f^{-1}(y),t)}{\partial y}(\mathbf{a}) = \frac{\partial F(f^{-1}(y),t)}{\partial x} \circ (f'(f^{-1}(y)))^{-1}(\mathbf{a})$$
$$= \frac{\partial F(f^{-1}(y),t)}{\partial x} \circ G(f^{-1}(y)) = \frac{\partial F(f^{-1}(y),t)}{\partial t}.$$

Hence we can apply Lemma 2, that is, there exists a differentiable function φ such that $F(f^{-1}(y),t) = \varphi(y + t\mathbf{a})$. This yields to the representation (12) of $F(x,t)$.

b) $F(x,0) = x$ implies $\varphi(f(x)) = x$ hence $\varphi = f^{-1}$. An easy computation shows that the function (13) satisfies also condition (4) and equation (5).

c) With respect to (12) we get $\partial F(x,t)/\partial t = \varphi'(f(x) + t\mathbf{a})(\mathbf{a})$. Hence condition (4) implies, in view of (7), $\varphi'(f(x))(\mathbf{a}) = (f'(x))^{-1}(\mathbf{a})$. Taking $y = f(x)$, i.e. $x = f^{-1}(y)$, we get $\varphi'(y)(\mathbf{a}) = f'(f^{-1}(y))^{-1}(\mathbf{a})$. This is $\varphi'(y)(\mathbf{a}) = (f(y)^{-1})'(\mathbf{a})$. Therefore $\varphi(y) - f^{-1}(y)$ is a solution of equation (10), which yields $\varphi(y) = f^{-1}(y) + \chi \circ \pi_{\mathfrak{X}}(y)$ due to (11). In this way we get the representation (14) ∎

Theorem 2: *The following statements hold.*

a) *A function F is a solution of the second Jabotinsky equation* (2),

$$\frac{\partial F(x,t)}{\partial t} = G(F(x,t)),$$

in a neighbourhood of $(x_0, 0) \in X \times \mathbb{K}$ *if and only if there exists a function* $k : U' \subseteq X \to X$ *(where* U' *is a neighbourhood of* x_0 *and* $k(U')$ *is contained in the domain of definition of* f*) such that*

$$F(x,t) = f^{-1}(k(x) + t\mathbf{a}). \tag{15}$$

b) *This function F satisfies the condition* (6) *if and only if* $k = f$, *i.e., if*

$$F(x,t) = f^{-1}(f(x) + t\mathbf{a}). \tag{16}$$

In this case the function (16) *is also a solution of the equations* (4) *and* (5).

c) *The solution* (15) *of equation* (2) *satisfies the condition* (4), *if and only if*

$$G(f^{-1}(k(x))) = G(x). \tag{17}$$

Thus, if G is injective, we have $k = f$, *hence*

$$F(x,t) = f^{-1}(f(x) + t\mathbf{a}). \tag{18}$$

In general, the function F even if it satisfies (17), *is not a solution of equation* (5).

Proof: a) Every function given by (15) is a solution of equation (2). Since G is supposed to be continuously differentiable, the Cauchy problem for equation (2) (see Dieudonné [7]), $F(x,t)|_{t=0} = \psi(x)$, for any given function ψ, has a unique solution which is given by (15), taking $k = f \circ \psi$.

Another way to prove that every solution of equation (2) has the form (15) is the following. Let F be a solution of equation (2). Then from (2) and property (7) there follows that $\partial F(x,t)/\partial t = f'(F(x,t))^{-1}(\mathbf{a})$. Thus $f'(F(x,t)) \circ \partial F(x,t)/\partial t = \mathbf{a}$ or $\partial f(F(x,t))/\partial t = \mathbf{a}$. This yields $f(F(x,t)) = t\mathbf{a} + k$, where k is depending on x, say $k = k(x)$. From this we get the stated representation of the solution of equation (2).

b) The initial condition (6) implies $f^{-1}(k(x)) = x$ hence $k(x) = f(x)$. The function (16) satisfies also condition (4) and equation (5).

c) In view of (15) we get $\partial F(x,t)/\partial t = f'(f^{-1}(k(x) + t\mathbf{a}))^{-1}(\mathbf{a})$, hence, by condition (4), $f'(f^{-1}(k(x)))^{-1}(\mathbf{a}) = f'(x)^{-1}(\mathbf{a})$. Thus, in view of (7), $G(f^{-1}(k(x))) = G(x)$ ∎

Theorem 3: *The following statements hold.*

a) *A function F is a solution of the third Jabotinsky equation* (3),

$$\frac{\partial F(x,t)}{\partial x} \circ G(x) = G(F(x,t))$$

in a neighbourhood of $(x_0,0) \in X \times \mathbb{K}$ if and only if there exists a function $\Lambda: \pi_{\mathfrak{X}} \circ f(U) \times \mathbb{K} \to X$, differentiable in its first variable, such that $f(x) + \Lambda(\pi_{\mathfrak{X}} \circ f(x), t)$ lies in the domain of definition of f^{-1}, and F has the form

$$F(x,t) = f^{-1}(f(x) + \Lambda(\pi_{\mathfrak{X}} \circ f(x), t)). \tag{19}$$

b) *This function F satisfies condition* (6) *if and only if $\Lambda|_{t=0} = 0$. But generally F, even with condition* (6), *is not a solution of equation* (5).

c) *The solution* (19) *of equation* (3) *satisfies both conditions* (4) *and* (6) *if and only if Λ is differentiable with respect to t, at $t = 0$, and $\Lambda|_{t=0} = 0$ and $(\partial\Lambda/\partial t)|_{t=0} = \mathbf{a}$. Even in the case that both conditions* (4) *and* (6) *are satisfied the solution F of equation* (3) *is in general not a solution of equation* (5).

Proof: a) Let F be given by (19). Thus

$$\begin{aligned}\frac{\partial \Lambda(\pi_{\mathfrak{X}} \circ f(x),t)}{\partial x} \circ G(x) &= \Lambda'(\pi_{\mathfrak{X}} \circ f(x),t) \circ \pi'_{\mathfrak{X}}(f(x)) \circ f'(x) \circ G(x)\\ &= \Lambda'(\pi_{\mathfrak{X}} \circ f(x),t) \circ \pi_{\mathfrak{X}} \circ f'(x) \circ G(x)\\ &= \Lambda'(\pi_{\mathfrak{X}} \circ f(x),t) \circ \pi_{\mathfrak{X}} \circ f'(x) \circ (f'(x))^{-1}(\mathbf{a})\\ &= \Lambda'(\pi_{\mathfrak{X}} \circ f(x),t) \circ \pi_{\mathfrak{X}}(\mathbf{a}) = 0.\end{aligned}$$

Now

$$\begin{aligned}\frac{\partial F(x,t)}{\partial x} \circ G(x) &= \Big(f'\big((f^{-1}(f(x) + \Lambda(\pi_{\mathfrak{X}} \circ f(x),t))\big)\Big)^{-1} \circ \Big(f'(x) + \frac{d\Lambda(\pi_{\mathfrak{X}} \circ f(x),t)}{dx}\Big) \circ G(x)\\ &= \big(f'(F(x,t))\big)^{-1} \circ f'(x) \circ (f'(x))^{-1}(\mathbf{a})\\ &= \big(f'(F(x,t))\big)^{-1}(\mathbf{a}) = G(F(x,t)).\end{aligned}$$

Therefore every function F given by (19) satisfies equation (3). Conversly, given any solution F of equation (3). To show that this solution has to be of the form (19), one cannot apply the standard existence and uniqueness theorems for the Cauchy problem, since it may be happen that (3) is a differential equation of singular type. One may proceed as follows. Equation (3) yields with respect to relation (7)

$$\frac{\partial F(x,t)}{\partial x} \circ (f'(x))^{-1}(\mathbf{a}) = \left(f'(F(x,t))\right)^{-1}(\mathbf{a}).$$

Putting $x = f^{-1}(y)$ and multiplying by $f'(F(x,t))$, one gets

$$f'(F(f^{-1}(y),t)) \circ \frac{\partial F(f^{-1}(y),t))}{\partial x} \circ (f'(f^{-1}(y)))^{-1}(\mathbf{a}) = \mathbf{a}.$$

Since $(f'(f^{-1}(y)))^{-1} = df^{-1}(y)/dy$, the chain rule yields $\left(\partial f(F(f^{-1}(y),t))/\partial y\right)(\mathbf{a}) = \mathbf{a}$. Therefore, $H(y) = y$ being a particular solution of $(\partial H(y)/\partial y)(\mathbf{a}) = \mathbf{a}$, Lemma 3 yields $f(F(f^{-1}(y),t)) = y + \Lambda(\pi_{\mathfrak{X}}(y),t)$, where the function Λ depends also upon the parameter t. Therefore $F(f^{-1}(y),t) = f^{-1}(y + \Lambda(\pi_{\mathfrak{X}}(y),t))$, and the substitution $y = f(x)$ gives the representation (19).

b) The initial condition (6) implies $f(x) + \Lambda(\pi_{\mathfrak{X}} \circ f(x),0) = f(x)$, hence $\Lambda(\pi_{\mathfrak{X}} \circ f(x),0) = 0$. But f being invertible and $\pi_{\mathfrak{X}}$ surjective yield $\Lambda|_{t=0} \equiv \mathbf{0}$.

c) Condition (4) yields

$$\left[f'\left(f^{-1}\left(f(x) + \Lambda(\pi_{\mathfrak{X}} \circ f(x),t)\right)\right)\right]^{-1} \circ \left.\frac{\partial \Lambda(\pi_{\mathfrak{X}} \circ f(x),t)}{\partial t}\right|_{t=0} = (f'(x))^{-1}(\mathbf{a}).$$

Since $\Lambda(\pi_{\mathfrak{X}} \circ f(x),t)|_{t=0} = \mathbf{0}$, one gets due to condition (6)

$$(f'(x))^{-1} \circ \left.\frac{\partial \Lambda(\pi_{\mathfrak{X}} \circ f(x),t)}{\partial t}\right|_{t=0} = (f'(x))^{-1}(\mathbf{a}),$$

thus $\left(\partial \Lambda(\pi_{\mathfrak{X}} \circ f(x),t)/\partial t\right)\Big|_{t=0} = \mathbf{a}$. Again, f invertible and $\pi_{\mathfrak{X}}$ surjective yields $(\partial\Lambda/\partial t)|_{t=0} = \mathbf{a}$. Examples of solutions of (3),(4) and (6) which are not solutions of (5) were given in Aczél and Gronau [2] ∎

Next we will take a second look at an example of the third Jabotinsky equation, which has been treated in Aczél and Gronau [2], with regard to "commuting mappings". In connection with this subject we refer also to Reich [14]. A family $\{F_t\}$ of mappings $F_t(x) = F(x,t)$ is said to be *commuting*, if it satisfies the equation

$$F(F(x,s),t) = F(F(x,t),s) \tag{20}$$

for all parameters s and t. As one can easily see, equation (3) together with initial condition (4) can be obtained from equation (20) and condition (6) by differentiating (20) with respect to s, putting $s = 0$ afterwards. Of course, every solution of equation (5) satisfies (20). The question in Aczél and Gronau [2] was, whether a function satisfying equation (3) with condition (4) and equation (6) would also fulfil equation (20). As a counter-example in $\mathbb{R}^n$ we there gave $G(x) = x$ and $F(x,t) = A(t)x$, where $A(t)$ is an $n \times n$ matrix with functions of t, satisfying at least for one pair (s,t) the inequality $A(s)A(t) \neq A(t)A(s)$.

Example: Let $G(x) = x$. We may take $\mathbf{a} = {}^t(1,\dots,1)$ and $U = \{x \in \mathbb{R}^n | x_i > 0 \text{ for } i = 1,\dots,n\}$, hence $f(x) = {}^t(\ln x_1,\dots,\ln x_n)$ and $f^{-1}(x) = {}^t(\exp x_1,\dots,\exp x_n)$. Every solution of equation (3) has, in view of (19) and Remark 2, a representation of the form

$$F(x,t) = \left(\exp(\ln x_i + \Lambda_i(\ln x_1 - \ln x_n,\dots,\ln x_{n-1} - \ln x_n,t))\right), \tag{21}$$

where Λ_i is the i-th component of Λ.

a) If Λ is independent of the y_i's, for example $\Lambda_i = \ln g_i(t)$, where the g_i's are arbitrary functions, one gets

$$F(x,t) = \big(g_i(t)x_i\big)_{i=1,\dots,n} = \mathrm{diag}\big(g_1(t),\dots,g_n(t)\big)x.$$

This function satisfies equation (20) even if the g_i's are chosen in that way that F does not satisfy (4) and (6). Generally, one can say that a solution (19) of equation (3) satisfies equation (20) if (but not only if) the function Λ is independent of the first $n-1$ arguments.

b) Choosing $\Lambda = (\Lambda_i)_{i=1,\dots,n}$ as

$$\Lambda_i(y_1,\dots,y_{n-1},t) = \ln\Big(\sum_{k=1}^{n-1} a_{ik}(t)\exp y_k + a_{in}(t) - y_i\Big) \quad \text{for } i = 1,\dots,n-1$$

and

$$\Lambda_n(y_1,\dots,y_{n-1},t) = \ln\Big(\sum_{k=1}^{n-1} a_{nk}(t)\exp y_k + a_{nn}(t)\Big)$$

one gets from (21)

$$F_i(x,t) = x_i\Big(\sum_{k=1}^{n-1} a_{ik}(t)x_k/x_n + a_{in}(t)\Big)\frac{x_n}{x_i} = \sum_{k=1}^{n-1} a_{ik}(t)x_k \quad \text{for } i = 1,\dots,n-1$$

and

$$F_n(x,t) = x_n\Big(\sum_{k=1}^{n-1} a_{nk}(t)x_k/x_n + a_{nn}(t)\Big),$$

hence $F_i(x,t) = \sum_{k=1}^{n-1} a_{ik}(t)x_k$ for $i = 1,\dots,n$, that is $F(x,t) = A(t)x$, where $A = (a_{ij})$ is an $n \times n$ matrix of functions a_{ij}. If one choses $A(0) = E$ and $(dA(t)/dt)|_{t=0} = E$ (the identity matrix) but $A(t)A(s) \neq A(s)A(t)$, then this yields a solution of equation (3) together with conditions (4) and (6), but this solution does not commute.

As a further statement we can give the following remark.

Remark 3 (on the uniqueness): **a)** The supposition of continuous differentiability of the function G in Lemma 1 is a sufficient condition to guarantee the existence of a C^1-diffeomorphism f, such that (7) holds. For all what follows to Lemma 1, only the existence of an invertible function f is necessary, where f and f^{-1} are differentiable and (7) holds. So, it is easy to see that the following statement on existence and uniqueness in the sense of Cauchy holds for equations (1) and (2).

Corollary: *Suppose that $f: U_0 \to f(U_0) \subseteq X$ (U_0 open connected in X) is invertible, f and f^{-1} are differentiable and (7) holds for $x \in U_0$. Then, equation (1) (resp. equation (2)) admits one and only one differentiable solution F which satisfies the initial condition $F(x,0) = \psi(x)$ for any given (in case of equation (1) differentiable) function ψ.*

b) For equation (3) a uniqueness condition in the sense of Cauchy cannot be given. This means that a solution F of equation (3) is not uniquely determined by the value $F(x,0)$, and /or, for a fixed $x_0 \in U_0$, by the value $F(x_0,t)$. But we can state that for a given solution F of equation (3), the function Λ: $\pi_{\mathfrak{X}} \circ f(U) \times X$ in the representation (19) is uniquely determined by this solution.

It should be pointed out that the above representation theorems may give some insight in the structure of the analytic iteration problem (as mentioned in the introduction). Especially equation (3) can be of advantage, since in this case no regularity of the solution F with respect to t is required. We give here an example for the one-dimensional real or complex case.

Example 2: Let X be $\mathbb{R}$ or $\mathbb{C}$. In this case the solution F of equation (3) is given by

$$F(x,t) = f^{-1}(f(x) + \Lambda(t)). \tag{22}$$

where the real or complex function Λ is only depending on t. The so defined function F is a solution of equation (5), if and only if Λ is additive, that is $\Lambda(s+t) = \Lambda(s) + \Lambda(t)$. The solution F of equation (5), given by (22), is continuous in t too, if and only if the one involved additive function Λ is continuous, that is, a linear function of the form $\Lambda(t) = \lambda t$, where λ is an arbitrary non-zero scalar.

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