

# On Structure Preserving Transformations in Hamiltonian Control Systems

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Hamiltonian control systems with natural output in the sense of [12] are considered. We investigate state-dependent co-ordinate transformations in the space of controls  $u$  and observables  $y$  which preserve the Hamiltonian structure. These transformations can be characterized and constructed by canonical transformations in the  $(y, u)$ -space. The results generalize known statements on Hamiltonian systems affine in the control and hold for general gradient-like control systems as well.

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## 0. Introduction

A good deal of problems encountered in current control theory belong to "systems shaping", that is, to gain a desired behaviour (like stability or decoupled disturbances) of a system by appropriately manipulating its structure. Most of the underlying concepts and methods come from linear control theory and they appear as more or less straightforward generalizations to nonlinear systems. Feedback transformations in the sense of state-dependent control transformations are at the heart of this set of problems. There are attempts to gain as large a nonlinear generalization as possible, but on the other hand very often it seems more promising to confine attention to some class of nonlinear systems which are characterized by certain structural peculiarities (analytical and/or geometrical) and then to utilize these peculiarities in solving a shaping problem. Preferred classes under current consideration are, e. g., bilinear systems, polynomial systems, systems affine in the control, Hamiltonian systems. Within the framework of geometric control theory any system is specified by certain geometric ingredients such as state space, a family of vector fields describing the dynamics, an output mapping. In a sense, each system is considered as a geometric object per se. In order to stay within a class of systems every transformation (interpreted as a mapping or as some change of co-ordinates in local description of geometric objects) applied to any system is subject to the restriction to preserve that structure which characterizes the class of systems under consideration.

In [11,12], van der Schaft characterizes feedback transformations which act in the class of affine Hamiltonian control systems. The primary aim of the present paper is to generalize his results to the class of general Hamiltonian control systems.

It is well known that Hamiltonian systems are a corner-stone of theoretical physics and, in particular, of analytical mechanics. They entered control theory in the context of Pontryagin's theory of optimal control (central part of necessary optimality conditions). Then, initiated by Brockett's philosophy [2] in 1977, Hamiltonian control systems, forming a distinct subclass of input-state-output systems, have gained an increasing interest in diverse control-theoretic investigations. We just mention as outstanding contributions

(i) van der Schaft's 1984 monograph [12], which treats Hamiltonian systems in a system-theoretic setting, thus coming back to the historically original notion of Hamiltonian systems with external forces, and investigates typical system-theoretic properties such as controllability, and

(ii) the 1987 monograph [4] by Crouch and van der Schaft, where Hamiltonian control systems are distinguished within the set of all control systems in close connection with the inverse problem of variational calculus (see Santilli [8]) and Jakubczyk's realization theory [5].

Disregarding here more general definitions in a fiber bundle setting, van der Schaft's introduction of Hamiltonian control systems can be sketched as follows. In the context of affine (with respect to control) input-state-output systems

$$\dot{x} = g^0(x) + \sum_{\mu=1}^m u_{\mu} g^{\mu}(x), \quad y = h(x), \quad m \leq n, \quad (1)$$

where  $g^0, g^{\mu}$  are given vector fields on a smooth  $n$ -dimensional manifold  $M$  (state space),  $u_{\mu}$  are  $\mathcal{R}$ -valued inputs, and  $h: M \rightarrow Y$  is an output mapping to some manifold  $Y$ ,  $y$  can be observed. Then  $M$  is supposed to be a symplectic manifold (e.g., of even dimension  $2n$ ) and  $g^0$  to be the Hamiltonian vector field generated by a Hamiltonian  $H^0: M \rightarrow \mathcal{R}, g^0 = X_{H^0}$ . Let  $Y = \mathcal{R}^m$ , let  $h = (H^1, \dots, H^m)$  be submersive (that is,  $h(M)$  is an open domain in  $Y$  or, equivalently,  $dh$  has maximal rank,  $m$ , everywhere). Finally, let  $g^{\mu} = -X_{H^{\mu}}$  (Hamiltonian vector fields generated by the output functions  $H^{\mu}$ ). Then (1) has the form

$$\dot{x} = X_{H^0}(x) - \sum_{\mu=1}^m u_{\mu} X_{H^{\mu}}(x), \quad y^{\mu} = H^{\mu}(x) \quad (2)$$

and is called an *affine Hamiltonian control system with natural output*. This is a straight generalization of the classical Hamiltonian system with external forces, written in canonical coordinates  $x = (q^1, \dots, q^n, p_1, \dots, p_n)$ :

$$\begin{aligned} \dot{q}^{\rho} &= H_{,p_{\rho}}^0(q, p), \quad \rho = 1, \dots, n \\ \dot{p}_{\rho} &= -H_{,q^{\rho}}^0(q, p) + \sum_{\mu=1}^m u_{\mu} \delta_{\rho}^{\mu} \\ y^{\mu} &= -q^{\mu}, \quad \mu = 1, \dots, m \end{aligned}$$

(a comma denotes partial derivatives). Here the state space is  $M = T^*Q$ , the cotangent bundle of the configuration manifold  $Q$ ,  $q^1, \dots, q^m$  are observed,  $u_1, \dots, u_m$  are the generalized forces considered as controls. Note that, with

$$H: M \times U \rightarrow \mathcal{R}, \quad H(x, u) = H^0(x) - \sum_{\mu=1}^m u_{\mu} H^{\mu}(x), \quad U = \mathcal{R}^m,$$

equation (2) becomes

$$\dot{x} = X_H(x, u), \quad y^{\mu} = -H_{,u_{\mu}}(x, u). \quad (2')$$

This can be generalized to a *general Hamiltonian control system with natural output* if the affine

dependence of  $H$  on the controls is dropped, that is, if  $H$  is taken as any (smooth) function  $M \times U \rightarrow \mathcal{R}$ .

### 1. Basic concepts

In what follows, Hamiltonian control systems will be considered. The aim is to describe transformations which, in a precise sense, preserve the structure of these systems. Since these transformations are to be seen as co-ordinate transformations in respective manifolds, all investigations will be local and are executed by means of classical tensor calculus as an appropriate tool.

Convention: Latin indices run from 1 to  $2n$ , Greek indices from 1 to  $m \leq 2n$ , respective summation conventions are accepted throughout. All geometric objects, functions etc. are supposed to be of class  $C^\infty$ . Notations like  $f(x)$  instead of  $f : x \rightarrow f(x)$  are sometimes preferred for the sake of brevity.

Keeping this in mind, a symplectic manifold  $M$  can be seen as (an open domain of)  $\mathcal{R}^{2n}$  endowed with a tensor field  $\Omega^{ij}$  which is

$$\text{regular} : \det(\Omega^{ij}(x)) \neq 0, x \in M \tag{3}_a$$

$$\text{skew} : \Omega^{ij} = -\Omega^{ji} \tag{3}_b$$

$$\text{Jacobi} : \Omega^{ijk} := \Omega^{ij}_{,r} \Omega^{rk} + \Omega^{ki}_{,r} \Omega^{rj} + \Omega^{jk}_{,r} \Omega^{ri} = 0. \tag{3}_c$$

Here  $x = (x^1, \dots, x^{2n})$  denotes arbitrary co-ordinates in  $M$ . There are *canonical co-ordinate systems* (mutually connected by *canonical transformations*, see Section 3.2) with respect to which

$$\Omega^{ij}(x) = \omega^{ij}, (\omega^{ij}) = \begin{pmatrix} O_n & I_n \\ -I_n & O_n \end{pmatrix}. \tag{4}$$

Any function  $H : M \rightarrow \mathcal{R}$  generates a Hamiltonian vector field  $X = X_H$  having co-ordinates  $X^i = \Omega^{ij} H_{,j}$ .

**Definition 1:** Let  $H$  be a given function  $M \times U \rightarrow \mathcal{R}$ , with open domains  $M \subset \mathcal{R}^{2n}$  and  $U \subset \mathcal{R}^m$ ,  $m \leq 2n$ , such that  $\text{rank}(H_{,u_\mu x^i}(x, u)) = m$  everywhere (that is,  $(H_{,u_\mu}(\cdot, u)) : M \rightarrow Y = \mathcal{R}^m$  is submersive), and let  $\Omega^{ij}$  describe a symplectic structure on  $M$ . Then

$$\dot{x}^i = \Omega^{ij} H_{,x^j}(x, u), y^\mu = -H_{,u_\mu}(x, u) \tag{5}_{a,b}$$

is called a *Hamiltonian control system* with state  $x$ , input  $u$  and output  $y$ . If, in particular,

$$H(x, u) = H^0(x) - u_\mu H^\mu(x), \tag{6}$$

then (5) is called an *affine Hamiltonian control system*.

Now (5) is in a form which is invariant with respect to arbitrary state transformations  $x^{i'} = \xi^{i'}(x)$ . Thus, primary interest is in co-ordinate transformations in  $U$  and/or  $Y$ , which may depend on the state  $x$ .

**Definition 2:** A state-dependent co-ordinate transformation in the control space  $U$ ,

$$u \rightarrow v : v_\mu = \bar{\alpha}_\mu(x, u) \quad \text{or} \quad u_\mu = \alpha_\mu(x, v) \quad (7)_{a,b}$$

is called a *feedback transformation*.

To be correct, the proper feedback transformation is (7)<sub>b</sub> :  $u$  is determined by a new control  $v$  while feeding back the state  $x$  to the input.

Due to the invertibility of  $\bar{\alpha}(x, \cdot)$ , the Jacobians  $(\bar{\alpha}_{\mu, u_\lambda})$  and  $(\alpha_{\mu, v_\lambda})$  are of maximal rank,  $m$ , everywhere. Obviously, if  $\alpha(\cdot, \cdot)$  is chosen arbitrarily, then  $H_{,x}(\cdot, \alpha(\cdot, v))$  is not a partial derivative anymore. Thus, in general, a feedback transformation will destroy the Hamiltonian structure of a control system.

**Definition 3:** A feedback transformation (7) is called *structure preserving for the Hamiltonian control system* (5), if the system with  $u$  replaced by  $\alpha(x, v)$  is again a Hamiltonian control system with the same symplectic structure; that is, if there exists a Hamiltonian  $K(x, v)$  such that

$$\Omega^{ij}(x) H_{,x^j}(x, \alpha(x, v)) = \Omega^{ij}(x) K_{,x^j}(x, v) \quad (8)$$

and the transformed system then is, with appropriate new observables  $z^\mu$ ,

$$\dot{x}^i = \Omega^{ij}(x) K_{,x^j}(x, v), \quad z^\mu = -K_{,v_\mu}(x, v). \quad (9)$$

The two Hamiltonian control systems (5) and (9) are called *feedback equivalent*.

**Remarks:** 1. The change of observables  $y \rightarrow z$  is intrinsically determined, this will become clear in Section 3.2. 2. In a larger sense  $\alpha$  could be called *Hamiltonian structure preserving* if  $\Omega^{ij}$  is allowed to change, that is, if there is a pair  $(\tilde{\Omega}^{ij}, K)$  such that  $\tilde{\Omega}^{ij}$  satisfies (3) and

$$\Omega^{ij}(x) H_{,x^j}(x, \alpha(x, v)) = \tilde{\Omega}^{ij}(x) K_{,x^j}(x, v).$$

To characterize such feedback transformations is still an open problem. Simple examples are considered in [9].

Since  $\Omega^{ij}$  is non-singular, it drops from (8), and  $\alpha$ , in the sense of Definition 3, is characterized by

$$H_{,x^j}(x, \alpha(x, v)) = K_{,x^j}(x, v) \quad \text{for some} \quad K(x, v). \quad (8)'$$

Thus, any conclusions concerning  $\alpha$  do not depend on the underlying symplectic structure, they are valid for any non-singular tensor  $\Omega^{ij}$  (on a manifold of any finite dimension) and cover, e. g., also gradient systems ( $\Omega^{ij}$  symmetric and positive definite). Structure preservation by  $\alpha$  is simply a matter of  $H$ . Its geometric meaning is therefore that: Given a family of exact 1-forms,  $dH(\cdot, u)$ ,  $\alpha$  has to be such that  $dH(\cdot, \alpha(\cdot, v))$  is a family of exact 1-forms again. The characterization (8)' follows immediately (integrability conditions, existence of  $K$ ).

**Lemma 1:** *If  $M$  is simply connected, then  $\alpha$  is structure preserving for  $H$  iff, for every  $v$ ,  $\alpha$  solves the quasilinear partial differential equations*

$$0 = H_{,x^i v_\mu}(x, \alpha(x, v)) \alpha_{\mu, x^j}(x, v) - H_{,x^j v_\mu}(x, \alpha(x, v)) \alpha_{\mu, x^i}(x, v). \quad (10)$$

Using this lemma, it is simple calculation to check the structure preservation by a given  $\alpha$ . In the affine case (6) it yields an explicit representation of such  $\alpha$  (see [1]), the general case will be treated in Section 3.

## 2. Output transformations in affine Hamiltonian control systems

It is a remarkable property of affine Hamiltonian control systems (2) that the input channels  $-X_{H^\mu}$  are determined by the output functions  $H^\mu$ . This strong interconnection has certain consequences.

Even for practical reasons it could be desirable to pass over from the observed quantities  $y^\mu$  to some others,  $z^\mu$ . (As an example one could think of a torque-controlled mathematical pendulum and wish to observe the horizontal elongation instead of the angle.) This then leads to the consideration of a co-ordinate transformation in the output space  $Y$ ,

$$T_Y : z^\mu = \varphi^\mu(y^1, \dots, y^m).$$

Thus, a change of the output functions is implied,

$$H^\mu(\cdot) \rightarrow K^\mu(\cdot) := \varphi^\mu(H^1(\cdot), \dots, H^m(\cdot)).$$

Structure preservation by  $T_Y$  (that is: the new output equations,  $z^\mu = K^\mu(x)$ , have to be part of an affine Hamiltonian control system) then requires a respective change of the input channels,  $X_{H^\mu} \rightarrow X_{K^\mu}$ , while the dynamics remains unchanged (same orbits). So for any  $x$  and any  $u$  there is a  $v = (v_1, \dots, v_m)$  such that

$$X_{H^0}(x) - u_\mu X_{H^\mu}(x) \doteq X_{H^0}(x) - v_\lambda X_{K^\lambda}(x).$$

Here  $X_{H^0}$  cancels out, then, writing the Hamiltonian vector fields co-ordinate-wise,  $\Omega^{ij}$  drops because of its non-singularity, and  $K^\lambda_{,i} = \varphi^\lambda_{,y^\mu} H^\mu_{,x^i}$  finally yields  $(u_\mu - v_\lambda \varphi^\lambda_{,y^\mu}) H^\mu_{,x^i} = 0$ . Since by Definition 1 the output functions  $H^\mu$  are independent, a transformation in  $U$ ,  $T_U : u_\mu = v_\lambda \varphi^\lambda_{,y^\mu}(y)$  follows.

**Observations:** *The above considerations give rise to the following statements.*

(i) *The co-ordinate transformation  $T_Y$  implies a linear transformation,  $T_U$ , of the input co-ordinates. Restricting  $y^\mu$  to the output values  $H^\mu(x)$ ,  $T_U$  becomes a linear output-feedback transformation (see Definition 2, dependence on state via output values).*

(ii) *The pair  $(T_Y, T_U)$  forms an extended point transformation on  $Y \times U$ , that is , a special canonical transformation ( see [3]). In particular, caused by  $T_Y$ ,  $u$  transforms like a covector. By this fact the two (up to now separate) spaces  $Y$  and  $U$  must be considered as constituting the external space  $W = T^*Y$  ( cotangent bundle of the output space) which in a natural way carries a symplectic structure (see [6, 12]),  $(y^\mu, u_\mu)$  could be seen as canonical co-ordinates with respect to that structure in  $W$ .*

**Remark:** This stringent cotangent bundle structure of the external space (with its natural symplectic structure) depends neither on the symplecticity of the state space nor on the Hamiltonian drift field  $X_{H^0}$ . It is solely a consequence of the fact that the input channels are in one-to-one linear correspondence with the output functions. The representation  $W = T^*Y$  is not new, it can be found in van der Schaft's papers, but there it appears simply as a convenient and "most natural" interpretation [14, 15] or it is a priori part of the definition of a Hamiltonian control system [11, 12].

It is interesting to see what happens if the output transformation is allowed to depend on the state:

$$T_Y : z^\mu = \varphi^\mu(y; x).$$

The implied change of output functions is

$$H^\mu(\cdot) \rightarrow K^\mu(\cdot) = \varphi^\mu(H^1(\cdot), \dots, H^m(\cdot); \cdot) \tag{11}$$

and, by the same reasoning as above, it follows that  $T_Y$  is structure preserving iff for each  $(x, u)$  there exists a  $v$  such that

$$0 = u_\mu H_{,x^\mu}^\mu(x) - v_\lambda K_{,x^\lambda}^\lambda(x). \tag{12}$$

This will specify the functions  $\varphi^\mu(y; \cdot)$ . The auxiliary transformation in  $M$ ,

$$x^i \rightarrow s^i : s^\mu = H^\mu(x) (\mu = 1, \dots, m) \quad \text{and} \quad s^{\bar{\mu}} = x^{\bar{\mu}} (\bar{\mu} = m + 1, \dots, 2n)$$

brings in some simplifications. Let  $\psi^\mu(y; s) := \varphi^\mu(y; x(s))$ , then (11) yields  $\Psi^\mu(s^\kappa, s^{\bar{\kappa}}) := \psi^\mu(s^\kappa; s) = K^\mu(x(s))$  and (12), written in the new co-ordinates  $s$ , splits into

$$0 = u_\mu - v_\lambda \Psi_{,s^\mu}^\lambda, \quad 0 = -v_\lambda \Psi_{,s^{\bar{\mu}}}^\lambda. \tag{13}_{a,b}$$

(i) Let  $T_Y$  be structure preserving. Then the linear equations (13)<sub>a</sub> have a solution  $v_\lambda$  at any  $u_\mu$ , hence  $(\Psi_{,s^\mu}^\lambda)$  is a non-singular matrix with, say,  $(\tilde{\Psi}_\lambda^\mu)$  as its inverse. Thus,  $v_\lambda = \tilde{\Psi}_\lambda^\mu u_\mu$  follows. The second part (13)<sub>b</sub> then writes  $0 = -u_\mu \tilde{\Psi}_\lambda^\mu \Psi_{,s^{\bar{\mu}}}^\lambda$  and, since this holds for arbitrary  $u_\mu$ , it yields  $0 = \Psi_{,s^{\bar{\mu}}}^\lambda$ . Thus  $\Psi^\lambda$  does not depend on  $s^{\bar{\mu}}$ , which means

$$\psi^\mu(s^\kappa; s) = \Psi^\mu(s^\kappa). \tag{14}$$

Now fix an arbitrary  $y \in Y$ . Then  $M_y = \{s | s^\mu = y^\mu\}$  is a  $(2n - m)$ -dimensional submanifold of  $M$ . Owing to (14),  $\psi^\mu(y; \cdot)$  is constant (with value  $\Psi^\mu(y)$ ) on  $M_y$ , therefore it does not depend on  $s^{m+1}, \dots, s^{2n}$ . Thus  $\psi^\mu(y; s) = \Phi^\mu(y; s^1, \dots, s^m)$  with some function  $\Phi^\mu$  and, finally,

$$\varphi^\mu(y; x) = \Phi^\mu(y; H^1(x), \dots, H^m(x)) \tag{15}$$

is necessary for  $T_Y$  to be structure preserving. Mind that  $\Phi^\mu(s^\kappa; s^\kappa) = \Psi^\mu(s^\kappa)$ , and  $\Psi^\mu$  has a non-singular Jacobian.

(ii) Let  $\varphi^\mu$  have the form given by (15). Then the new output functions are  $K^\mu(\cdot) = \Phi^\mu(H^\kappa(\cdot); H^\kappa(\cdot)) = \Psi^\mu(H^\kappa(\cdot))$  yielding  $K_{,x^\lambda}^\lambda = \Psi_{,s^\mu}^\lambda H_{,x^\lambda}^\mu$ . Therefore (12) will be satisfied by  $u_\mu = \Psi_{,s^\mu}^\lambda v_\lambda$  or  $v_\lambda = \tilde{\Psi}_\lambda^\mu u_\mu$ , respectively. Hence (15) also suffices for  $T_Y$  to preserve structure.

**Theorem 1:** A state dependent output transformation  $T_Y : z^\mu = \varphi^\mu(y; x)$  preserves the structure of the affine Hamiltonian control system

$$\dot{x} = X_{H^0}(x) - u_\mu X_{H^\mu}(x), \quad y^\mu = H^\mu(x), \quad \text{rank}(H_{,x^\mu}^\mu) = m \quad \text{everywhere,}$$

if and only if

(i)  $\varphi^\mu(y; \cdot) = \Phi^\mu(y; H^1(\cdot), \dots, H^m(\cdot))$  with some functions  $\Phi^\mu$  such that  $\Psi^\mu, \Psi^\mu(y) = \Phi^\mu(y; y)$  have non-singular Jacobians, and

(ii)  $T_Y$  is accompanied with the output-feedback transformation  $T_U : u_\mu = \Psi_{,y^\mu}^\lambda(H^1(x), \dots, H^m(x)) v_\lambda$ .

Then  $(T_Y, T_U)$  takes the system to

$$\dot{x} = X_{H^0}(x) - v_\lambda X_{K^\lambda}(x), \quad z^\lambda = K^\lambda(x) := \Psi^\lambda(H^1(x), \dots, H^m(x)).$$

**Remark:** Typically, in all transformation formulas the state  $x$  appears "packet-wise" by  $H^1(x), \dots, H^m(x)$ . This fact will again be encountered in the following.

### 3. Input transformations in general Hamiltonian control systems

In this section Hamiltonian control systems in the sense of Definition 1 are considered and their structure preserving feedback transformations (Definition 3) are characterized in a way which allows for construction of such transformations. Clearly, this characterization will be independent of the underlying symplectic structure of  $M$  (remind Lemma 1) but equally it does not depend on the particular form  $-H_{,u}$  of the natural output functions. In order to put this latter fact into evidence, systems with no output will be considered first.

#### 3.1 The main theorem

Before going into details, recall the notion of Poisson bracket: Given two functions  $A, B$  depending on  $(y, u) \in \mathcal{R}^{2m}$ , then their *Poisson bracket*  $\{A, B\}$  (needed here in a narrow sense) is  $\{A, B\} = A_{,y\rho} B_{,u\rho} - B_{,y\rho} A_{,u\rho}$  (mind summation  $\rho = 1, \dots, m$ ).

**Theorem 2:** *Let  $H(x, u)$  be a Hamiltonian with  $\text{rank}(H_{,x,u}(x, u)) = m$ , let  $F : u_\mu = \alpha_\mu(x, v)$  be a feedback transformation ( $F^{-1} : v_\mu = \bar{\alpha}_\mu(x, u)$ ). Then the following statements are equivalent.*

- (a) *F is structure preserving for H.*
- (b) *There are m functions  $V_\mu(y, u)$  having Poisson brackets*

$$\{V_\mu, V_\lambda\} = 0 \tag{16}$$

*such that  $F^{-1}$  has the representation*

$$\bar{\alpha}_\mu(x, u) = V_\mu(-H_{,u}(x, u), u). \tag{17}$$

*(In case  $m = 1$  the bracket condition (16) is void.)*

**Remarks:** (a) On account of the invertibility of  $\bar{\alpha}_\mu(x, \cdot)$  the equations  $v_\mu = V_\mu(-H_{,u}(x, u), u)$  are solvable for  $u$ . This implies that the functions  $V_\mu$  are independent:  $\text{rank}(V_{\mu,y\rho}, V_{\mu,u\rho}) = m$ .

(b) Following Theorem 2 one can set up a structure preserving feedback transformation by prescribing  $m$  functions  $V_\mu$  having mutual Poisson brackets zero and non-singular Jacobian of  $V_\mu(-H_{,u}(x, \cdot), \cdot)$  and then solving the equations mentioned under (a).

(c) Equally, Theorem 2 describes solutions of the quasilinear partial differential equations (10) stressed in Lemma 1. Each  $m$ -tuple of functions  $V_\mu$  with properties as in Theorem 2 gives rise to a family  $\{\alpha(\cdot, v)\}_{v \in \pi}$  of solutions which are implicitly given by the equations mentioned under (a).

(d) The occurrence of  $-H_{,u}$  within (17) is an intrinsic matter of the differential equations (5)<sub>a</sub> and their structure preservation under feedback transformation, primarily it has nothing to do with the choice of output functions. Clearly, it is just this fact which, later on, makes Theorem 2 fit to Hamiltonian systems with natural output as well.

(e) In [13], there is a Remark concerning something like Theorem 2. But apparently it contains some confusion with the affine case where  $H_{,u}$  occurring in (17), does not depend on  $u$  (see the Remark in Section 3.4). Structure preserving feedback transformations for affine Hamiltonian control systems are considered by van der Schaft in several of his papers, see, for example, [11, 12, 14, 15]. His main tool is to see a Hamiltonian system as describing a Lagrangian submanifold of a certain symplectic manifold.

thereby making explicit use of the symplecticity of the state space, but this does not hit the core of the problem (remind the Remarks preceding Lemma 1). His result will of course be rediscovered as a special case of the above theorem (Section 3.4).

**Proof of Theorem 2/Sufficiency:** Let the functions  $V_\mu$  have vanishing Poisson brackets and let  $\alpha(x, v)$  be the solution of the equations  $v_\mu = V_\mu(-H_{,v_\mu}(x, u), u)$ . Then  $v_\mu = V_\mu(-H_{,v_\mu}(x, \alpha(x, v)), \alpha(x, v))$  is an identity with respect to  $x$  and  $v$ . Differentiation of this identity with respect to  $v_\lambda$  or  $x^i$  gives rise to the auxiliary relations

$$\delta_\mu^\lambda = A_\mu^\nu \alpha_{\nu, v_\lambda} \tag{18}$$

with

$$A_\mu^\nu = -V_{\mu, y^\rho}(y, u) H_{, v_\rho v_\nu}(x, u) + V_{\mu, v_\nu}(y, u), \tag{19}$$

where here and in the following  $u, y$  have to be replaced by  $\alpha(x, v), -H_{,v_\mu}(x, \alpha(x, v))$ , respectively, and

$$0 = A_\mu^\nu \alpha_{\nu, x^i} - V_{\mu, y^\rho}(y, u) H_{, v_\rho x^i}(x, u). \tag{20}$$

Now (18) indicates the non-singularity of  $(A_\mu^\nu)$  and  $(\alpha_{\nu, v_\lambda})$ , thus (20) solves for  $\alpha_{\nu, x^i}$ ,

$$\alpha_{\nu, x^i} = \alpha_{\nu, v_\lambda} V_{\lambda, y^\rho} H_{, v_\rho x^i}. \tag{20}'$$

Following Lemma 1, it is to be shown that  $h_{ij} := H_{, x^i v_\rho} \alpha_{\rho, x^j}$  is symmetric,  $h_{ij} = h_{ji}$ . Since (20)' yields  $h_{ij} = H_{, x^i v_\rho} \alpha_{\rho, v_\lambda} V_{\lambda, y^\sigma} H_{, v_\sigma x^j}$ , it is sufficient to show the symmetry of  $k_{\rho\sigma} := \alpha_{\rho, v_\lambda} V_{\lambda, y^\sigma}$  or, utilizing the non-singularity of  $(A_\mu^\nu)$ , the symmetry of  $l_{\kappa\lambda} := A_\lambda^\rho A_\rho^\sigma k_{\rho\sigma}$ . Using (19) it is straight calculation to find  $l_{\kappa\lambda} - l_{\lambda\kappa} = \{V_\kappa, V_\lambda\} = 0$  ■

To prove the necessity of Theorem 2, let  $u = \alpha(x, v), v = \bar{\alpha}(x, u)$  be a structure preserving feedback transformation for  $H$ . Then it has to be shown that there are  $m$  functions  $V_\mu$  with zero valued Poisson brackets such that  $\bar{\alpha}$  has the structure given by (17). This will be done by construction.

Consider the auxiliary equations

$$y^\mu = -H_{, v_\mu}(x, u). \tag{21}$$

Owing to the supposed regularity  $\text{rank}(H_{, v_\mu x^i}) = m$  they can be solved for (perhaps after relabelling)  $x^1, \dots, x^m$ :

$$x^\kappa = X^\kappa(y, u, \bar{x}), \quad \kappa = 1, \dots, m, \tag{22}$$

where  $\bar{x}$  stands for  $x^{\bar{\kappa}}, \bar{\kappa} = m + 1, \dots, 2n$ . Then let

$$\bar{V}_\mu(y, u; \bar{x}) := \bar{\alpha}_\mu(X, (y, u; \bar{x}), \bar{x}, u). \tag{23}$$

**Proposition 1:** In (23),  $\bar{x}$  is a dummy parameter, so in fact  $\bar{V}_\mu(y, u; \bar{x}) = V_\mu(y, u)$ .

**Proposition 2:** *The so-defined functions  $V_\mu$  have Poisson brackets zero.*

In order to prove these propositions, some auxiliary relations will be needed.

(i) Since  $\alpha$  is structure preserving for  $H$  there is a function  $K(x, v)$  such that

$$K_{,i} (x, v) = H_{,i} (x, \alpha(x, v)), K_{,i} (x, \tilde{\alpha}(x, u)) = H_{,i} (x, u). \tag{24}_a$$

This yields by differentiation  $K_{,i v_\rho} (x, v) = H_{,i v_\rho} (x, \alpha(x, v)) \alpha_{\rho, v_\rho} (x, v)$ , showing  $K_{,i v_\rho}$  to be of the same rank as provided for  $H_{,i v_\rho}$ . Replacing  $v$  by  $\tilde{\alpha}(x, u)$  yields

$$K_{,i v_\rho} (x, \tilde{\alpha}(x, u)) = H_{,i v_\rho} (x, u) \alpha_{\rho, v_\rho} (x, \tilde{\alpha}(x, u)). \tag{25}$$

From identity (24)<sub>b</sub> follows, by appropriate differentiation,

$$K_{,i v_\rho} (x, \tilde{\alpha}(x, u)) \tilde{\alpha}_{\mu, v_\rho} (x, u) = H_{,i v_\rho} (x, u) \tag{26}_a$$

$$K_{,i v_\rho} (x, \tilde{\alpha}(x, u)) + K_{,i v_\rho} (x, \tilde{\alpha}(x, u)) \tilde{\alpha}_{\mu, v_\rho} (x, u) = H_{,i v_\rho} (x, u). \tag{26}_b$$

Alternating  $i$  and  $j$ , the latter identity gives

$$0 = K_{,i v_\rho} (x, \tilde{\alpha}(x, u)) \tilde{\alpha}_{\mu, v_\rho} (x, u) - K_{,j v_\rho} (x, \tilde{\alpha}(x, u)) \tilde{\alpha}_{\mu, v_\rho} (x, u) \tag{26}_c$$

which is the analogue to (11):  $\tilde{\alpha}$  is structure preserving for  $K$ .

(ii) The equivalent equations (21) and (22) yield the identity  $x^\kappa = X^\kappa(-H_{,i} (x, u), u; \bar{x})$ . Differentiating this identity by  $u_\rho$  and then replacing again  $-H_{,i}$  by  $y$ , it is easy to find the Poisson brackets ( $\bar{x}$  as a parameter)

$$\{X^\kappa, X^\lambda\} = 0. \tag{27}$$

Differentiation of the above identity by  $x^\lambda$  yields

$$\delta_\lambda^\kappa = -X_{,y^\rho}^\kappa (y, u; \bar{x}) H_{,v_\rho} (X, \bar{x}, u). \tag{28}$$

(iii) Finally, differentiation of the identity  $-y^\mu = H_{,v_\mu} (X, \bar{x}, u)$  by  $x^\kappa$  gives

$$0 = H_{,v_\mu} (X, \bar{x}, u) X_{,x^\kappa}^\lambda + H_{,v_\mu} (X, \bar{x}, u). \tag{29}$$

**Proof of Proposition 1:** It has to be shown that, starting with (23),

$$\tilde{V}_{\mu, x^\kappa} (y, u; \bar{x}) = \tilde{\alpha}_{\mu, x^\kappa} (X, \bar{x}, u) X_{,x^\kappa}^\lambda (y, u; \bar{x}) + \tilde{\alpha}_{\mu, x^\kappa} (X, \bar{x}, u)$$

is zero. If this equation is multiplied by  $K_{,x^\lambda} (X, \bar{x}, \tilde{\alpha}(X, \bar{x}, u))$ , which has maximal rank  $m$ , then (26)<sub>c</sub>, (25), and (29) can be utilized in this order to find easily  $K_{,x^\lambda} \tilde{V}_{\mu, x^\kappa} = 0$ , whence  $\tilde{V}_{\mu, x^\kappa} = 0$  ■

**Proof of Proposition 2:** Again starting with (23) and now using  $\tilde{V}_\mu (y, u; \bar{x}) = V_\mu (y, u)$ , one finds that the derivatives  $V_{\mu, y^\rho} = \tilde{\alpha}_{\mu, x^\kappa} X_{,y^\rho}^\kappa$  and  $V_{\nu, u_\rho} = \tilde{\alpha}_{\nu, x^\lambda} X_{,u_\rho}^\lambda + \tilde{\alpha}_{\nu, u_\rho}$  (where the derivatives of  $\tilde{\alpha}$  have the argument  $(X, \bar{x}, u)$ ) yield the brackets

$$\{V_\mu, V_\nu\} = \tilde{\alpha}_{\mu, x^\kappa} \tilde{\alpha}_{\nu, x^\lambda} \{X^\kappa, X^\lambda\} + X_{,y^\rho}^\kappa [\tilde{\alpha}_{\mu, x^\kappa} \tilde{\alpha}_{\nu, u_\rho} - \tilde{\alpha}_{\nu, x^\kappa} \tilde{\alpha}_{\mu, u_\rho}].$$

Here the first term vanishes on account of (27). But so does also the second one, which is easy to see after multiplication by  $K_{x^i, v_\mu} K_{x^j, v_\nu} (X, \bar{x}, \bar{\alpha}(X, \bar{x}, u))$  and inspection of the identities (26)<sub>a</sub>, (28), and (26)<sub>b</sub>.

**Proof of Theorem 2/Necessity:** The two propositions above complete the proof of the necessity: relation (23) now reads  $\bar{\alpha}_\mu(x, u) = V_\mu(-H_{,u}(x, u), u)$ , and the  $V_\mu$  have Poisson brackets zero ■

If  $\alpha(x, v)$  describes a structure preserving feedback transformation for  $H$ , then the Hamiltonian of the transformed system is given by the line integral  $K(x, v) = \int H_{,x^i}(x, \alpha(x, v)) dx^i$ . Lemma 1 ensures its integrability, thus  $K$  is well defined up to an additive term  $k(v)$  which does not enter the transformed differential equations  $\dot{x}^i = \Omega^{ij} K_{,x^j}$ . There is an alternative form for  $K$  which will be exploited below.

### 3.2 The main theorem for systems with natural output

It is well known (see, for example, [3,6]) that any system of independent functions  $V_\mu(y, u)$  having Poisson brackets zero determines a system of canonically conjugate functions  $Z^\mu(y, u)$  which satisfy

$$\{Z^\mu, V_\nu\} = \delta^\mu_\nu, \{Z^\mu, Z^\nu\} = 0. \tag{30}$$

These functions  $Z^\mu$  are well defined (by (30), seen as partial differential equations) to within an additive term  $\zeta_{,v_\mu}(V(y, u))$ , where  $\zeta(v_1, \dots, v_m)$  is an arbitrary function. Then

$$(y, u) \rightarrow (z, v): z^\mu = Z^\mu(y, u), v_\mu = V_\mu(y, u) \tag{31}$$

is a canonical transformation in  $\mathcal{R}^{2m}$ .

Recall some basic facts on canonical transformations needed below in this pure analytical fashion: The transformation (31) is canonical

- (i) iff  $\{Z^\mu, Z^\nu\} = 0, \{Z^\mu, V_\nu\} = \delta^\mu_\nu, \{V_\mu, V_\nu\} = 0$  (Poisson bracket characterization),
- (ii) iff there exists a function  $S(y, u)$  such that

$$y^\rho du_\rho - Z^\rho dV_\rho = dS \tag{32}$$

(exact 1-form characterization). This differential relation is equivalent to

$$-Z^\rho V_{\rho, v_\mu} = S_{,v_\mu}, y^\mu - Z^\rho V_{\rho, y_\mu} = S_{,y_\mu}. \tag{32}'$$

$S$  is uniquely determined (up to an additive constant) by the canonical transformation  $(Z, V)$ , it is called *coupling function*.

Let now  $v = \bar{\alpha}(x, u)$  be structure preserving for  $H$ , let  $\bar{\alpha}$  correspond to  $V_\mu$  as in Theorem 2, let  $Z^\mu$  be the canonical conjugates of  $V_\mu$  and  $S$  the coupling function of the canonical transformation made up by  $(Z^\mu, V_\mu)$ . Then there is a useful representation of the new Hamiltonian.

**Lemma 2:** *The new Hamiltonian is given by*

$$K(x, v) = H(x, \alpha(x, v)) + S(-H_{,u}(x, \alpha(x, v)), \alpha(x, v)). \tag{33}$$

**Proof:** It is straight calculation, using (32)' and the identity  $V_\mu(-H_{,u}(x, \alpha(x, v)), \alpha(x, v)) = v_\mu$ , to find  $K_{,x^i}(x, v) = H_{,x^i}(x, \alpha(x, v))$  ■

**Remark:** The ambiguity of the canonical conjugates  $Z^\mu$  (additive terms  $\zeta_{,v_\mu}(V(y, u))$ ) implies, via (32), a corresponding additive term  $-\zeta(V(y, u))$  to  $S$ . Thus (33) shows, again,  $K$  to be unique up to additive  $-\zeta(v)$  (remind  $V(-H_{,u}(x, \alpha(x, v)), \alpha(x, v)) \equiv v$  because of (17)).

The most important feature of  $K$  is now disclosed by the following calculation using (33) and (32)' with  $y = -H_{,u}(x, u)$  and  $u = \alpha(x, v)$ :

$$\begin{aligned} K_{,v_\mu}(x, v) &= [H_{,v_\rho}(x, u) - S_{,y^\sigma}(y, u) H_{,v_\sigma u_\rho}(x, u) + S_{,v_\rho}(y, u)] \alpha_{\rho, v_\mu} \\ &= [-y^\rho + Z^\kappa V_{\kappa, y^\sigma}(y, u) H_{,v_\sigma u_\rho}(x, u) + y^\rho - Z^\kappa V_{\kappa, v_\rho}(y, u)] \alpha_{\rho, v_\mu} \\ &= -Z^\kappa(y, u) [V_\kappa(y, u)]_{,v_\mu}. \end{aligned}$$

Now, because of  $V_\kappa(-H_{,u}(x, \alpha(x, v)), \alpha(x, v)) = v_\kappa$ , there follows

$$K_{,v_\mu}(x, v) = -Z^\mu(-H_{,u}(x, \alpha(x, v)), \alpha(x, v)). \tag{34}$$

The crucial point in considering systems with output now is the following: Each feedback transformation which is structure preserving for  $H$  (that is, for the differential equations (5)<sub>a</sub> disregarding output) is, as outlined above, connected with a canonical transformation (31). Theorem 2 tells that  $v = \bar{\alpha}(x, u)$  is given by restricting the  $v$ -part of this transformation to  $y = -H_{,u}(x, u)$ , which is just the natural output (5)<sub>b</sub> belonging to the differential equations (5)<sub>a</sub>. The above equation (34) means that nothing else but the natural output of the transformed system is achieved by restricting the  $z$ -part of the canonical transformation to the natural output of the original system (and expressing  $u$  by the new control  $v$ ). This observation throws a new light on output equations of the form  $y^\mu = -H_{,u_\mu}(x, u)$ : they are natural under a transformation-theoretical point of view since they now show up as being intrinsically determined by the differential equations themselves.

Altogether, it has been shown that the transition of a Hamiltonian control system with natural output to another system of this type via feedback transformation is governed by a canonical transformation in the input-output space  $Y \times U$  (which therefore, again, has to be seen as  $W = T^*Y$ ).

**Theorem 3:** *The Hamiltonian control systems with natural output,*

$$\begin{aligned} \dot{x}^i &= \Omega^{ij}(x) H_{,x^j}(x, u), \quad y^\mu = -H_{,u_\mu}(x, u) \\ \dot{x}^i &= \Omega^{ij}(x) K_{,x^j}(x, v), \quad z^\mu = -K_{,v_\mu}(x, v) \end{aligned}$$

are equivalent via feedback transformation  $u_\mu = \alpha_\mu(x, v), v_\mu = \bar{\alpha}_\mu(x, u)$ , iff there is, in the external space  $W = T^*Y$ , a canonical transformation  $(y, u) \rightarrow (z, v) : z^\mu = Z^\mu(y, u), v_\mu = V_\mu(y, u)$  such that

$$\begin{aligned} \bar{\alpha}_\mu(x, u) &= V_\mu(-H_{,u}(x, u), u), \\ K_{,v_\mu}(x, v) &= Z^\mu(-H_{,u}(x, u), u) \quad \text{with } u = \alpha(x, v). \end{aligned}$$

### 3.3 Special feedback transformations

Every canonical transformation (31) can be composed of an elementary canonical transformation (which is, apart from some changes in sign, a permutation of the  $2m$  variables  $y^\mu, u_\mu$ ) and a second canonical transformation which is featured by maximal rank of some  $m \times m$  submatrix of its Jacobian [3]. Following [7], emphasis is on four types of canonical transformations

$$\begin{aligned} \text{type 1 : } \text{rank}(Z_{,u_\nu}^\mu) &= m, & \text{type 2 : } \text{rank}(V_{\mu,u_\nu}) &= m \\ \text{type 3 : } \text{rank}(Z_{,y^\nu}^\mu) &= m, & \text{type 4 : } \text{rank}(V_{\mu,y^\nu}) &= m. \end{aligned}$$

Each of these transformations can be described by a generating function which can be used profitably for the construction of structure preserving feedback transformations. If all possibilities are displayed, it turns out that type 4 is the simplest case while type 2 is most important.

**Type 4 transformations.** Owing to the above rank condition the equations  $v_\mu = V_\mu(y, u)$  can be solved for  $y$ :  $y^\mu = g^\mu(u, v)$ . The identity  $v_\mu = V_\mu(g(u, v), u)$  then implies  $\delta_\mu^\lambda = V_{\mu,y^\kappa} g_{,v_\lambda}^\kappa$ , showing  $(g_{,v_\lambda}^\kappa)$  to be non-singular, and  $0 = V_{\mu,y^\kappa} g_{,v_\lambda}^\kappa + V_{\mu,u_\lambda}$  (appropriate arguments). Using the coupling function  $S(y, u)$  of the canonical transformation, let  $G^4(u, v) := -S(g(u, v), u)$ . Then (32)' yields  $G_{,v_\mu}^4(u, v) = -g^\mu(u, v)$ , showing  $(G_{,v_\mu}^4)$  to be non-singular, and  $G_{,v_\mu}^4(u, v) = Z^\mu(g(u, v), u)$ . Hence the canonical transformation is implicitly given by

$$y^\mu = -G_{,v_\mu}^4(u, v), \quad z^\mu = G_{,v_\mu}^4(u, v). \tag{35}$$

Vice versa, any given function  $G^4(u, v)$  with maximal rank,  $m$ , of  $(G_{,v_\mu}^4(u, v))$  generates, via (35), a type 4 canonical transformation.

Let now  $G^4(u, v)$  be any type 4 generator. Then the equations

$$-H_{,v_\mu}(x, u) = -G_{,v_\mu}^4(u, v), \tag{36}$$

where  $H$  is a given Hamiltonian, can be solved for  $v$ :  $v_\mu = \bar{\alpha}_\mu(x, u)$ . This solution is certainly of the form  $V_\mu(-H_{,u_\nu}(x, u), u)$ , where  $V_\mu$  is part of a canonical transformation (generated by  $G^4$ ). The identity  $H_{,v_\mu}(x, u) = G_{,v_\mu}^4(u, \bar{\alpha}(x, u))$  implies, with appropriate arguments of functions,  $H_{,v_\mu u_\lambda} = G_{,v_\mu v_\lambda}^4 + G_{,v_\mu v_\rho}^4 \bar{\alpha}_{,v_\lambda}^\rho$ , so the local invertibility of  $\bar{\alpha}(x, \cdot)$  is guaranteed by non-singularity of  $(H_{,v_\mu v_\lambda} - G_{,v_\mu v_\lambda}^4)$ . Then Theorem 2 exhibits  $\bar{\alpha}$  to describe a structure preserving feedback transformation for  $H$ , the new Hamiltonian is given by (33).

**Theorem 4:** *Let  $H(x, u)$  be a given Hamiltonian,  $G^4(u, v)$  a given function, let*

$$\text{rank}(G_{,v_\mu v_\lambda}^4(u, v)) = \text{rank}(H_{,v_\mu u_\lambda}(x, u) - G_{,v_\mu v_\lambda}^4(u, v)) = m \quad \text{for all } x, u, v.$$

*Then the solution  $v_\mu = \bar{\alpha}_\mu(x, u)$  of the equations (36) defines a structure preserving feedback transformation for  $H$ . With its inverse,  $u_\mu = \alpha_\mu(x, u)$ , the transformed Hamiltonian is*

$$K(x, v) = H(x, \alpha(x, v)) - G^4(\alpha(x, v), v).$$

**Type 2 transformations.** Similar reasoning as above now leads to the implicit form of the canonical transformation  $u_\mu = G_{,y^\mu}^2(y, v)$ ,  $z^\mu = G_{,v_\mu}^2(y, v)$  with a generator  $G^2(y, v)$  having  $\text{rank}(G_{,y^\mu v_\lambda}^2(y, v)) = m$  given by  $G^2(y, v) := -S(y, g(y, v)) + y^\rho g_\rho(y, v)$ , where  $g_\rho$  solves the equations  $v_\mu = V_\mu(y, u)$ . Then there holds

**Theorem 5:** *Let  $H(x, u)$  be a given Hamiltonian,  $G^2(y, v)$  a given function, let*

$$\text{rank}(G_{y,\mu,\lambda}^2) = \text{rank}(\delta_\mu^\nu + G_{y,\mu,\nu,\rho}^2 H_{\mu,\rho,u_\nu}) = m \text{ for all } x, y, u, v.$$

Then the solution  $v_\mu = \bar{\alpha}_\mu(x, u)$  of the equations  $u_\mu = G_{y,\mu}^2(-H_{,\mu}(x, u), v)$  defines a structure preserving feedback transformation for  $H$ . With its inverse,  $u_\mu = \alpha_\mu(x, v)$ , the transformed Hamiltonian is

$$K(x, v) = H(x, \alpha(x, v)) - G^2(-H_{,\mu}(x, \alpha(x, v)), v) - \alpha_\mu(x, v) H_{,\mu}(x, \alpha(x, v)).$$

### 3.4 Application to affine systems

Let  $H$  be an affine Hamiltonian,  $H(x, u) = H^0(x) - u_\mu H^\mu(x)$ , with  $\text{rank}(H_{,\mu}^\mu(x)) = m$ . Then Theorem 2 characterizes a structure preserving feedback transformation by  $v_\mu = \bar{\alpha}_\mu(x, u) = V_\mu(y, u)$  with  $y^\mu = -H_{,\mu}^\mu(x, u) = H^\mu(x)$  now not depending on  $u$  any more. The invertibility of  $\bar{\alpha}(x, \cdot)$  therefore implies  $\text{rank}(V_{\mu,\nu,\lambda}(y, u)) = m$ . Thus, in this case, the canonical transformation governing the feedback transformation is necessarily of type 2 and Theorem 5 applies. This amounts to

**Theorem 6:** A feedback transformation  $u_\mu = \alpha_\mu(x, v)$  is structure preserving for

$$H : H(x, u) = H^0(x) - u_\mu H^\mu(x), \text{rank}(H_{,\mu}^\mu(x)) = m,$$

iff there is a function  $G(y, v)$  with  $\text{rank}(G_{y,\mu,\lambda}(y, v)) = m$  such that

$$\alpha_\mu(x, v) = G_{y,\mu}(H^1(x), \dots, H^m(x), v). \tag{37}$$

Then the transformed Hamiltonian is

$$K(x, v) = H^0(x) - G(H^1(x), \dots, H^m(x), v). \tag{38}$$

In this affine case  $\alpha(\cdot, v)$  is (natural) output-feedback (see Observation (i) in Section 2). In general, the new Hamiltonian (38) is not affine any more. The transformed system is again affine with respect to  $v$ , iff so is the generating function  $G, G(y, u) = G^0(y) + G^\rho(y) v_\rho$ . In this case, (38) yields the natural output  $z^\mu = G^\mu(H^1(x), \dots, H^m(x))$  of the transformed system, which again fits to the scheme of output transformations dealt with in Section 2. The feedback transformation then is

$$u_\mu = G_{y,\mu}^0(y) + v_\rho G_{y,\mu}^\rho(y) \text{ with } y = (H^1(x), \dots, H^m(x)),$$

and this is exactly van der Schaft's "Hamiltonian feedback" [11,12].

**Remark:** In [13,p. 4] van der Schaft claims, in a non-affine context, that a Hamiltonian control system after feedback is again Hamiltonian iff (adapted to the present notation)  $\alpha_\mu(x, v)$  has the form  $\alpha_\mu = G_{y,\mu}(-H_{,\mu}, v)$ . Formally, this coincides with the above representation (37) but it can hold in the affine case only since otherwise  $H_{,\mu}$  actually depends on  $u$  and the above formula does not give a representation of  $\alpha_\mu$ .

Equation (38) describes the general form of a Hamiltonian which can be gained by a structure preserving feedback transformation applied to an affine Hamiltonian system. Vice versa, a Hamiltonian control system with some Hamiltonian  $K(x, v)$  can be transformed to an affine

Hamiltonian system iff  $K(x, v)$  has the analytic structure given by (38). How to check this structure and how to find an affinizing feedback transformation is considered in [10].

#### 4. Concluding remarks

The starting point of this paper was to generalize known results about structure preserving feedback transformations for affine Hamiltonian control systems to non-affine systems. This is a self-contained mathematical problem which, moreover, has some relevance in control theory since there are Hamiltonian control systems which, in a natural way, cannot be described by Hamiltonians affine in the controls. The answer to the problem is given by the main Theorem 2. But this theorem indeed covers more than Hamiltonian control systems: since the tensor  $\Omega^{ij}$  describing the symplectic structure of the state space drops right at the beginning of all investigations, Theorem 2 is in fact related to the bigger class of gradient-like control systems in  $\mathbb{R}^n$ ,  $n \in \mathcal{N}$ ,

$$\dot{x}^i = S^{ij}(x) H_{,x^j}(x, u), \text{ rank}(S^{ij}) = n \text{ everywhere,}$$

where the tensor  $S$ , besides its non-degeneracy, does not undergo any restrictions and thus need not allow for a normal form like (4) anymore. Even for these systems it remains true that, seen in the context of feedback transformations, the "naturalness" of outputs  $y^\mu = -H_{,x^\mu}$  is an intrinsic matter of the differential equations.

Problems still under consideration are concerned with degenerate systems of the above kind,  $\text{rank}(S^{ij}) < n$ . Respective results then could possibly allow for an approach to Poisson control systems, mentioned in [4], which generalize (but are still "near to") Hamiltonian control systems.

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### Book review

A. PIETSCH (ed.): **D. HILBERT and E. SCHMIDT - Integralgleichungen und Gleichungen mit unendlich vielen Unbekannten** (Teubner-Archiv zur Mathematik: Vol. 11). Leipzig: B.G. Teubner Verlagsges. 1989, 316 pp., 12 fig.

In a time of a renaissance of boundary integral methods it is good to remember the origins of the theory. At the beginning of our century the method of reduction of boundary value problems to linear integral equations was not only applied to ordinary differential equations and to the Laplace equation, but already to the equations of the linear elasticity theory. In this connection H. Weyl recognized in 1915 that boundary integral equations can appear, which do not lead to a regular integral equation for the unknown distribution. Today such integral equations are no more a barrier, since these are singular integral equations, though this theory reached a certain completion not until the sixties. Of immediate interest is the analytic