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On the Ranges of Realizations in Distribution Spaces

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The paper deals with the closedness of ranges and the surjectivity of realizations related to linear partial differential operators $L(x,D)$. A characterization (with the help of a certain coercivity condition) of the surjectivity of the maximal realization $L'^{\#}_{p,k,G}$ in $B_{p,k}(G) = \{u \in D'(G) | u = f_u|_G$ for some $f_u \in B_{p,k}\}$ is established. Here $B_{p,k}$ ($p \in (0,1)$, $k \in K$) is the Hörmander space. Furthermore, the closedness of the range $R(\Lambda^{\sim}_{p,k}(G))$ corresponding to the minimal realization $\Lambda^{\sim}_{p,k}(G)$ in local Hörmander spaces $B^{loc}_{p,k}(G) = \{u \in D'(G) | \Psi u \in B_{p,k}$ for any $\Psi \in C_0^{\infty}(G)\}$ is considered.

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1. Introduction

Let $L(x,D)$ be a linear partial differential operator with $C^{\infty}(\mathbb{R}^n)$-coefficients. Further more, let G be an open set in $\mathbb{R}^n$. Choose $p \in (1,\infty)$ and $k \in K$, where K is the Hörmander class of weight functions k: $\mathbb{R}^n \to \mathbb{R}$. We shall deal with the closedness of the range $R(\Lambda^{\sim}_{p,k}(G))$ of the minimal realization $\Lambda^{\sim}_{p,k}(G)$ in the local Hörmander space $B^{loc}_{p,k}(G)$. The closedness of $R(\Lambda^{\sim}_{p,k}(G))$ is closely related to the theory of $L(x,D)$-convex sets (cf. [3, pp. 41 - 59], [8, pp. 57 - 120], [5, pp. 358 - 371] and [6]). This can be seen, when we consider, for example, the operators $L(D)$ with constant coefficients. In this case the closedness of $R(\Lambda^{\sim}_{p,k}(G))$ implies that $R(\Lambda^{\sim}_{p,k}(G)) = B^{loc}_{p,k}(G)$ and so also for the maximal realization in $B^{loc}_{p,k}(G)$, say $\Lambda'^{\#}_{p,k}(G)$, one has the equality $R(\Lambda'^{\#}_{p,k}(G)) = B^{loc}_{p,k}(G)$ (cf. [6] and note that $N(\Lambda^{\#}_{p',1/k^{\vee}}(G)) \cap E'(G) = \{0\}$, that is, the distributional equation $L(D)u = 0$, $u \in E'(G) \cap B^{loc}_{p',1/k^{\vee}}(G)$ holds if and only if $u = 0$). The surjectivity of $\Lambda'^{\#}_{p,k}(G)$ implies that G is $L(G)$-convex (cf. [3, Theorem 10.6.6]) and on the contrary (cf. [3, Theorem 10.6.7]). Especially, one finds that, if $R(\Lambda^{\sim}_{p,k}(G))$ is closed in $B^{loc}_{p,k}(G)$ with a fixed pair $(p,k) \in (1,\infty) \times K$, then $R(\Lambda^{\sim}_{q,k'}(G))$ is closed for any pair $(q,k') \in (1,\infty) \times K$.

The surjectivity of the maximal realization $L'^{\#}_{p,k,G}$ in $\mathbb{B}_{p,k}(G) := \{u \in D'(G) | u = f_u|_G$ for some $f_u \in B_{p,k}\}$ can be characterized by means of the validity of the inequality

$$\| L'(x,D)\varphi \|_{p',1/k^{\vee}} \geq c \|\varphi\|_{p',1/k^{\vee}} , \quad \varphi \in C_0^{\infty}(G)$$

(cf. Theorem 2.2). Also the closedness of $R(L'^{\#}_{p,k,G})$ can be characterized in an easy way.

The surjectivity of the operator $\Lambda^{\sim}_{p,k}(G)$ (which in many cases is equal to $\Lambda'^{\sim}_{p,k}(G)$) and the closedness of $R(\Lambda^{\sim}_{p,k}(G))$ is much more difficult to check. In Theorem 3.1 we show a necessary criterion that the range $R(\Lambda^{\sim}_{p,k}(G))$ is not closed. Furthermore, in Theorem 3.4 we establish a sufficient condition that $R(\Lambda^{\sim}_{p,k}(G))$ is not closed (under suitable circumstances). Theorem 3.6 shows that in some cases our theory will give a characterization for the closedness of $R(\Lambda^{\sim}_{p,k}(G))$. The basic idea is to study the existence of the distributional solutions v for the equation $L'(x,D)v = g$, where $v \in E'(\bar{G})$, $g \in E'(G)$ and $\operatorname{supp} v \cap \partial G \neq \Phi$, where ∂G is the boundary of G, and Φ denotes the empty set.

2. Definitions and preliminaries

2.1. For the (unexplained) notations and definitions concerning the distribution theory and its related topics, we refer to the monographs [2, 3]. Let G be an open set in $\mathbb{R}^n$ and let $p \in (1, \infty)$ and $k \in K$. We recall that $B_{p,k}$ is a Banach space and $B^{\text{loc}}_{p,k}(G)$ is a Frechet space. For $p < \infty$, the space C^∞_0 is dense in $B_{p,k}$ and $C^\infty_0(G)$ is dense in $B^{\text{loc}}_{p,k}(G)$. The notation $B^c_{p,k}(G)$ means the intersection $B_{p,k} \cap E'(G)$. The completion of $C^\infty_0(G)$ in $B_{p,k}$ is denoted by $B_{p,k}(G)$. Then one sees that $B_{p,k}(G) \subset B_{p,k} \cap E'(\bar{G})$. Here $E'(A)$ (where $A \subset \mathbb{R}^n$) is the set of distributions $u \in E'(\mathbb{R}^n)$ such that $\operatorname{supp} u \subset A$. The set $\{u \in B_{p,k} \mid \operatorname{supp} u \subset A\}$ is denoted by $B^{\circ}_{p,k}(A)$. Finally, we denote by $\mathbb{B}_{p,k}(G)$ the set of distributions $u \in D'(G)$ such that $u = f_{u|G}$ for some distribution $f_u \in B_{p,k}$, where $f_{u|G}$ denotes the restriction of f_u to G. One sees that $\mathbb{B}_{p,k}(G)$ is isomorphic with the factor space $B_{p,k}/B^{\circ}_{p,k}(\mathbb{R}^n \setminus G)$ and we transfer the topology of this factor space to $\mathbb{B}_{p,k}(G)$ in the canonical way (note that $B^{\circ}_{p,k}(\mathbb{R}^n \setminus G)$ is closed in $B_{p,k}$, since $\mathbb{R}^n \setminus G$ is closed in $\mathbb{R}^n$). Furthermore, one sees that for $p \in (1,\infty)$ the spaces $B_{p',1/k^\vee}(G)$ and $\mathbb{B}_{p,k}(G)$ are in duality with respect to the extension of the bilinear form

$$\lambda : C^\infty_{(0)}(G) \times C^\infty_0(G)\,,\ \lambda(\varphi,\psi) = \int_{\mathbb{R}^n} \varphi(x)\psi(x)\,dx\,.$$

Here $C^\infty_{(0)}(G)$ denotes the subspace of functions ψ in $C^\infty(G)$ such that there exists $f_\psi \in C^\infty_0$ with $\psi = f_{\psi|G}$. Note that $C^\infty_{(0)}(G)$ is dense in $\mathbb{B}_{p,k}(G)$, $p < \infty$. We also write $\mathbb{B}^\infty(G) = \bigcap_{p,k}\{u \in D'(G) \mid u = f_{u|G}$ with some $f_u \in B^{\text{loc}}_{p,k}(\mathbb{R}^n)\}$.

2.2. Let $L(x,D) = \sum_{|\sigma|\le r} a_\sigma(x) D^\sigma$ be a linear partial differential operator with $\mathbb{B}^\infty(G)$-coefficients. The formal transpose $\sum_{|\sigma|\le r} (-D)^\sigma (a_\sigma(x)(\cdot))$ is denoted by $L'(x,D)$. Let $L_{p,k,G}$ ($p \in (1,\infty)$; $k \in K$) be a linear operator $B_{p,k}(G) \to B_{p,k}(G)$ such that

$$D(L_{p,k,G}) = C^\infty_0(G)\,,\quad L_{p,k,G}\,\varphi = L(x,D)\varphi\,.$$

Then $L_{p,k,G}$ is closable in $B_{p,k}(G)$: Let $\{\varphi_n\} \subset C^\infty_0(G)$ be a sequence and let $g \in B_{p,k}(G)$ be an element such that $\|\varphi_n\|_{p,k} \to 0$ and $\|L_{p,k,G}\,\varphi_n - g\|_{p,k} \to 0$ as $n \to \infty$.

Then one has for any $\Phi \in C_0^\infty$

$$g(\Phi) = \lim_n (L_{p,k,G}\, \varphi)(\Phi) = \lim_n \varphi_n(L'(x,D)\Phi) = 0, \tag{2.1}$$

where we utilized the fact that for $\varphi \in C_0^\infty(G)$ and $u \in \mathbb{B}_{p',1/k^\vee}(G)$ the inequality

$$|u(\varphi)| \le |||u|||_{p',1/k^\vee} \, \|\varphi\|_{p,k} \tag{2.2}$$

holds. Here $|||\cdot|||_{p',1/k^\vee}$ denotes the $\mathbb{B}_{p',1/k^\vee}(G)$-norm. In the last step of (2.1) we observed that $L'(x,D)\Phi|_G \in \mathbb{B}^\infty(G)$ for any $\Phi \in C_0^\infty$, since $a_\alpha \in \mathbb{B}^\infty(G)$. Due to (2.1) one gets $g = 0$, and so $L_{p,k,G}$ is closable in $B_{p,k}(G)$. The smallest closed extension of $L_{p,k,G}$ is denoted by $L^\sim_{p,k,G}$.

Furthermore, we define a linear operator $\mathbb{L}'^{\#}_{p,k,G}$ by

$$D(\mathbb{L}'^{\#}_{p,k,G}) = \left\{ u \in \mathbb{B}_{p,k}(G) \,\middle|\, \begin{array}{l} \text{there exists } f \in \mathbb{B}_{p,k}(G) \text{ such that} \\ u(L'(x,D)\varphi) = f(\varphi)\ \forall\ \varphi \in C_0^\infty(G) \end{array} \right\},$$

$$\mathbb{L}'^{\#}_{p,k,G}\, u = f.$$

Due to (2.2) one sees that $\mathbb{L}'^{\#}_{p,k,G}$ is a closed operator. In the case when $G = \mathbb{R}^n$, one sees that $\mathbb{B}_{p,k}(G) = B_{p,k} = B_{p,k}(G)$, and we write $L_{p,k,\mathbb{R}^n} = L_{p,k}$ and so on.

As we mentioned above the spaces $\mathbb{B}_{p,k}(G)$ and $B_{p',1/k^\vee}(G)$ are in duality with respect to λ. Explicitly, this means that there exists an isometrical isomorphism

$$J_{p,k} : \mathbb{B}_{p,k}(G) \to B^*_{p',1/k^\vee}(G)\ ,\ (J_{p,k}\, U)(\varphi) = U(\varphi)\ \ \forall\, \varphi \in C_0^\infty(G),$$

and similarly there exists an isimetrical isomorphism

$$j_{p',1/k^\vee} : B_{p',1/k^\vee}(G) \to \mathbb{B}^*_{p,k}(G)\ ,\ (j_{p',1/k^\vee}\, v)(\varphi|_G) = v(\varphi)\ \ \forall\, \varphi \in C_0^\infty.$$

Here $*$ refers to the dual space. In [7] we have explicitly shown the existence of $J_{p,k}$ and $j_{p',1/k^\vee}$.

Let $L^*_{p',1/k^\vee,G}$ be the dual operator of the (densily defined) operator $L_{p',1/k^\vee,G}$. Then one easily sees

Lemma 2.1: *Suppose that $L(x,D)$ has $\mathbb{B}^\infty(G)$-coefficients and that $p \in (1,\infty)$, $k \in K$. Then the relation*

$$\mathbb{L}^{\#}_{p,k,G} = J^{-1}_{p,k}\ \ (L^*_{p',1/k^\vee,G})\ \ J_{p,k} \tag{2.3}$$

holds.

We verify the next existence result of solutions.

Theorem 2.2: *Suppose that $L(x,D)$ has $\mathbb{B}^\infty(G)$-coefficients and that $p \in (1,\infty)$, $k \in K$. Then the range $R(\mathbb{L}'^{\#}_{p,k,G})$ is the whole space $\mathbb{B}_{p,k}(G)$ if and only if there exists a* *constant $c > 0$ such that*

$$\| L'(x,D)\varphi \|_{p',1/k^{\vee}} \geq c \|\varphi\|_{p',k^{\vee}} \quad \textit{for all } \varphi \in C_0^{\infty}(G). \tag{2.4}$$

Proof: Suppose first that (2.4) holds. Then the range $R(L'^{\sim}_{p',1/k^{\vee},G})$ is closed in $B_{p',1/k^{\vee}}(G)$ and the kernel $N(\cdot)$ is trivial, i.e. $N(L'^{\sim}_{p',1/k^{\vee},G}) = \{0\}$. Hence one has

$$R(L'^{*}_{p',1/k^{\vee},G}) = R(L'^{\sim *}_{p',1/k^{\vee},G}) = B^{*}_{p',1/k^{\vee}}(G).$$

Since, by (2.3), $\mathbb{L}'^{\#}_{p,k,G} = J^{-1}_{p,k} \circ (L'^{*}_{p',1/k^{\vee},G}) \circ J_{p,k}$, where $J_{p,k}$ is a bijection, one gets that $R(\mathbb{L}'^{\#}_{p,k,G}) = \mathbb{B}_{p,k}(G)$.

Suppose that $R(\mathbb{L}'^{\#}_{p,k,G}) = \mathbb{B}_{p,k}(G)$. Then, by (2.3), $R(L'^{\sim *}_{p',1/k^{\vee},G}) = B^{*}_{p',1/k^{\vee}}(G)$. Hence the range $R(L'^{**}_{p',1/k^{\vee},G}) = R(L'^{\sim}_{p',1/k^{\vee},G})$ is closed and for the kernel we have the equality $N(L'^{\sim}_{p',1/k^{\vee},G}) = N(L'^{**}_{p',1/k^{\vee},G}) = \{0\}$ (cf. [4, p. 168 and 234]). Hence (by the Closed Graph Theorem) the inverse $L'^{\sim -1}_{p',1/k^{\vee},G}$ is continuous, which implies the validity of (2.4) ∎

Remark 2.3 : a) Theorem 2.2 says that, when (2.4) holds, then the distributional equation $L(x,D)u = f$, $u \in \mathbb{B}_{p,k}(G)$ is solvable for any $f \in \mathbb{B}_{p,k}(G)$. As well known, there are several kind of algebraic criteria under which (2.4) is valid. The criterions in question are often independent of the underlying open set G. **b)** The operator $\mathbb{L}'^{\#}_{p,k,G}$ is called *maximal realization of $L(x,D)$ in $\mathbb{B}_{p,k}(G)$* and the operator $L'^{\sim}_{p',1/k^{\vee},G}$ is called the *minimal realization of $L'(x,D)$ in $B_{p',1/k^{\vee}}(G)$*.

The closedness of $R(\mathbb{L}'^{\#}_{p,k,G})$ can be characterized by

Theorem 2.4: *The range $R(\mathbb{L}'^{\#}_{p,k,G})$ is closed in $\mathbb{B}_{p,k}(G)$ if and only if the range $R(L'^{\sim}_{p',1/k^{\vee},G})$ is closed in $B_{p',1/k^{\vee}}(G)$.*

Proof: From Lemma 2.1 we obtain that $\mathbb{L}'^{\#}_{p,k,G} = J^{-1}_{p,k} \circ (L'^{*}_{p',1/k^{\vee},G}) \circ J_{p,k}$. Thus the range $R(\mathbb{L}'^{\#}_{p,k,G})$ is closed if and only if $R(L'^{*}_{p',1/k^{\vee},G})$ is closed in $B^{*}_{p',1/k^{\vee}}(G)$. Furthermore, the range $R(L'^{*}_{p',1/k^{\vee},G})$ is closed if and only if the range $R((L'^{*}_{p',1/k^{\vee},G})^{*})$ is closed in $(B^{*}_{p',1/k^{\vee}}(G))^{*}$ (cf. [4, p. 234]). Since $B_{p',1/k^{\vee}}(G)$ is a reflexive space for $p \in (1,\infty)$, one sees that $(L'^{*}_{p',1/k^{\vee},G})^{*} = \varkappa \circ L'^{\sim}_{p',1/k^{\vee},G} \circ \varkappa^{-1}$, where $\varkappa : B_{p',1/k^{\vee}}(G) \to (B^{*}_{p',1/k^{\vee}}(G))^{*}$ is the canonical isomorphism (cf. [4, p. 168]). Hence $R((L'^{*}_{p',1/k^{\vee},G})^{*})$ is closed if and only if $R(L'^{\sim}_{p',1/k^{\vee},G})$ is closed ∎

With the same kind of conclusions as made in the proof of the previous theorem one gets : The range $R(\mathbb{L}^{\sim}_{p,k,G})$ is closed in $\mathbb{B}_{p,k}(G)$ if and only if the range $R(L^{\#}_{p',1/k^{\vee},G})$ is closed in $B_{p',1/k^{\vee}}(G)$, where $\mathbb{L}^{\sim}_{p,k,G}$ is the minimal realization of $L(x,D)$ in $\mathbb{B}_{p,k}(G)$ and where $L^{\#}_{p',1/k^{\vee},G}$ is the maximal realization of $L'(x,D)$ in $B_{p',1/k^{\vee}}(G)$.

Let Q be the factor space $B_{p',1/k^\vee}(G)/N(L'^{\sim}_{p',1/k^\vee,G})$ (with the usual norm topology). Denote the norm in Q by $\|\cdot\|^\sim$. Then the range $R(L'^{\sim}_{p',1/k^\vee,G})$ is closed if and only if, with some $c > 0$,

$$\|L'(x,D)\varphi\|_{p',1/k^\vee} \geq c\|\varphi\|^\sim \text{ for all } \varphi \in C_0^\infty(G). \tag{2.7}$$

The estimate (2.4) implies that of (2.7).

2.3. Assume that $L(x,D)$ has $C_0^\infty(G)$-coefficients. Then the

minimal realization $\Lambda^\sim_{p,k}(G)$ *in* $B^{\text{loc}}_{p,k}(G)$

and the

maximal realization $\Lambda'^{\#}_{p,k}(G)$ *in* $B^{\text{loc}}_{p,k}(G)$

of $L(x,D)$ can be defined (cf. [6]; the definitions go analogously to $L^\sim_{p,k,G}$ and $\mathbb{L}'^{\#}_{p,k,G}$). Furthermore, the maximal realization $\Gamma'^{\#}_{p,k}(G)$ of $L(x,D)$ in $B^c_{p,k}(G)$ is analogously defined. The operator $\Gamma^\sim_{p,k}(G)$ is defined by

$$D(\Gamma^\sim_{p,k}(G)) = \left\{ v \in B^c_{p,k}(G) \,\middle|\, \begin{array}{l} \exists \text{ a sequence } \{\varphi_n\} \subset C_0^\infty(G) \text{ and a } g \in B^c_{p,k}(G) \\ \text{such that } \varphi_n \to v \text{ and } L(x,D)\varphi_n \to g \text{ in } B^c_{p,k}(G) \end{array} \right\}$$

$$\Gamma^\sim_{p,k}(G)v = g.$$

In the next chapter we shall consider the closedness of $R(L^\sim_{p,k}(G))$ (in $B^{\text{loc}}_{p,k}(G)$), which is much more complicated to check than the closedness of $R(\mathbb{L}^{\sim\#}_{p,k,G})$ or $R(\mathbb{L}^\sim_{p,k,G})$ (in $\mathbb{B}_{p,k}(G)$). In the study one must take into account the geometry of G and the characteristic curves with respect to $L'(x,D)$ (cf. Theorems 3.3, 3.4 and 3.6).

To preparate the investigations we present the following lemmas for $p \in (1,\infty)$ and $k \in K$ (cf. [6]).

Lemma 2.5: *The range* $R(\Lambda^\sim_{p,k}(G))$ *is closed in* $B^{\text{loc}}_{p,k}(G)$ *if and only if the range* $R(\Gamma^{\#}_{p',1/k^\vee}(G))$ *is closed in* $B^c_{p',1/k^\vee}(G)$.

Lemma 2.6: *The relation* $R(\Lambda^\sim_{p,k}(G)) = B^{\text{loc}}_{p,k}(G)$ *holds if and only if*

(i) $R(\Gamma^{\#}_{p',1/k^\vee}(G))$ *is closed in* $B^c_{p',1/k^\vee}(G)$ (ii) $N(\Gamma^{\#}_{p',1/k^\vee}(G)) = \{0\}$.

Remark 2.7: a) Similarly one has the following:

1° The range $R(\Lambda'^{\#}_{p,k}(G))$ is closed in $B^{\text{loc}}_{p,k}(G)$ if and only if the range $R(\Gamma'^{\sim}_{p',1/k^\vee}(G))$ is closed in $B_{p',1/k^\vee}(G)$.

2° The relation $R(\Lambda'^{\#}_{p,k}(G)) = B^{\text{loc}}_{p,k}(G)$ holds if and only if

(iii) $R(\Gamma'^{\sim}_{p',1/k^\vee}(G))$ is closed in $B^c_{p',1/k^\vee}(G)$. **(iv)** $N(\Gamma'^{\sim}_{p',1/k^\vee}(G)) = \{0\}$.

The proofs of 1 and 2 goes analogously to the considerations expressed in [6] (cf. also [8, pp. 49 - 51]).

b) We recall that the closedness of a subspace H in $B^c_{p',1/k^\vee}(G)$ means that $H \cap \overset{\circ}{B}_{p',1/k^\vee}(K)$ is closed in $B_{p',1/k^\vee}$ for any compact set $K \subset G$. Furthermore, the closedness of $F := H \cap \overset{\circ}{B}_{p',1/k^\vee}(K)$ in (a normed space) $B_{p',1/k^\vee}$ means that F is sequentially closed.

3. On the closedness of $R(\Lambda_{p,k}^{\sim}(G))$

3.1. We assume everywhere in this chapter that the operator $L(x,D)$ has C^{∞}-coefficients. For the first instance we establish

Theorem 3.1: *Suppose that G is bounded. Furthermore, assume that (with $c > 0$) the estimate (2.4) is true,*

$$\Gamma_{p',1/k^{\vee}}^{\#}(G) = \Gamma_{p',1/k^{\vee}}^{\prime\sim}(G), \tag{3.1}$$

and that the range $R(\Lambda_{p,k}^{\sim}(G))$ is not closed in $B_{p,k}^{\mathrm{loc}}(G)$. Then there exists elements $v \in B_{p',1/k^{\vee}}(G)$ and $g \in B_{p',1/k^{\vee}}^{c}(G)$ such that (recall that $L_{p',1/k^{\vee}}^{\#} := L_{p',1/k^{\vee},\mathbb{R}^n}$)

$$L_{p',1/k^{\vee}}^{\#} v = g \quad \text{and} \quad \operatorname{supp} v \cap \partial G \neq \Phi. \tag{3.2}$$

Proof: Since we assume that $R(\Lambda_{p,k}^{\sim}(G))$ is not closed, we obtain from Lemma 2.5 that $R(\Gamma_{p',1/k^{\vee}}^{\#}(G))$ is not closed in $B_{p',1/k^{\vee}}^{c}(G)$. Due to Remark 2.7/b) we see that there exists a compact set $K \subset G$ such that $F := R(\Gamma_{p',1/k^{\vee}}^{\#}(G)) \cap B_{p',1/k^{\vee}}^{\circ}(K)$ is not closed in $B_{p',1/k^{\vee}}$. Hence one finds an element $g \in \bar{F} \setminus F$. Choose a sequence $\{g_n\} \subset F$ such that $\|g_n - g\|_{p',1/k^{\vee}} \to 0$ as $n \to \infty$. Since $g_n \in B_{p',1/k^{\vee}}^{\circ}(K)$ and since $B_{p',1/k^{\vee}}^{\circ}(K)$ is closed in $B_{p',1/k^{\vee}}$, one sees that $g \in B_{p',1/k^{\vee}}^{\circ}(K)$. Thus g does not belong to $R(\Gamma_{p',1/k^{\vee}}^{\#}(G))$.

The assumptions (2.4), (3.1) imply that

$$\left\|\Gamma_{p',1/k^{\vee}}^{\#}(G)v\right\|_{p',1/k^{\vee}} \geq c\|v\|_{p',1/k^{\vee}} \text{ for all } v \in D(\Gamma_{p',1/k^{\vee}}^{\#}(G)).$$

Choose $v_n \in D(\Gamma_{p',1/k^{\vee}}^{\#}(G))$ such that $\Gamma_{p',1/k^{\vee}}^{\#}(G)v_n = g_n$. Then $\{v_n\}$ is a Cauchy sequence in $B_{p',1/k^{\vee}}$. Choose $v \in B_{p',1/k^{\vee}}$ with $\|v_n - v\|_{p',1/k^{\vee}} \to 0$. Then $v \in B_{p',1/k^{\vee}}(G)$ (since $B_{p',1/k^{\vee}}(G)$ is closed in $B_{p',1/k^{\vee}}$). Furthermore, one sees that, for all $\Phi \in C_0\ (G)$,

$$g(\Phi) = \lim_n (\Gamma_{p',1/k^{\vee}}^{\#}(G)v_n)(\Phi) = \lim_n v_n(L(x,D)\Phi) = v(L(x,D)\Phi)$$

and so $L_{p',1/k^{\vee}}^{\#} v = g$. In addition,

$$v \in B_{p',1/k^{\vee}}(G) \subset E'(\bar{G}) \text{ and } g \in B_{p',1/k^{\vee}}^{\circ}(K) \subset B_{p',1/k^{\vee}}^{c}(G).$$

Suppose that $\operatorname{supp} v \cap \partial G = \Phi$. Then one has

$$\operatorname{supp} v \subset \bar{G} \cap (\mathbb{R}^n \setminus \partial G) = (G \cup \partial G) \cap (\mathbb{R}^n \setminus \partial G) = G.$$

Hence in this case $v \in D(\Gamma_{p',1/k^{\vee}}^{\#}(G))$ and $\Gamma_{p',1/k^{\vee}}^{\#}(G)v = g$. This contradicts the fact that $g \notin R(\Gamma_{p',1/k^{\vee}}^{\#}(G))$. Thus $\operatorname{supp} v \cap \partial G \neq \Phi$, which finishes the proof ∎

Remark 3.2: **a)** Suppose that $L(D) = \sum_{|\sigma| \leq r} a_\sigma D^\sigma$ has constant coefficients and that G is bounded. Then (2.4) and (3.1) are valid. **b)** Suppose that $\Lambda_{p',1/k^{\vee}}^{\sim}(G) = \Lambda_{p',1/k^{\vee}}^{\prime\#}(G)$ (this relation holds for any first-order operator $L(x,D) = \sum_{|\sigma| \leq 1} a_\sigma(x) D^\sigma$ in the case when $p = 2$ and $k = 1$ (cf. [1]). Then one easily sees that (2.4) is valid. **c)** The assumption (2.4) can be replaced by the weaker estimate $\|L'(x,D)\varphi\|_{p',1/k^{\vee}} \geq c\|\varphi\|^{\sim}$ for all $\varphi \in C_0^{\infty}(G)$ (recall that $\|\varphi\|^{\sim} := \inf\{\|\varphi - u\|_{p',1/k^{\vee}} \mid u \in N(L_{p',k^{\vee},G}^{\prime\sim})\}$.

3.2. In the sequel our aim is to seek criteria under which the existence of elements $v \in B_{p',1/k^\vee}(G)$ and $g \in B^c_{p',1/k^\vee}(G)$ satisfying (3.2) implies that $R(\Lambda_{p,k}^{\sim}(G))$ is not closed in $B^{loc}_{p,k}(G)$.

Assume that h is a real-valued function in $C^1(\mathbb{R}^n)$. Furthermore, assume that there exist points x_1 and x_2 with

$$h(x_1) < 0 \text{ and } h(x_2) > 0 \tag{3.3}$$

and

$$(\nabla h)(x) \neq 0 \text{ for any } x \in h^{-1}(0). \tag{3.4}$$

We suppose that $G = h^{-1}(-\infty,0)$. Then $\mathbb{R}^n \setminus \bar{G} = h^{-1}(0,\infty)$ and $\partial G = h^{-1}(0)$. In addition, by (3.3) one has $G \neq \Phi$ and $\partial G = \Phi$. The boundary $\partial G = h^{-1}(0)$ is by (3.4) a regular hypersurface in $\mathbb{R}^n$.

The next theorem yields information about the points, where ∂G can touch supp v (cf. (3.2)).

Theorem 3.3: *Let $G = h^{-1}(-\infty,0)$ as above and let $L(x,D)$ be a partial differential operator with (in $\mathbb{R}^n$) real-analytic coefficients a_σ. Suppose that there exists elements $v \in B_{p',1/k^\vee}(G)$ and $g \in B^c_{p',1/k^\vee}(G)$ such that (3.2) holds. Then one has*

$$L_r(x,(\nabla h)(x)) = 0 \quad \forall\, x \in \operatorname{supp} v \cap \partial G, \text{ where } L_r(x,\xi) = \textstyle\sum_{|\sigma|=r} a_\sigma(x)\xi^\sigma.$$

Proof: a) Suppose that $x \in \operatorname{supp} v \cap \partial G$. Since supp g is a compact subset of G, there exists a constant $d > 0$ such that $\operatorname{dist}(\operatorname{supp} g,\{x\}) \geq d$ and then $L^{\#}_{p',1/k^\vee} v = 0$ in $B(x,d) := \{y \in \mathbb{R}^n \mid |x-y| < d\}$, where $\operatorname{dist}(A,B)$ is the distance between A and B.

b) Suppose that $L_r(x,(\nabla h)(x)) \neq 0$. Then there exists a number $\varepsilon \in (0,d)$ such that $L_r(x,(\nabla h)(x)) \neq 0$ on that patch $U_x := B(x,\varepsilon) \cap \partial G$. Then the patch U_x is a regular C^1-surface and U_x is non-characteristic with respect to $L'(x,D)$ (note that $L'_r(x,\xi) = (-1)^{-r} \times L_r(x,\xi)$). In addition, since $\mathbb{R}^n \setminus \bar{G} = h^{-1}(0,\infty)$, one sees that $L^{\#}_{p',1/k^\vee} v = 0$ in $\{y \in \mathbb{R}^n \mid h(y) > 0\}$. Thus $v = 0$ in some neighbourhood of x (cf. [2, Theorem 8.6.5]), which is a contradiction, because $x \in \operatorname{supp} v$ ∎

A partial converse of Theorem 3.1 is obtained by

Theorem 3.4: *Suppose that $G = h^{-1}(-\infty,0)$, where h obeys (3.3) - (3.4), and that $L(x, D)$ has (in $\mathbb{R}^n$) real-analytic coefficients. Furthermore, assume that*

$$N(L^{\#}_{p',1/k^\vee k_r}) \cap E'(\mathbb{R}^n) = \{0\} \tag{3.5}$$

and that for any $x \in \partial G$ there exists a constant $\varepsilon_x > 0$ such that

$$L_r(y,(\nabla h)(y)) \neq 0 \quad \text{on} \quad (B(x,\varepsilon_x)\setminus\{x\}) \cap \partial G, \tag{3.6}$$

where we denote $k_s(\xi) = (1+|\xi|^2)^{s/2}$). Then the existence of elements $v \in B_{p',1/k^\vee}(G)$

and $g \in B^{c}_{p',1/k^{\vee}}(G)$ *satisfying* (3.2) *implies that the range* $R(\Lambda^{\sim}_{p,kk_r}(G))$ *is not closed in* $B^{loc}_{p,kk_r}(G)$.

Proof: Due to Lemma 2.5 it suffices to verify that the range $R(\Gamma_{p',1/k^{\vee}k_r}(G))$ is not closed in $B^{c}_{p',1/k^{\vee}k_r}(G)$, that is, $R(\Gamma^{\#}_{p',1/k^{\vee}k_r}(G)) \cap B^{o}_{p',1/k^{\vee}k_r}(K)$ is not closed with some compact $K \subset G$.

Suppose that $v \in B_{p',1/k^{\vee}}(G)$ and $g \in B^{c}_{p',1/k^{\vee}}(G)$ are such that $L^{\#}_{p',1/k^{\vee}} v = g$ and $\operatorname{supp} v \cap \partial G \neq \emptyset$. Choose $x \in \operatorname{supp} v \cap \partial G$. Due to the assumption there exists $\varepsilon_x > 0$ such that $L_r(y,(\nabla h)(y)) \neq 0$ on $(B(x,\varepsilon_x)\setminus\{x\}) \cap \partial G$. Hence, similarly to the proof/part b) of Theorem 3.3, one obtains that $v = 0$ in some neighbourhood U_x of $(B(x,\varepsilon_x)\setminus\{x\}) \cap \partial G$. Choose $\vartheta \in C_o^{\infty}(B(x,\delta))$ such that $\vartheta(x) \equiv 1$ in $B(x,\delta/2)$, where $\delta := \{\varepsilon_x/2, \operatorname{dist}(\operatorname{supp} g, \partial G)\} > 0$. Define distributions w_n by $w_n(\varphi) = (\vartheta v)(\varphi(\cdot) + 1/n(\nabla h)(x))$ (we translate ϑv in the direction of the vector $-1/n(\nabla h)(x)$; note that $-(\nabla h)(x)$ is pointing to G and that $(\nabla h)(x)$ is the normal of ∂G at x). Then one sees that, for n large enough, the inclusions

$$w_n \in B^{c}_{p',1/k^{\vee}k_r}(G) \quad \text{and} \quad g_n := \Gamma^{\#}_{p',1/k^{\vee}k_r}(G) w_n \in B^{c}_{p',1/k^{\vee}k_r}(G)$$

hold. Note that $g_n = (L^{\#}_{p',1/k^{\vee}k_r} w)_n$, where $w = \vartheta v$ and where $(L^{\#}_{p',1/k^{\vee}k_r} w)_n$ is similarly defined (via translation) as w_n. Furthermore, one finds that $g_n \to g := L^{\#}_{p',1/k^{\vee}k_r}(\vartheta v)$ in $B^{c}_{p',1/k^{\vee}k_r}(G)$. Since $N(L^{\#}_{p',1/k^{\vee}k_r}) \cap E'(\mathbb{R}^n) = \{0\}$ and since $\vartheta v \notin B^{c}_{p',1/k^{\vee}k_r}(G)$, one sees that $R(\Gamma^{\#}_{p',1/k^{\vee}k_r}(G))$ is not closed in $B^{c}_{p',1/k^{\vee}k_r}(G)$ ∎

Remark 3.5: a) Suppose that $L(D)$ has constant coefficients. Then the relation (3.5) is valid. **b)** The condition (3.6) is in many particular cases superflous, as we shall make explicit below (Theorem 3.6). **c)** Also the Theorem 2.5 in [5,p. 367] can be applied (as above) to the study of the closedness of $R(\Lambda^{\sim}_{p,kk_r}(G))$. **d)** In the case of constant coefficients, the assertion in Theorem 3.4 can be replaced by the following: The range $R(\Lambda^{\sim}_{p,k}(G))$ is not closed for any $(p,k) \in (1,\infty) \times K$ (cf. the Introduction). **e)** Suppose that G and $L(x,D)$ are as in Theorem 3.3 and that $L_r(x,(\nabla h)(x)) \neq 0$ for all $x \in \partial G$. Then there do not esxist elements $v \in B_{p',1/k^{\vee}}(G)$ and $g \in B^{c}_{p',1/k^{\vee}}(G)$ such that (3.2) is valid (cf. Theorem 3.3). **f)** We also remark that (under the assumptions of Theorem 3.4) the points where $\operatorname{supp} v$ can touch ∂G are isolated points of ∂G (cf. the proof of Theorem 3.4)

3.3. We consider some examples.

A. Let $L(x,D) = -i(x_1 D_1 + x_2 D_2)$ and $h(x_1,x_2) = 1 - x_1^2 - x_2^2$. Then one sees that $G = B(0,1)$, $(\nabla h)(x) = (-2x_1, -2x_2) \neq 0$ for any $x \in h^{-1}(0)$ and $h(0,0) < 0$, $h(2,0) > 0$. In addition, one gets

$$L_r(x,(\nabla h)(x)) = L_r(x,(-2x_1, -2x_2)) = -2i(x_1^2 + x_2^2) = -2i$$

for $x \in h^{-1}(0)$ and

$$\operatorname{Re}(L(x,D)\varphi,\varphi)_o = 1/2\big((-i(x_1D_1 + x_2D_2)\varphi,\varphi)_o + i\overline{(\varphi, D_1(x_1\varphi) + D_2(x_2\varphi))_o}\big) = \|\varphi\|_o^2$$

for all $\varphi \in C_0$ (here $(\varphi,\check{\psi})_0 = \int_{\mathbb{R}^n} \varphi(x)\overline{\check{\psi}(x)}\,dx$). In virtue of Theorem 3.1, Remark 3.2/b and Remark 3.5/e one sees that the range $R(\Lambda^{\sim}_{2,1}(G))$ is closed in $L_2^{\mathrm{loc}}(G)$.

B. Suppose that $L(x,D)$ is as in Example A and that $G = \mathbb{R}^2 \setminus B(0,1)$. Then due to Remark 3.5/e one sees that there do not exist elements $v \in B_{p',1/k^{\vee}}(G)$ and $g \in B_{p',1/k^{\vee}}(G)$ such that (3.3) is valid. It is remarkable to note that, when $L(D)$ has constant coefficients, when $L(D)$ is non-elliptic and when $G = \mathbb{R}^2 \setminus \overline{B}(0,1)$, there exist elements $v \in C_0^{\infty}(\mathbb{R}^n)$, $g \in C_0^{\infty}(\mathbb{R}^n)$ such that $L'(D)v = g$ (in $\mathbb{R}^2$), $\operatorname{supp} v \subset \overline{G}$ and $\operatorname{supp} v \cap \partial G \neq \emptyset$ (cf. [2: Theorem 8.6.7] and Figure 1). Hence by Theorem 3.4 one gets that the range $R(\Lambda^{\sim}_{p,k}(G))$ is

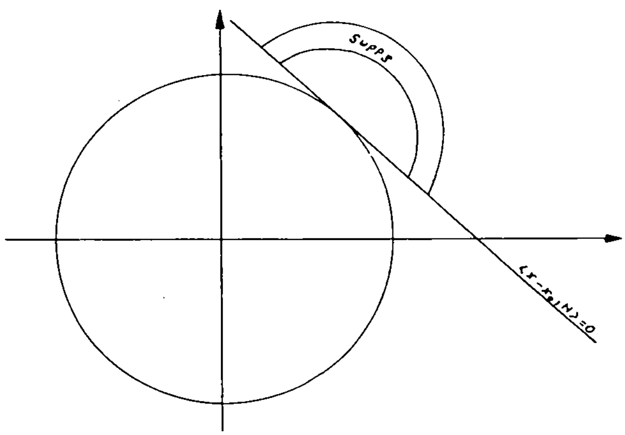

Figure 1

not closed in $B_{p,k}^{\mathrm{loc}}(G)$. Here one must note that any $v \in C_0^{\infty}$ with $\operatorname{supp} v \subset \overline{G}$ belongs to $B_{p',1/k^{\vee}}(G)$ (since in this case $B^{\cdot}_{p',1/k^{\vee}}(G) = B^{\circ}_{p',1/k^{\vee}}(G)$). In addition, one sees that (3.5) - (3.6) are valid.

3.4. We finally consider the operator $L(D)$ with constant coefficients. Suppose (as above) that $h \in C^1(\mathbb{R}^n)$ is such that (3.3) - (3.4) hold and that $G = h^{-1}(-\infty,0)$. Denote $S(0,1) = \{x \in \mathbb{R}^n \mid |x| = 1\}$. We show the following

Theorem 3.6: *Let $G = h^{-1}(-\infty,0)$ be a bounded set and let $L(D)$ be an operator with constant coefficients such that the set $C = \{N \in S(0,1) \mid L_r(N) = 0\}$ is finite. Then the range $R(\Lambda^{\sim}_{p,k}(G))$ is closed if and only if there do not exist elements $v \in B_{p',1/k^{\vee}}(G)$ and $g \in B^{c}_{p',1/k^{\vee}}(G)$ such that $L^{\#}_{p',1/k^{\vee}}\, v = g$ and $\operatorname{supp} v \cap \partial G \neq \emptyset$ (see (3.2)).*

Proof: A. Suppose that $R(\Lambda^{\sim}_{p,k}(G))$ is not closed. Then, due to Theorem 3.1, the required elements v and g exist (cf. Remark 3.2/a). Hence it suffices to show that, if the range $R(\Lambda^{\sim}_{p,k}(G))$ is closed, then the elements v and g satisfying (3.2) do not exist.

B. Suppose that the range $R(\Lambda^{\sim}_{p,k}(G))$ is closed. Then $R(\Lambda^{\sim}_{p,kk_r}(G))$ is also closed and so $R(\Gamma^{\#}_{p',1/k^{\vee}k_r}(G))$ is closed in $B^{c}_{p',1/k^{\vee}k_r}(G)$ (cf. Introduction and Lemma 2.5). Assume that v and g owning (3.2) exist. This leads to a contradiction as follows: Due to Theorem 3.3

$$L_r(x,(\nabla h)(x)) = 0 \quad \text{for all } x \in \operatorname{supp} v \cap \partial G. \tag{3.7}$$

Choose $x_0 \in \operatorname{supp} v \cap \partial G$ and define $F_{x_0} = \{x \in \partial G \mid (\nabla h)(x) = (\nabla h)(x_0)\}$. Let C_{x_0} be the connected component of F_{x_0} containing x_0. Since C is finite there exists (by (3.8)) a constant $\varepsilon > 0$ such that $v = 0$ in $G_\varepsilon = \{x \in \partial G \setminus C_{x_0} \mid \operatorname{dist}(x, C_{x_0}) < \varepsilon\}$. Define numbers d and δ by $d = \operatorname{dist}(\operatorname{supp} f, \partial G) > 0$ and $\delta = \min\{d, \varepsilon\}$. The set $U_\delta = \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, C_{x_0}) < \delta\}$ is open and $F_\delta = \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, C_{x_0}) \le \delta/2\}$ is a compact set of U_δ. Choose a function $\vartheta \in C_0^\infty(U_\delta)$ such that $\vartheta(x) = 1$ in F_δ and define $w_n(\psi) = (\vartheta v)(\psi((\cdot) + 1/_n N_0))$, where $N_0 = (\nabla h)(x_0)$. Then one sees that, for n large enough, one has $w_n \in E'(G)$ and

$$\Gamma^{\#}_{p',1/k^\vee k_r}(G) w_n \to L^{\#}_{p',1/k^\vee k_r}(\vartheta v) = \sum_\alpha ((-1)^{|\alpha|}/\alpha!) D^\alpha \vartheta L^{(\alpha)}(D) v$$

in $B_{p',1/k^\vee k_r}(G)$. Hence $R(\Gamma^{\#}_{p',1/k^\vee k_r}(G))$ is not closed (recall: $N(L^{\#}_{p',1/k^\vee k_r}) \cap E'(\mathbb{R}^n) = \{0\}$), which is a contradiction ∎

Remark 3.7 : a) The boundedness of G in theorem 3.6 is not essential (which fact we shall not deal with in detail). **b)** One sees that the *heat operator* $L(D) = -iD_n - \sum_{j=1}^{n-1} D_j^2$ and the *wave operator* $D_1^2 - D_2^2$ satisfy, for example, the assumptions of Theorem 3.6.

REFERENCES

[1] FRIEDRICHS, K.: *On the identity of weak and strong extensions of differential operators*. Trans. Amer. Math. Soc. **55** (1944), 132 - 151.

[2] HÖRMANDER, L.: *The Analysis of Linear Partial Differential Operators* I. Berlin - Heidelberg - New York : Springer - Verlag 1983.

[3] HÖRMANDER, L.: *The Analysis of Linear Partial Differential Operators* II. Berlin - Heidelberg - New York : Springer - Verlag 1983.

[4] KATO, T.: *Perturbation Theory for Linear Operators*. Berlin - Heidelberg - New York : Springer - Verlag 1966.

[5] TAYLOR, M.: *Pseudodifferential Operators*. Princeton, N.J.: Princeton Univ. Press 1981.

[6] TERVO, J.: *On Solvability of Linear Partial Differential Equations in Local Spaces* $B^{loc}_{p,k}(G)$. Z. Anal. Anw. **7** (1988), 99 - 112.

[7] TERVO, J.: *On the existence and regularity of solutions of linear equations in spaces* $H_k(G)$. Ann. Pol. Math. (to appear).

[8] TREVES, F.: *Locally Convex Spaces and Linear Partial Differential Equations*. Berlin - Heidelberg - New York : Springer - Verlag 1967.

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