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A Note on Benford's Law for Second Order Linear Recurrences with Perodical Coefficients

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The solutions of second order linear recurrences with periodical coefficients are shown to obey Benford's law.

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1. Let (u_n) be a sequence of real numbers which satisfy a *second order linear recurrence*

$$u_{n+2} = a_{n+2} u_{n+1} + b_{n+2} u_n , \quad n \geq 0, \tag{1}$$

where (a_n) and (b_n) are given real sequences with a common *period* r, i.e., $a_{n+r} = a_n$ and $b_{n+r} = b_n$.

Linear recurrences of such type arise, e.g., in the theory of continued fractions. If ω is a quadratic irrational, then the numerators and the denominators of the n-th convergent p_n/q_n of the continued fraction of ω fulfil the relation (1) with $b_n = 1$ and a periodic positive integer sequence (a_n) (cf., e.g., [4: Satz 1/p.5 and Satz 28 /p.52]). Linear recurrences with periodical coefficients are also treated in [1: p.148–150].

The sequence (u_n) is said to obey *Benford's law* if

$$N^{-1} \# \{ n : 1 \leq n \leq N , 1 \leq \text{mantissa of } u < x \} \to \lg x \quad (N \to \infty)$$

for $1 < x < 10$. Here $\lg x = \log_{10} x$ and $\# A$ is the number of the elements of A. More specially this means that

$$N^{-1} \# \{ n : 1 \leq n \leq N , u_n \text{ has leading digit } k \} \to \lg(k+1) - \lg k ,$$

for $k = 1,\ldots,9$ (cf., e.g., [9] for a survey on Benford's law). The sequence (u_n) obeys Benford's law if and only if the sequence $(\lg|u_n|)$ is uniformly distributed mod 1, put $\lg 0 = 1$ in this connection. For the uniform distribution mod 1 of sequences cf., e.g., [6].

In case of $r = 1$, i.e., $a_n = a$ and $b_n = b$, the solutions (u_n) of (1) obey Benford's law under weak suppositions. This case was extensively treated in the literature (cf., e.g., [3] for a survey and [8]). In the following we shall show that the solutions (u_n) of (1) obey Benford's law also in the case $r > 1$ by reducing this case to that of $r = 1$. As a corollary we obtain that the numerators (p_n) and the denominators (q_n) of the n-th convergent p_n/q_n of the continued fraction of a quadratic irrational ω obey Benford's law. Previous proofs of this fact can be found in [2], where a result from [5] is applied, and in [10]. Our result generalizes this assertion on continued fractions.

2. We put $z_{n,i} = u_{nr+i}$ for $0 \le i \le r-1$. Furthermore we introduce the shift operator S and write $z_{n+1,i} = Sz_{n,i}$. Then the linear recurrence (1) is equivalent to the system

$$\begin{array}{llr}
c_1 z_{n,0} + a_2 z_{n,1} - z_{n,2} & = 0, & \\
c_2 z_{n,0} + b_3 z_{n,1} + a_3 z_{n,2} - z_{n,3} & = 0, & \\
\vdots & \vdots & (2) \\
c_{r-2} z_{n,0} \qquad + b_{r-1} z_{n,r-3} + a_{r-1} z_{n,r-2} - z_{n,r-1} & = 0, & \\
c_{r-1} z_{n,0} \qquad + b_r z_{n,r-2} + a_r z_{n,r-1} & = 0, & \\
c_r z_{n,0} - S z_{n,1} \qquad + b_1 z_{n,r-1} & = 0, &
\end{array}$$

where $c_1 = b_2, c_2 = \dots = c_{r-2} = 0, c_{r-1} = -S, c_r = a_1 S$. Now we multiply the r-th equation in this system by the signed minor of the element c_r of the matrix of coefficients of this system. After summing up all the arising equations we arrive at

$$\begin{vmatrix}
b_2 & a_2 & -1 & & & 0 \\
 & b_3 & a_3 & -1 & & \\
 & & \ddots & \ddots & \ddots & \\
0 & & & b_{r-1} & a_{r-1} & -1 \\
-S & & & & b_r & a_r \\
a_1 S & -S & & & & b_1
\end{vmatrix} z_{n,0} = 0 .$$

By expanding the determinant with respect to powers of S we obtain finally

$$z_{n+2,0} = D_r z_{n+1,0} + E_r z_{n,0} , \tag{3}$$

where

$$E_r = (-1)^{r-1} b_1 \cdots b_r \quad \text{and} \quad D_r = a_1 A_r + b_1 A_{r-1} + B_r ,$$

with

$$A_r = \begin{vmatrix}
a_2 & -1 & & & 0 \\
b_3 & a_3 & -1 & & \\
 & \ddots & \ddots & \ddots & \\
 & & b_{r-1} & a_{r-1} & -1 \\
0 & & & b_r & a_r
\end{vmatrix}, \qquad B_r = \begin{vmatrix}
b_2 & 0 & & & 0 \\
b_3 & a_3 & -1 & & \\
 & \ddots & \ddots & \ddots & \\
 & & b_{r-1} & a_{r-1} & -1 \\
0 & & & b_r & a_r
\end{vmatrix} . \tag{4}$$

The determinants A_r and B_r are so-called *continuants* (cf.[7:p.8]).If we multiply the equations of (2) with the signed minors of the i-th column of the matrix of coefficients of (2) and after that sum up,then we obtain

$$z_{n+2,i} = D_r z_{n+1,i} + E_r z_{n,i} \tag{5}$$

for $0 \le i \le r-1$ in generalization of (3).

3. Now we are in a position to formulate our main result.

Theorem 1: *Let λ_1, λ_2 be real roots of the common characteristic equation*

$$\lambda^2 = D_r \lambda + E_r \tag{6}$$

of the linear recurrences (5);assume $|\lambda_1| \ge |\lambda_2|$.If $\lg |\lambda_1|$ is irrational and $u_n \neq 0$ for $n \ge n_0$,then the sequence (u_n) obeys Benford's law.

Proof: Since the sequences $(z_{n,i})$ fulfil (5),they all obey Benford's law as well-known (cf.,e.g.,[3]).But then the sequence (u_n) obeys Benford's law, likewise ∎

Remarks: **1.**If we set $A_0 = 0, A_1 = 1, B_0 = 1, B_1 = 0$,then we get from (4)

$$A_i = a_i A_{i-1} + b_i A_{i-2} \quad , \quad B_i = a_i B_{i-1} + b_i B_{i-2} \tag{7}$$

for $i \ge 2$. Thus we find that $A_i, B_i \ge 1, D_i \ge 3$ if a_i, b_i are positive integers. **2.**In order to derive yet another expression for D_r we rewrite (7) in the form

$$\begin{pmatrix} A_i & A_{i-1} \\ B_i & B_{i-1} \end{pmatrix} = \begin{pmatrix} A_{i-1} & A_{i-2} \\ B_{i-1} & B_{i-2} \end{pmatrix} \begin{pmatrix} a_i & 1 \\ b_i & 0 \end{pmatrix}$$

(cf.[7:p.13]).Then we obtain

$$D_r = \operatorname{trace} \begin{pmatrix} a_1 & 1 \\ b_1 & 0 \end{pmatrix} \begin{pmatrix} A_r & A_{r-1} \\ B_r & B_{r-1} \end{pmatrix} = \operatorname{trace} \prod_{i=1}^{r} \begin{pmatrix} a_i & 1 \\ b_i & 0 \end{pmatrix}$$

(cf.[2:Equality (2.7)]). **3.**The proof of Theorem 1 works also if some $b_i = 0$ and therefore $E_r = 0$.In this case $D_r \neq 0$ must hold since otherwise $u_n = 0$ for $n \ge 2r$. **4.**The line of reasoning can easily be generalized to linear recurrences of higher order.

4. From the main result we conclude the following

Corollary : *Let p_n/q_n be the n-th convergent of the continued fraction of a quadratic irrational ω.Then the sequences (p_n) and (q_n) obey Benford's law.*

Proof: The sequences (p_n) and (q_n) satisfy recurrences of type (1) with $b_n = 1$ and positive integers a_n .The case $r = 1$ being treated in [3: Theorem 3.1] we can restrict to that of $r > 1$.But then $D_r \ge 3$ according to Remark 1. Therefore the roots

$$\lambda_1 = \left(D_r + \sqrt{D_r^2 - 4(-1)^r}\right)/2 \ , \quad \lambda_2 = \left(D_r - \sqrt{D_r - 4(-1)^r}\right)/2$$

of (6) are real and irrational. If $\lambda_1 = 10^{p/q}$ with integers p and q, then $\lambda_1^q = 10^p$ must be an integer which is an obvious contradiction, apply the binomial theorem ∎

5. Finally we shortly treat the case of complex conjugate roots of (6), i.e., $D_r^2 - 4E_r < 0$. Let $\beta = \sqrt{E_r}$, $\cos 2\pi\gamma = D_r/2\beta$.

Theorem 2 : *If* $1, \lg \beta$, *and* γ *are linearly independent over the rationals and* $u_n \neq 0$ *for* $n \geq n_0$, *then the sequence* (u_n) *obeys Benford's law.*

Proof : Apply Corollary 2 in [8] or Theorem 2.1 in [3] ∎

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