

On Imbeddings between Weighted Orlicz Spaces

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Es werden Bedingungen für Gewichte ϱ und σ untersucht, die notwendig und hinreichend für die Gültigkeit der Einbettung für die Orlicz-Räume $L_{\varrho,\sigma} \hookrightarrow L_{p,\sigma}$ sind. Wir benutzen zwei Methoden: die erste ist ein Limitprozeß, der von den L_p -Räumen zur Einbettung von gewichteten Zygmund-Klassen in den gewichteten Raum L_1 führt, die zweite Methode beruht auf der Dualität in gewichteten Orlicz-Räumen, dem Index der Youngschen Funktion und gibt einen allgemeinen Einbettungssatz.

Найдены необходимые и достаточные условия на весовые функции ϱ и σ для того, чтобы существовало вложение пространств Орлича с весом $L_{\varrho,\sigma} \hookrightarrow L_{p,\sigma}$. Применяются два метода: первый является предельным процессом и ведет от L_p -пространств к вложению весовых классов Зигмунда в весовое пространство L_1 , второй подход применяет понятие двойственности в весовых пространствах Орлича, индекса Н-функции и дает общую теорему вложения.

Necessary and sufficient conditions are given for weight functions ϱ and σ , which guarantee validity of the imbedding between the Orlicz spaces $L_{\varrho,\sigma} \hookrightarrow L_{p,\sigma}$. We make use of two methods: the first of them is a limit procedure leading to the imbedding of weighted Zygmund classes into weighted L_1 , the second method employs the concept of duality in weighted Orlicz spaces, index of a Young function and results in a general imbedding theorem.

1. Introduction and preliminaries

This paper deals with necessary and sufficient conditions for imbeddings between weighted Orlicz spaces. Actually, it is an amalgam of results for weighted Lebesgue spaces (the particular case of continuous weights was solved by AVANTAGGIATI [1] and the general problem with measures by KABAILA [3]), ideas about limit treatment of necessary and sufficient conditions on weight functions [5, 9], and inequalities linking modulars and norms. We get a general imbedding theorem with no restrictions on the growth of the Young functions involved.

In [5, 9] we have considered two weight weak type inequalities for the maximal operator in Zygmund classes $L(1 + \log^+ L)^K$. It turns out that the necessary and sufficient condition on weights can be obtained by considering a limit case of the Muckenhoupt A_p condition. Rather surprisingly, an analogous procedure applied to the condition

$$\int_{\Omega} (\sigma(x)/\varrho(x))^{p/(p-1)} \varrho(x) dx < \infty, \quad 1 < p < \infty,$$

which is necessary and sufficient for the imbedding $L_{p,\varrho} \hookrightarrow L_{1,\sigma}$ turns out to provide us with a necessary and sufficient condition for the imbedding $L(1 + \log^+ L)_\varrho^K$ into $L_{1,\sigma}$. (As usual, \hookrightarrow stands for continuous imbedding.) This is the main result of Section 2.

In Section 3 we present a necessary and sufficient condition for the imbedding between weighted Orlicz spaces $L_{Q,\varrho} \hookrightarrow L_{P,\sigma}$. Roughly speaking, if QP^{-1} is a Young function and N is the complementary function to QP^{-1} , then $L_{Q,\varrho} \hookrightarrow L_{P,\sigma}$ iff

$$\int_{\Omega} N(\mu\sigma(x)/\varrho(x)) \varrho(x) dx < \infty \quad \text{for some } \mu > 0.$$

An alternative technique of indices of Young functions is employed in Section 4. We obtain various relations concerning imbeddings in question under certain restrictive assumptions (as Δ_2, Δ_3 , conditions on indices etc.).

Throughout the paper, Ω will be a measurable subset of \mathbb{R}^n . The symbols ϱ and σ will denote *weights* in Ω , i.e. measurable and a.e. positive functions in Ω , $L_{p,\varrho}$ will stand for the usual weighted Lebesgue space with norm $\|\cdot\|_{p,\varrho}$. The reader is supposed to be familiar with the basic facts from the theory of modular and Orlicz spaces (we refer, e.g., to [8, 4]). Here, we shall work with (real) weighted Orlicz spaces in the following sense: For any Young function M , we consider the *modular*

$$m_{\varrho}(f, M) = \int_{\Omega} M(|f(x)|) \varrho(x) dx,$$

the *weighted Orlicz space* $L_{M,\varrho}$ is the linear hull of the class of functions with finite modular. This space is endowed with the *Luxemburg norm*

$$\|f\|_{M,\varrho} = \inf \{ \lambda > 0; m_{\varrho}(f/\lambda, M) \leq 1 \}.$$

Recall that the Young function M satisfies the Δ_2 condition (sometimes we write $M \in \Delta_2$) if $M(2t) \leq CM(t)$ for large t . We shall say that M satisfies the Δ_2 condition globally on $(0, \infty)$ if $M(2t) \leq CM(t)$ for each $t \geq 0$. If N is the *complementary Young function* with respect to M , i.e.

$$N(t) = \sup_{\tau \geq 0} (t\tau - M(\tau)), \quad t \geq 0,$$

then

$$t \leq M^{-1}(t) N^{-1}(t) \leq 2t, \quad t \geq 0, \tag{1.1}$$

and the spaces $L_{M,\varrho}$ and $L_{N,\varrho}$ are naturally associated with the duality

$$\langle f, g \rangle = \int_{\Omega} f(x) g(x) \varrho(x) dx, \quad f \in L_{M,\varrho}, \quad g \in L_{N,\varrho},$$

giving birth to the so-called *Orlicz norm*; on $L_{M,\varrho}$ it is

$$\|f\|_{M,\varrho} = \sup \left\{ \int_{\Omega} f(x) g(x) \varrho(x) dx; m_{\varrho}(g, N) \leq 1 \right\}.$$

It is easy to check (quite analogously as in the case of non-weighted spaces) that

$$\|f\|_{M,\varrho} \leq \|f\|_{M,\varrho} \leq 2\|f\|_{M,\varrho}.$$

We shall denote by $E_{M,\varrho}(\Omega)$ the closure in $L_{M,\varrho}(\Omega)$ of the set of all bounded measurable functions with bounded support in Ω . Then $L_{N,\varrho}$ is the dual space of $E_{M,\varrho}$.

Let us still recall that a Young function M satisfies the Δ_3 condition ($M \in \Delta_3$) iff there is $K > 0$ such that $tM(t) \leq M(Kt)$ for large t . If $M \in \Delta_3$, then $N \in \Delta_2$ and

$$N(t) \leq CtM^{-1}(Ct), \quad t \geq T, \tag{1.2}$$

for some $C, T \geq 0$.

The basic properties of weighted spaces used here do not go beyond the limits of an easy verification.

We only mention several useful assertions on general modular spaces (for the concept see, e.g., [8]). The modulars in question need not be necessarily strictly convex so that any weighted L_1 space is included.

Let (X_i, m_i) be a modular space with the (Luxemburg) norm $\|\cdot\|_i$ ($i = 1, 2$) and let $T: X_1 \rightarrow X_2$ be a sublinear operator. It is clear that the modular inequality $m_2(Tf) \leq C m_1(f)$, $f \in X_1$, implies its norm counterpart $\|Tf\|_2 \leq C \|f\|_1$, $f \in X_1$. The converse is not true, however, we have (the proof is easy and therefore omitted)

Lemma 1.1: *Let T, X_1, X_2 be as above. Then the following two assertions are equivalent:*

- (i) *There exists $C \geq 1$ such that $\|Tf\|_2 \leq C \|f\|_1$.*
- (ii) *There exists $K \geq 1$ such that $m_2(Tf/K) \leq 1$ provided $m_1(f) \leq 1$.*

Moreover, the best constants in (i) and (ii) coincide.

Later, we shall also make use of

Lemma 1.2: *Let T, X_1, X_2 be as above. Let h be a positive function defined on $(0, \infty)$ and bounded in some right neighbourhood of zero. If*

$$\|Tf\|_2 \leq h(m_1(f)), \quad f \in X_1, \tag{1.3}$$

then there is $C \geq 1$ such that for all f

$$\|Tf\|_2 \leq C \|f\|_1. \tag{1.4}$$

Proof: Suppose (1.4) is not true. Then there is a sequence $\{f_n\} \subset X_1$ such that $\|f_n\|_1 \rightarrow 0$ and $\|Tf_n\|_2 \rightarrow \infty$. For large n , $m_1(f_n) \leq \|f_n\|_1 \leq 1$, whence $m_1(f_n) \rightarrow 0$, which contradicts with (1.3) ■

As currently adopted, different constants are denoted by the same letters if no confusion can arise.

2. The limiting approach

In what follows, p and p' are conjugate exponents, i.e. $p \in (1, \infty)$, $p' = p/(p - 1)$. Let us mention the elementary and frequently used formulas

$$p'/p = p' - 1 = 1/(p - 1).$$

We start with a special case of a weighted imbedding theorem which appeared in AVANTAGGIATI's paper [1] for continuous weights and generally for measures in KABAILA [3]. For the sake of completeness, we present a proof based on the latter paper, which will, additionally, provide us with a relation between important constants.

Theorem 2.1: *The following statements are equivalent:*

- (i) $\int_{\Omega} |f(x)| \sigma(x) dx \leq C_p(\sigma, \varrho) \left(\int_{\Omega} |f(x)|^p \varrho(x) dx \right)^{1/p}$;
- (ii) $B_p(\sigma, \varrho) = \int_{\Omega} (\sigma(x)/\varrho(x))^{p'} \varrho(x) dx < \infty$.

Moreover, the best $C_p(\sigma, \varrho)$ in (i) equals $(B_p(\sigma, \varrho))^{1/p'}$.

Proof: Let (i) hold. Putting $f(x) = (\sigma(x)/\varrho(x))^{1/(p-1)}$ we get

$$\begin{aligned} \int_a^\infty \left(\frac{\sigma(x)}{\varrho(x)}\right)^{p'} \varrho(x) dx &= \int_a^\infty |f(x)| \sigma(x) dx \\ &\leq C_p(\sigma, \varrho) \left(\int_a^\infty |f(x)|^p \varrho(x) dx\right)^{1/p} \\ &= C_p(\sigma, \varrho) \left(\int_a^\infty \left(\frac{\sigma(x)}{\varrho(x)}\right)^{p'} \varrho(x) dx\right)^{1/p}. \end{aligned}$$

The last term is finite as for each $g \in L_p(\varrho)$

$$\begin{aligned} \left| \int_a^\infty g(x) \frac{\sigma(x)}{\varrho(x)} \varrho(x) dx \right| &= \left| \int_a^\infty g(x) \sigma(x) dx \right| \\ &\leq \int_a^\infty |g(x)| \sigma(x) dx \leq C_p(\sigma, \varrho) \left(\int_a^\infty |g(x)|^p \varrho(x) dx\right)^{1/p}, \end{aligned}$$

so that the function σ/ϱ represents a bounded linear functional on $L_{p,\varrho}$ and belongs to $L_{p',\varrho}$. This yields $\int_a^\infty (\sigma(x)/\varrho(x))^{p'} \varrho(x) dx \leq C_p(\sigma, \varrho)^{p'}$, hence $B_p(\sigma, \varrho) \leq C_p(\sigma, \varrho)^p$.

Conversely, if $B_p(\sigma, \varrho)$ is finite, then

$$\begin{aligned} \int_a^\infty |f(x)| \sigma(x) dx &= \int_a^\infty |f(x)| \sigma(x) \varrho^{1/p}(x) \varrho^{-1/p}(x) dx \\ &\leq \|f\|_{p,\varrho} \left(\int_a^\infty (\sigma(x)/\varrho(x))^{p'} \varrho(x) dx\right)^{1/p'} = (B_p(\sigma, \varrho))^{1/p'} \|f\|_{p,\varrho}. \end{aligned}$$

Therefore, (i) holds and $C_p(\sigma, \varrho) \leq (B_p(\sigma, \varrho))^{1/p'}$ ■

Let $1 \leq K < \infty$. We introduce the Young functions

$$\Phi_K(t) = \begin{cases} t^2, & 0 \leq t \leq 1, \\ t(1 + \log t)^K, & t > 1, \end{cases}$$

$$F_K(t) = \sum_{j=2}^\infty \frac{t^j}{j!} \left(\frac{1}{j-1}\right)^{j(K-1)}, \quad t \geq 0.$$

We shall denote by Ψ_K and G_K the complementary Young functions to Φ_K and F_K , respectively.

The rather lengthy and tedious arithmetical proof of the following lemma is postponed to Appendix.

Lemma 2.2: *There exists a constant $\beta_K \geq 1$ such that*

$$F_K(t) > \exp(t^{1/K}/\beta_K) - t^{1/K}/\beta_K - 1, \quad t \geq 1.$$

It is worth to notice that the function $\exp(t^{1/K}) - t^{1/K} - 1$ is equivalent to Ψ_K for large values of t . Indeed, it is easy to verify that $\Phi_K(t)/t < C\Phi_K(\sqrt{t})/\sqrt{t}$, $t \geq 1$, with, say, $C = 3^K$. Thus, by [4, Theorem 6.8], Ψ_K satisfies the Δ_3 condition and, on using (1.2), Ψ_K^{-1} is equivalent to $(1 + \log t)^K$ for large t .

Lemma 2.3: The functions F_K, Φ_K, Ψ_K satisfy

- (i) $\Phi_K(t) \geq t$ for $t \geq 1$;
- (ii) $2F_K(t) \geq t$ for $t \geq 1$;
- (iii) $\Psi_K(t) \leq F_K(t)$ for $0 \leq t \leq 1$;
- (iv) $G_K(t) \leq C_K \Phi_K(t)$ for $t \geq 1$;
- (v) $\Phi_K(t) \leq C_K t^p (p - 1)^{-K}$ for $t \geq 0$ and $p \in (1, 2)$.

Proof: The statements (i)–(iii) follow immediately. We prove (iv). Note that $F_K \in \Delta_3$, thus, in virtue of (1.2), $G_K(t) \leq CtF_K^{-1}(Ct)$ for some $C \geq 1$ and all large t . Making use of Lemma 2.2 we have $F_K(t) \geq \alpha_K \exp(t^{1/K} \beta_K)$ with some $\alpha_K > 0, \beta_K$ from Lemma 2.2, and for every $t \geq 1$, so that passing to the inverse functions,

$$G_K(t) \leq Ct\beta_K^K \log^K(Ct/\alpha_K) \leq C_K t \log^K t \leq C_K \Phi_K(t) \text{ for } t \geq 1.$$

As to (v), it is easy to check that $\max_{t>0} \Phi_K(t) t^{-p} \leq K^K (p - 1)^{-K}$, so that we can put $C_K = K^K$ ■

Lemma 2.4: There exists a constant δ_K such that the (modified) Young inequality $ab \leq \delta_K (F_K(a) + \Phi_K(b))$ holds for all $a, b \geq 0$.

Proof: We shall make use of Lemma 2.3/(i)–(iv). If $a, b \leq 1$, then $ab \leq \Phi_K(a) + \Psi_K(b) \leq \Phi_K(a) + F_K(b)$. If $a < 1, b \geq 1$, then $ab < b \leq 2F_K(b) \leq \Phi_K(a) + 2F_K(b)$. If $a \geq 1, b < 1$, then $ab < a \leq \Phi_K(a) + F_K(b)$. Finally, if $a, b > 1$, then, according to the Young inequality, $ab \leq G_K(a) + F_K(b) \leq C_K \Phi_K(a) + F_K(b)$. Putting $\delta_K = \max\{2, C_K\}$ concludes the proof ■

Lemma 2.5: There exists a constant θ_K such that

$$\|f\|_{\Phi_{K,\varrho}} \leq \frac{\theta_K}{(p - 1)^K} \|f\|_{p,\varrho}, \quad f \in L_{p,\varrho}, \quad p \in (1, 2).$$

Proof: The thesis follows directly from Lemma 2.3/(v) and from the definition of the Luxemburg norm ■

Theorem 2.6: The following conditions on σ, ϱ are equivalent:

- (i) $L_{\Phi_{K,\varrho}}(\Omega) \hookrightarrow L_{1,\sigma}(\Omega)$;
- (ii) $C_p(\sigma, \varrho) \leq C(p - 1)^{-K}, \quad p \in (1, 2)$;
- (iii) $(\sigma|\varrho) \in L_{F_K,\varrho}(\Omega)$.

Proof: We show (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i). Let (i) hold. Then, according to Lemma 2.5, $\|f\|_{1,\sigma} \leq C_K \theta_K / (p - 1)^K \|f\|_{p,\varrho}, p \in (1, 2)$.

Now suppose that (ii) is valid. Employing the relation between the best constants from Theorem 2.1 (j' is the conjugate exponent to j) we get

$$\begin{aligned} \int_{\Omega} F_K \left(\mu \frac{\sigma(x)}{\varrho(x)} \right) \varrho(x) dx &= \sum_{j=2}^{\infty} \frac{\mu^j}{j!} \left(\frac{1}{j - 1} \right)^{j(K-1)} B_j(\sigma, \varrho) \\ &\leq \sum_{j=2}^{\infty} \frac{\mu^j}{j!} \left(\frac{1}{j - 1} \right)^{j(K-1)} C^j \left(\frac{1}{j' - 1} \right)^{Kj} \\ &\leq \sum_{j=2}^{\infty} \frac{\mu^j}{j!} \left(\frac{1}{j - 1} \right)^{j(K-1)} C^j j! \left(\frac{1}{j' - 1} \right)^{j(K-1)} \\ &= \sum_{j=2}^{\infty} \frac{1}{j!} (C\mu j)^j, \end{aligned}$$

the last sum being finite provided $0 < \mu < (Ce)^{-1}$.

If (iii) is true, then the modified Young inequality (Lemma 2.4) implies

$$\begin{aligned} \int_{\Omega} |f(x)| \sigma(x) dx &= \int_{\Omega} |f(x)| \frac{1}{\mu} \frac{\mu \sigma(x)}{\varrho(x)} \varrho(x) dx \\ &\leq \frac{\delta_K}{\mu} \int_{\Omega} \Phi_K(|f(x)|) \varrho(x) dx + \frac{\delta_K}{\mu} \int_{\Omega} F_K \left(\mu \frac{\sigma(x)}{\varrho(x)} \right) \varrho(x) dx \\ &= C_{K,\mu} \Phi_K(|f(x)|) \varrho(x) dx + C_{K,\mu} B_{\log,K}(\sigma, \varrho). \end{aligned}$$

Application of Lemma 1.2 (with $h(t) = C_{K,\mu}(t + B_{\log,K}(\sigma, \varrho))$) leads to the imbedding (i) ■

Remark 2.7: By duality argument, we have also characterized those weights ϱ and σ for which $\|f/\varrho\|_{\Psi_{K,\varrho}} \leq C \|f/\sigma\|_{\infty}$.

For $K = 1$ and Ω bounded, $L_{\Phi_{K,\varrho}}$ is the weighted Zygmund class $L(1 + \log^+ L)_{\varrho}$ consisting of all functions f with $\int_{\Omega} |f(x)| (1 + \log^+ |f(x)|) \varrho(x) dx < \infty$. Let us state the following particular

Corollary 2.8: Let Ω be bounded and $\varrho \in L_1(\Omega)$. Then the weighted Zygmund class $L(1 + \log^+ L)_{\varrho}$ is imbedded into $L_{1,\sigma}$ iff $\int_{\Omega} \exp(\mu \sigma(x)/\varrho(x)) \varrho(x) dx < \infty$ for some $\mu > 0$.

3. General imbeddings

In the rest of the paper, P and Q are Young functions.

Definition 3.1: For a couple of Young functions P, Q we introduce the set $Y(P, Q) = \{K > 0; Q(KP^{-1}) \text{ is equivalent to some Young function}\}$. For $K \in Y(P, Q)$, the Young function equivalent to $Q(KP^{-1})$ will be denoted $Q(KP^{-1})$ again. Recall that two functions F, G are equivalent iff $\alpha^{-1}F(\alpha^{-1}t) \leq G(t) \leq \alpha F(\alpha t)$ for all t and some $\alpha \geq 1$ independent of t .

Remark 3.2: Note that if at least one of the functions P, Q satisfies the Δ_2 condition globally on $(0, \infty)$, then $Y(P, Q)$ is either empty, or equal to $(0, \infty)$.

Proposition 3.3: Suppose that QP^{-1} is a Young function. Then the following statements are equivalent:

- (i) $\|f\|_{P,\sigma} \leq \|f\|_{Q,\varrho}$ for all f ,
- (ii) $\|f\|_{1,\sigma} \leq \|f\|_{Q(KP^{-1}),\varrho}$ for all f .

Proof: According to Lemma 1.1, (i) is subsequently equivalent to the implications $m_{\varrho}(f, Q) \leq 1 \Rightarrow m_{\sigma}(f, P) \leq 1$, and, on taking $g = P(f)$, $m_{\varrho}(g, QP^{-1}) \leq 1 \Rightarrow \int g \sigma \leq 1$. The repeated use of Lemma 1.1 completes the proof ■

Realizing that $K\|f\|_{Q,\varrho} = \|f\|_{Q_K,\varrho}$, where Q_K is the Young function given by $Q_K(t) = Q(Kt)$, we easily obtain

Corollary 3.4: Let $K \in Y(P, Q)$. Then

$$\|f\|_{P,\sigma} \leq K \|f\|_{Q,\varrho} \forall f \Rightarrow \|f\|_{1,\sigma} \leq \|f\|_{Q(KP^{-1}),\varrho} \forall f.$$

Conversely, if $K \in Y(P, Q)$, then for every $C \geq 1$

$$\|f\|_{1,\sigma} \leq C \|f\|_{Q(KP^{-1}),\varrho} \forall f \Rightarrow \|f\|_{P,\sigma} \leq KC \|f\|_{Q,\varrho} \forall f.$$

Corollary 3.5: *Let at least one of the functions P, Q satisfy the Δ_2 condition globally on $(0, \infty)$. Then $L_{Q,\varrho} \hookrightarrow L_{P,\sigma}$ iff $L_{QP^{-1},\varrho} \hookrightarrow L_{1,\sigma}$ provided QP^{-1} is a Young function.*

In the sequel, we will assume that QP^{-1} is a Young function. If $K \in Y(P, Q)$, we denote by N_K the function complementary to $Q(KP^{-1})$. We shall study relations between the imbedding $L_{Q,\varrho} \hookrightarrow L_{P,\sigma}$ and the condition

$$\int_{\Omega} N_K(\mu\sigma(x)/\varrho(x)) \varrho(x) dx < \infty \text{ for some } \mu > 0.$$

Proposition 3.6: *Let $\int_{\Omega} N_K(\mu\sigma(x)/\varrho(x)) \varrho(x) dx$ be finite for some positive K, μ . Then $L_{Q,\varrho} \hookrightarrow L_{P,\sigma}$.*

Proof: Making use of the Young inequality, we have

$$\begin{aligned} \|f\|_{P,\sigma} &= K \|f/K\|_{P,\sigma} \leq K \left(1 + \mu^{-1} \int_{\Omega} P(|f(x)|/K) (\sigma(x)/\varrho(x)) \mu \varrho(x) dx \right) \\ &\leq K + (K/\mu) \int_{\Omega} Q(|f(x)|) \varrho(x) dx + (K/\mu) \int_{\Omega} N_K(\mu\sigma(x)/\varrho(x)) \varrho(x) dx \\ &= C + (K/\mu) m_{\varrho}(f, Q), \end{aligned}$$

and in virtue of Lemma 1.2 we arrive at the desired imbedding ■

Proposition 3.7: *Let the imbedding $L_{Q,\varrho} \hookrightarrow L_{P,\sigma}$ hold and suppose $K \in Y(P, Q)$ for some $K \geq \sup \{ \|f\|_{P,\sigma} / \|f\|_{Q,\varrho}; f \neq 0 \}$. Then $\int_{\Omega} N_K(\mu\sigma(x)/\varrho(x)) \varrho(x) dx < \infty$ for some $\mu > 0$.*

Proof: Assume $g \in E_{Q(KP^{-1}),\varrho}(\Omega)$. Then, by Corollary 3.4,

$$\left| \int_{\Omega} g(x) (\sigma(x)/\varrho(x)) \varrho(x) dx \right| \leq \|g\|_{1,\sigma} \leq \|g\|_{Q(KP^{-1}),\varrho},$$

in other words, $(\sigma/\varrho) \in L_{N_K,\varrho}$ ■

Let us summarize the results of this section.

Theorem 3.8: *Let at least one of the functions P, Q satisfy the Δ_2 condition. Then the following statements are equivalent:*

- (i) $L_{Q,\varrho}(\Omega) \hookrightarrow L_{P,\sigma}(\Omega)$;
- (ii) $\int_{\Omega} N(\mu\sigma(x)/\varrho(x)) \varrho(x) dx < \infty$ for some $C, \mu > 0$.

Theorem 3.9: *Suppose that $Y(P, Q)$ contains a sequence $K_n \nearrow \infty$. Then the following statements are equivalent:*

- (i) $L_{Q,\varrho}(\Omega) \hookrightarrow L_{P,\sigma}(\Omega)$;
- (ii) $\int_{\Omega} N_C(\mu\sigma(x)/\varrho(x)) \varrho(x) dx < \infty$ for some $C, \mu > 0$.

Remark 3.10: The duality argument applied to Theorems 3.8 and 3.9 gives the characterization of ϱ and σ such that $\|f/\varrho\|_{\tilde{Q},\varrho} \leq C \|f/\sigma\|_{\tilde{P},\sigma}$, where \tilde{P} and \tilde{Q} are the complementary Young functions to P, Q , respectively.

4. Some remarks on an alternative approach (employing indices)

First, we recall the concept of the lower and upper index of a Young function. Actually, it is a certain refinement of the Δ_2 condition. The existence of finite indices for functions satisfying Δ_2 can be proved e.g. on the base of the theory of submulti-

plicative functions. We refer to MATUSZEWSKA and ORLICZ [7], GUSTAVSSON and PEETRE [2], and the comprehensive paper by MALIGRANDA [6]:

Definition 4.1: Let $\Phi \in \Delta_2$ globally on $(0, \infty)$ and

$$h(\lambda) = h_\Phi(\lambda) = \sup_{t>0} \Phi(\lambda t)/\Phi(t), \quad \lambda \geq 0.$$

We define the lower index of Φ as $i(\Phi) = \lim_{\lambda \rightarrow 0+} \log h(\lambda)/\log \lambda$ and the upper index of Φ as $I(\Phi) = \lim_{\lambda \rightarrow \infty} \log h(\lambda)/\log \lambda$.

It follows easily from the definition that for each $\varepsilon > 0$ there exists $C_\varepsilon \geq 1$ such that

$$\begin{aligned} \Phi(\lambda t) &\leq C_\varepsilon \max \{ \lambda^{i(\Phi)-\varepsilon}, \lambda^{I(\Phi)+\varepsilon} \} \Phi(t), \quad t \geq 0, \quad \lambda \geq 0, \\ \min \{ \mu^{i(\Phi)-\varepsilon}, \mu^{I(\Phi)+\varepsilon} \} \Phi(t) &\leq C_\varepsilon \Phi(\mu t), \quad t \geq 0, \quad \mu \geq 0. \end{aligned}$$

The following assertion linking modulars and norms in weighted Orlicz spaces is an immediate consequence of the properties of indices and the Luxemburg norm.

Proposition 4.2: Assume $\|f\|_{P,\sigma} \leq C\|f\|_{Q,\sigma}$ and let $\varepsilon > 0$.

(i) If $P \in \Delta_2$ globally on $(0, \infty)$, then

$$m_\sigma(f, P) \leq C_\varepsilon \max \{ \|f\|_{Q,\sigma}^{i(P)-\varepsilon}, \|f\|_{Q,\sigma}^{I(P)+\varepsilon} \}.$$

(ii) If $Q \in \Delta_2$ globally on $(0, \infty)$, then

$$\min \{ \|f\|_{P,\sigma}^{i(Q)-\varepsilon}, \|f\|_{P,\sigma}^{I(Q)+\varepsilon} \} \leq C_\varepsilon m_\sigma(f, Q).$$

(iii) If $P, Q \in \Delta_2$ globally on $(0, \infty)$, then

$$m_\sigma(f, P) \leq C_\varepsilon \max \{ (m_\sigma(f, Q))^{(i(P)/I(Q))-\varepsilon}, (m_\sigma(f, Q))^{(I(P)/I(Q))+\varepsilon} \}.$$

Proposition 4.3: The following two statements are true:

(i) Let $i(Q) > \alpha > 0$ ($i(Q) = \infty$ is admissible here). If $\|f\|_{Q,\sigma} > 1$, then we have $\|f\|_{Q,\sigma} \leq C_\alpha (m_\sigma(f, Q))^{1/\alpha}$.

(ii) Let $I(Q) < \beta$ and $\|f\|_{Q,\sigma} \leq 1$. Then $\|f\|_{Q,\sigma} \leq C_\beta (m_\sigma(f, Q))^{1/\beta}$.

Proof: (i) If $\|f\|_{Q,\sigma} > 1$, then obviously $m_\sigma(f, Q) \geq \|f\|_{Q,\sigma}$. Therefore suppose that $m_\sigma(f, Q) < \infty$. The definition of $i(Q)$ guarantees the existence of $\lambda_0 \in (0, 1)$ such that $h_Q(\lambda) \leq \lambda^\alpha$ for $\lambda \in (0, \lambda_0)$. Set $C_\alpha = \lambda_0^{-1}$. We get

$$\int_\Omega Q \left(\frac{|f(x)|}{C_\alpha (m_\sigma(f, Q))^{1/\alpha}} \right) \varrho(x) dx \leq \lambda_0^\alpha < 1.$$

(ii) Under our assumption we have $Q(\lambda t) \leq \lambda^\beta Q(t)$, $t \geq 0$, $\lambda \geq \lambda_\beta$, for some $\lambda_\beta > 1$. Therefore

$$\begin{aligned} \int_\Omega Q \left(\lambda_\beta^{1-\beta} \frac{|f(x)|}{(m_\sigma(f, Q))^{1/\beta}} \right) \varrho(x) dx &\leq \lambda_\beta^{-\beta} \int_\Omega Q \left(\lambda_\beta \frac{|f(x)|}{(m_\sigma(f, Q))^{1/\beta}} \right) \varrho(x) dx \\ &\leq \lambda_\beta^{-\beta} \left(\frac{\lambda_\beta}{(m_\sigma(f, Q))^{1/\beta}} \right)^\beta m_\sigma(f, Q) = 1. \end{aligned}$$

It suffices to choose $C_\beta = \lambda_\beta^{\beta-1}$. ■

Remark 4.4: If $\varrho \in L_1(\Omega)$, $Q \in \Delta_3$, and $P \in \Delta_2$ globally on $(0, \infty)$, then the necessity of the condition $(\sigma/\varrho) \in L_{N,\varrho}$ for the imbedding $L_{Q,\varrho} \hookrightarrow L_{P,\sigma}$ can be proved directly without use of the representation theorem. Actually, easily $QP^{-1} \in \Delta_3$, thus $N \in \Delta_2$. Combined with (1.2) and Proposition 4.2/(i), this gives

$$\begin{aligned} \int_{\Omega} N\left(\frac{\sigma(x)}{\varrho(x)}\right) \varrho(x) dx &\leq C_{\varrho}(\Omega) + C \int_{\Omega} PQ^{-1}\left(\frac{\sigma(x)}{\varrho(x)}\right) \sigma(x) dx \\ &\leq C_{\varrho}(\Omega) + C_* \max\{\|Q^{-1}(\sigma/\varrho)\|_{Q,\varrho}^{(P)-\varepsilon}, \|Q^{-1}(\sigma/\varrho)\|_{Q,\varrho}^{(P)+\varepsilon}\} \\ &\leq C_{\varrho}(\Omega) + C_* \left(1 + \int_{\Omega} QQ^{-1}(\sigma(x)/\varrho(x)) \varrho(x) dx\right)^{(P)+\varepsilon} \\ &= C_{\varrho}(\Omega) + C_*(1 + \sigma(\Omega))^{(P)+\varepsilon}. \end{aligned}$$

All above quantities are finite; $\varrho \in L_1(\Omega)$ was an assumption and the integrability of σ over Ω follows from membership of constant functions in $L_{Q,\varrho}(\Omega)$.

Remark 4.5: If $P, Q \in \Delta_2$ and $I(P) < i(Q)$, we can get a quantitative relation for

$$B = \int_{\Omega} N(\sigma(x)/\varrho(x)) \varrho(x) dx.$$

Indeed, setting $H(t) = P^{-1}(N, (t)/t)$ it is easy to check that $QH(t) \leq N(t)$, $t \geq 0$; to clarify this we invoke (1.1). Choose $\varepsilon > 0$ and α, β in such a manner that $I(P) + \varepsilon < \alpha < i(Q)$, $I(Q) < \beta$. For k natural put

$$\Omega_k = \{x \in \Omega; |x| < k, \sigma(x) < k, \varrho(x) > k^{-1}\}, \quad \Omega_0 = \bigcup_k \Omega_k,$$

and

$$B_k = \int_{\Omega_k} N(\sigma(x)/\varrho(x)) \varrho(x) dx = \int_{\Omega_k} PH(\chi_{\Omega_k}(x) \sigma(x)/\varrho(x)) \sigma(x) dx.$$

Applying Proposition 4.2/(i) and Proposition 4.3 we can continue

$$\begin{aligned} B_k &\leq C_* \max\{\|H(\chi_{\Omega_k} \sigma/\varrho)\|_{Q,\varrho}^{(P)-\varepsilon}, \|H(\chi_{\Omega_k} \sigma/\varrho)\|_{Q,\varrho}^{(P)+\varepsilon}\} \\ &\leq C_* \max\{C_{\alpha}(m_{\varrho}(\chi_{\Omega_k} \sigma/\varrho, QH))^{(I(P)+\varepsilon)/\alpha}, C_{\beta}(m_{\varrho}(\chi_{\Omega_k} \sigma/\varrho, QH))^{(i(P)-\varepsilon)/\beta}\} \\ &\leq C_* \max\{C_{\alpha} B_k^{(I(P)+\varepsilon)/\alpha}, C_{\beta} B_k^{(i(P)-\varepsilon)/\beta}\}. \end{aligned}$$

Now, $B_k < \infty$, both the exponents $(I(P) + \varepsilon)/\alpha$ and $(i(P) - \varepsilon)/\beta$ are smaller than 1, and $|\Omega \setminus \Omega_0| = 0$. Thus, $B \leq \max\{(C_{\alpha} C_{\alpha})^{\alpha}/(\alpha - I(P) - \varepsilon), (C_{\beta} C_{\beta})^{\beta}/(\beta - i(P) + \varepsilon)\}$. Note that $i(Q) > I(P)$ implies $i(QP^{-1}) > 1$, hence $N \in \Delta_2$ and therefore the constant μ in the argument of N can simply be omitted.

Remark 4.6: Let us notice that Theorem 2.6 is partly covered by Theorem 3.9. On the other hand, the procedure described in Remark 4.4 cannot be applied in that case.

Appendix. This "arithmetic supplement" is added for the sake of completeness.

Proof of Lemma 2.3: We shall show the existence of a constant $\beta (= \beta_K)$ such that

$$\sum_{j=2}^{\infty} \frac{t^{jK}}{\beta^j j!} \leq \sum_{j=2}^{\infty} \frac{t^j}{j!} \left(\frac{1}{j-1}\right)^{j(K-1)}, \quad t \geq 1,$$

in several steps. As usual, $[\cdot]$ denote the integer part. Let us write

$$\sum_{j=2}^{\infty} \frac{t^{jK}}{\beta^j j!} = \sum_{j=2}^{[2K]} + \sum_{l=3}^{\infty} \sum_{j=(l-1)K+1}^{[lK]} = S_2 + \sum_{l=3}^{\infty} S_l.$$

Step 1 (an estimate of S_2):

$$S_2 = \sum_{j=2}^{[2K]} \frac{t^j/K}{\beta^j j!} \leq \frac{t^2}{([2K] + 1)!} \frac{[2K] + 1}{\beta^{[2K]}} \times (1 + \beta[2K] + \beta^2[2K]([2K] - 1) + \dots + \beta^{[2K]-2}[2K]! 2^{-1}).$$

The last sum contains $[2K] - 1$ terms and the largest one equals $2^{-1}[2K]! \beta^{[2K]-2}$. Thus

$$S_2 \leq \frac{t^2}{([2K] + 1)!} \frac{([2K] - 1) ([2K] + 1)!}{2\beta^2} \leq \frac{t^2}{([2K] + 1)!}$$

for sufficiently large β .

Step 2 (an estimate of $S_l, l \geq 3$):

$$S_l = \sum_{j=([l-1]K+1)}^{[lK]} \frac{t^j/K}{\beta^j j!} \leq \frac{t^l}{([lK] + 1)!} \frac{[lK] + 1}{\beta^{[lK]}} \times \left(1 + \beta[lK] + \beta^2[lK]([lK] - 1) + \dots + \beta^{[lK]-([l-1]K)-1} \frac{[lK]!}{([([l-1]K) + 1)!} \right).$$

Now, there are $[lK] - [([l-1]K)]$ terms in the last sum and the last is the largest of them. As $[lK] - [([l-1]K)] \leq K + 1$ we obtain

$$\begin{aligned} S_l &\leq \frac{t^l}{([lK] + 1)!} \frac{K + 1}{\beta^{[lK]}} \frac{([lK] + 1)!}{([([l-1]K) + 1)!} \beta^{[lK]-([l-1]K)-1} \\ &\leq \frac{t^l}{([lK] + 1)!} \frac{K + 1}{\beta^{([l+1]K)+1}} (lK + 1)^{K+1} \\ &\leq \frac{t^l}{([lK] + 1)!} \left(\frac{(lK + 1)^{1+1/K} (K + 1)^{1/K}}{\beta^{l-1}} \right)^K \end{aligned}$$

The last ratio is smaller than 1 provided β is sufficiently large (uniformly with respect to l), again. Hence, for such $\beta, S_l \leq t^l/([lK] + 1)!$

Step 3: Combining the estimates obtained we get for β large enough

$$\sum_{j=2}^{\infty} \frac{t^j/K}{\beta^j j!} \leq \sum_{j=2}^{\infty} \frac{t^j}{(jK + 1)!}, \quad t \geq 1.$$

Easily,

$$\frac{1}{(jK + 1)!} = \frac{1}{j!(j + 1) \dots (jK + 1)} \leq \frac{1}{j!} \left(\frac{1}{j - 1} \right)^{jK+1-j} \leq \frac{1}{j!} \left(\frac{1}{j - 1} \right)^{jK-j},$$

and, finally,

$$\sum_{j=2}^{\infty} \frac{t^j/K}{\beta^j j!} \leq \sum_{j=2}^{\infty} \frac{t^j}{j!} \left(\frac{1}{j - 1} \right)^{j(K-1)}, \quad t \geq 1,$$

the desired inequality ■

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