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On a Class of Generalized K-Entropies and Bernoulli Shifts

T. DE PALY

Die Arbeit bringt die Konstruktion eines für die Diskussion um die Verallgemeinerung der K-Entropie wichtigen Beispiels. Für den Fall, daß der Prozeß durch einen Bernoulli-shift und eine nur aus Zylindermengen bestehende Verteilung beschrieben werden kann, wird eine Formel zur Berechnung der verallgemeinerten Entropie angegeben. In Verbindung damit wird auf neuartige Weise ein Optimierungsproblem über der Menge aller Wahrscheinlichkeitsvektoren gleicher Entropie behandelt.

В работе строится пример, важный для обсуждения некоторых обобщений K-энтропии, предложенных в более ранней работе. Вычислены обобщенные энтропии для того случая, когда процесс задается сдвигом Бернулли и распределением, состоящим только из цилиндрических множеств. Кроме того, новым методом решена некоторая оптимизационная задача на множестве всех вероятностных векторов заданной энтропии.

The paper presents the construction of an example important for the discussion of some generalizations of the K-entropy. These generalizations have been introduced in a previous paper. A formula for the generalized entropies of a process is calculated for the case where the process is given by a Bernoulli shift and a partition consisting only of cylinder sets. Furthermore, a special optimization problem on the set of all probability vectors of a given entropy is solved using a new method. The results of these computations are combined to the cited example.

1. Introduction

In [1] we introduced a new class of isomorphy invariants for dynamical systems. This class is a generalization of the dynamical entropy (K-entropy). Besides the construction of the generalized entropies, the paper [2] contains the derivation of some general properties of them. The present paper is devoted to a particular topic connected with the investigation of the new invariants, namely the construction of an example sharply illustrating the complicated character of the generalized entropies.

To this end we derive a formula for the generalized relative entropies of a transformation T with respect to a partition $\underline{C}$ in the case where T is a Bernoulli shift and $\underline{C}$ consists only of cylinder sets. As a second step we solve an optimization problem of a somewhat special kind. The concave functionals to be maximalized are not Gateaux-differentiable at the maximum points, and the region the supremum is taken over is not convex. But these apparently unpleasant properties assure us that the problem can be solved in an explicit form. The proofs of the solution of the optimization problem are worked out by using simple arguments from the order-structure of states [4]. If we combine the results we get the desired example.

2. Notations and definitions

A *dynamical system* is an aggregate $(X, \mathfrak{B}, \mu, T)$ where $(X, \mathfrak{B}, \mu)$ is a Lebesgue space and T is an automorphism of $(X, \mathfrak{B}, \mu)$.

If $\underline{C}$ is a partition of X into measurable sets we call the pair $(\underline{C}/T)$ a *process* in $(X, \mathfrak{B}, \mu)$.

Definition 1: Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system, and let $g\colon [0, 1] \to \mathbf{R}$ be a real, bounded, concave function of the closed unit interval with $g(0) = 0$. $\underline{C}$, $\underline{D}$ are finite partitions of $(X, \mathfrak{B})$. We define:

$$\text{i) } G(\underline{C}/\underline{D}) = \sum_j \mu(D_j) \sum_i g\big(\mu(C_i/D_j)\big) \tag{2.1}$$

where

$$\mu(A/B) = \frac{\mu(A \cap B)}{\mu(B)} \quad \forall A \in \mathfrak{B},\ \forall B \in \mathfrak{B} \text{ s.t. } \mu(B) > 0 \tag{2.2}$$

and

$$\mu(B)\, g\big(\mu(A/B)\big) = 0 \ \forall A \in \mathfrak{B}, \quad \forall B \in \mathfrak{B} \text{ s.t. } \mu(B) = 0. \tag{2.3}$$

C_i and D_j denote the elements of the partitions $\underline{C}$ and $\underline{D}$, respectively.

$$\text{ii) } G(\underline{C}/T) = \lim_n G\Big(\underline{C}/\bigvee_{i=1}^{n} T^{-i}\underline{C}\Big) \tag{2.4}$$

where $\bigvee_{i=1}^{n} T^{-i}\underline{C}$ denotes the common refinement of the partitions $\{T^{-i}\underline{C}\}$.

$$\text{iii) } G(T) = \sup_{\underline{C}} G(\underline{C}/T) \tag{2.5}$$

where the supremum is taken over all finite partitions.

Remarks: i) All the functionals $G(T)$ defined above are isomorphy invariants of dynamical systems (i.e. dynamical invariants).

ii) If we insert the special strongly concave function

$$h(x) = \begin{cases} -x \log x & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$$

into the Definitions 1(i, ii, iii), then we get the definitions of the *relative entropy* of $\underline{C}$ with respect to $\underline{D}$, the *entropy of the process* $(\underline{C}/T)$, and the *dynamical entropy* of T, respectively.

For the solution of the optimization problem in Sect. 5 we will use some simple arguments from the theory of the order-structure of states [4]. The definitions and results needed for our special problem are listed below.

Let $x = (x_i)$, $y = (y_i)$ $(i = 1, 2, \ldots)$ be two probability vectors. We say that x is *more mixed than* y $(x \succ y)$ iff for all $j = 1, 2, \ldots$ the sum of the j greatest components of x is not greater than the corresponding sum for y. In other words, let $\bar{x} = (\bar{x}_i)$, $\bar{y} = (\bar{y}_i)$ be reorderings of the components of x and y, respectively, such that

$$\bar{x}_1 \geqq \bar{x}_2 \geqq \cdots \geqq \bar{x}_n \geqq \cdots; \qquad \bar{y}_1 \geqq \bar{y}_2 \geqq \cdots \geqq \bar{y}_n \geqq \cdots.$$

Then we write

$$x \succ y \ \text{ iff } \sum_{i=1}^{j} \bar{x}_i \leqq \sum_{i=1}^{j} \bar{y}_i \qquad (j = 1, 2, \ldots). \tag{2.6}$$

We write $x \simeq y$ (*x is mixing equivalent to y*) iff $x \succ y$ and $y \succ x$.

The following assertion is a well-known result: Let $x = (x_i)_{i=1}^{\infty}$, $y = (y_i)_{i=1}^{\infty}$ be two probability vectors. Then

$$\text{i) } x \succ y \text{ iff } \sum g(x_i) \geqq \sum g(y_i) \tag{2.7}$$

for all concave functions $g: [0, 1] \to \mathbf{R}$.

ii) Let g be strongly concave, and let $x \succ y$ but $x \not\sim y$. Then

$$\sum g(x_i) > \sum g(y_i). \tag{2.8}$$

Throughout this paper we are concerned with the following special class of dynamical systems (*Bernoulli systems*):

Let $Y = \{0, 1, \ldots, n-1\}$ be equipped with the σ-algebra of all subsets and with the measure μ^0 given by $\mu^0(i) = p_i \geqq 0$ $(i \in Y)$, $\sum p_i = 1$. We define $X = Y^{\mathbf{Z}}$ ($\mathbf{Z}$ being the set of all integers) as the direct product space with the measure μ on the σ-algebra $\mathfrak{B}$ generated by the cylinder sets. A cylinder set is defined as follows:

$$A_{i_1 \ldots i_k}^{y_{i_1} \ldots y_{i_k}} = \{x = (x_i)_{-\infty}^{\infty} \in X : x_{i_j} = y_{i_j} \quad (j = 1, \ldots, k)\} \tag{2.9}$$

and

$$\mu(A_{i_1 \ldots i_k}^{y_{i_1} \ldots y_{i_k}}) = \prod_{j=1}^{k} p_{y_{i_j}}. \tag{2.10}$$

The automorphism of $(X, \mathfrak{B}, \mu)$ is given by the shift

$$T: X \to X, \qquad Tx = x' \quad \text{and} \quad x_i' = x_{i-1} \qquad (i = 0, \pm 1, \pm 2, \ldots). \tag{2.11}$$

The system $(X, \mathfrak{B}, \mu, T)$ is called the $(p_0, \ldots, p_{n-1})$-*Bernoulli system.*

To deal with the class of Bernoulli systems we need the following well known definitions of *independence of partitions*:

Let $(X, \mathfrak{B}, \mu, T)$ be a dynamical system, and let $\underline{C}$, $\underline{D}$, $\underline{E}$ be finite partitions. $\underline{C}$ is said to be

i) *independent* of $\underline{D}$ (written $\underline{C} \perp \underline{D}$) iff $\forall i, j$

$$\mu(C_i \cap D_j) = \mu(C_i)\, \mu(D_j);$$

ii) an *independent partition* for T iff for all $n = 1, 2, \ldots$

$$\underline{C} \perp \bigvee_{i=1}^{n} T^{-i}\underline{C};$$

iii) *independent* of $\underline{D}$ related to $\underline{E}$ (written $\underline{C} \perp^{\underline{E}} \underline{D}$) iff on any atom E_k of the partition $\underline{E}$ the partitions of E_k induced by $\underline{C}$ and $\underline{D}$, respectively, are independent, i.e. $\forall i, j, k$

$$\mu(C_i \cap D_j / E_k) = \mu(C_i / E_k)\, \mu(D_j / E_k).$$

C_i and D_j denote the atoms of the partitions $\underline{C}$ and $\underline{D}$, respectively.

Remark: Bernoulli systems are exactly those dynamical systems which have an independent generator for T. (A *generator* for T is a partition $\underline{C}$ such that the σ-algebra generated by $\{T^{-i}C\}$ $(i = 0, \pm 1, \pm 2, \ldots)$ is $\mathfrak{B}$ (up to measure zero).)

3. The problem

The entropy of a Bernoulli system is known to be $H(T) = \sum h(p_i) = -\sum p_i \log p_i$. In [2] we showed that for all ergodic systems with the dynamical entropy $H(T) = s$ and for all generalized dynamical entropies

$$X_g(s) := \sup_{\bar{q} \in \mathscr{S}} \sum_i g(q_i) \leqq G(T) \tag{3.1}$$

where $\mathscr{S}$ denotes the set of all finite probability vectors $\bar{q} = (q_i)$ s.t. $\sum h(q_i) = s$. $X_g(s)$ is the supremum of $G(\underline{C}/T)$ over all processes $(\underline{C}/T)$ constructed with a finite independent partition.

From now on we will suppose that T is a Bernoulli shift with the entropy $H(T) = s$. Bernoulli systems of finite entropy are characterized by the statement that they have an independent finite generator for T. According to Ornstein's theorem, for any finite probability vector $\bar{q} \in \mathscr{S}$ there is an independent generator $\underline{C} = (C_1, \ldots, C_m)$ s.t. $\mu(C_i) = q_i$ $(i = 1, 2, \ldots, m)$. But even in the case of Bernoulli systems, equality does not hold in (3.1) in general. This will be shown by the example to be constructed with the help of the results of the following sections.

The example is the $(p, 1-p)$-Bernoulli system, and the dynamical invariants which give $X_g\big(s = h(p) + h(1-p)\big) < G(T)$ are formed with the special concave functions

$$g_r(x) = \begin{cases} x & \text{for } 0 \leqq x \leqq r \\ r & \text{for } \quad x > r \end{cases} \qquad (0 < r < 1) \tag{3.2}$$

and are denoted by G_r.

In Section 4 (Equ. 4.5) we see that for the partition $\underline{C} = (A_0{}^0, A_{01}^{10}, A_{01}^{11})$ (which is not independent, but is generating), the generalized entropies of the process $(\underline{C}/T)$ are

$$\begin{aligned} G(\underline{C}/T) = F_g(p) := p\big[g(p) + g\big(p(1-p)\big) + g\big((1-p)^2\big)\big] \\ + p(1-p)\, g(1) + (1-p)^2\, [g(p) + g(1-p)]. \end{aligned} \tag{3.3}$$

If $p = \dfrac{1}{2}$ then

$$G_r(\underline{C}/T) = \begin{cases} (r+3)/4 & \text{for } 1/2 \leqq r \leqq 1, \\ (5r+1)/4 & \text{for } 1/4 \leqq r \leqq 1/2, \\ 9r/4 & \text{for } \qquad r \leqq 1/4. \end{cases} \tag{3.4}$$

Now we use the solution of the problem $X_r(s) = \sup_{\bar{p} \in \mathscr{S}} \sum g_r(p_i)$ and refer to the notation introduced at the beginning of Section 5 (Equ. 5.2, 5.3).

If $s = 2h\left(\dfrac{1}{2}\right) = \log 2$, then $a_1 = \dfrac{1}{2}$ and $a_2 < \dfrac{1}{8}$, so for $r = \dfrac{1}{4}$ we find $n = 2$.

Therefore we have $X_{1/4}(\log 2) = 2 \cdot \dfrac{1}{4} + b$, where b is the solution of the equation $h(b) + h\left(\dfrac{3}{4} - b\right) = \dfrac{1}{2} \log 2$. The calculation gives $b \leqq 0.04$ and therefore

$$X_{1/4}(\log 2) \leqq 0.54. \tag{3.5}$$

Equ. (3.3) provides us with $G_{1/4}(\underline{C}/T) = 0.5625 > X_{1/4}(\log 2)$. A sharper analysis shows that such an inequality holds for any r s.t. $a_2 < r < k$, where k is the solution of $h(k) + h\big((1-3k)/4\big) + h\big((3-k)/4\big) = \log 2$.

If $r \notin (a_2, k)$, then we find other Bernoulli systems and other partitions of type (4.1) to construct analogous examples (see Theorem 1). So the constructed example shows that we cannot restrict ourselves to the independent generators if we want to compute the G_r-invariants for Bernoulli systems. This is a little surprising, because the independent processes are the characteristic ones in the Bernoullian case. Therefore, serious difficulties arise in connection with the calculation of the generalized dynamical entropies for other dynamical systems [1—2].

4. Generalized process-entropies for a particular type of process in Bernoulli systems

In this section we compute an explicit formula for the generalized process-entropies $G(\underline{C}/T)$ (see Def. 1(ii)) in the case where T is a Bernoulli shift and the partition $\underline{C}$ consists only of cylinder sets (c.f. Equ. 2.9). We consider the $(p_0, \ldots, p_{n-1})$-Bernoulli system and the partition

$$\underline{C} = \left({}_1A^{v_{i_{11}}\cdots v_{i_{k1}}}_{i_{11}\cdots i_{k1}}, \ldots, {}_jA^{v_{i_{1j}}\cdots v_{i_{kj}}}_{i_{1j}\cdots i_{kj}}, \ldots, {}_mA^{v_{i_{1m}}\cdots v_{i_{km}}}_{i_{1m}\cdots i_{km}}\right) \tag{4.1}$$

the elements of which are numbered by the left lower index. A collection of pairwise disjoint sets of the form (4.1) is a partition if and only if for any right lower index occuring on the cylinder sets of $\underline{C}$, all elements of the set $Y = \{0, 1, \ldots, n-1\}$ appear at least once as the corresponding upper index on some cylinder set of $\underline{C}$. For instance, in the case of the (p_0, p_1)-Bernoulli system $\underline{C} = ({}_1A_0{}^0, {}_2A^{10}_{01}, {}_3A^{110}_{012}, {}_4A^{111}_{012})$ is a partition of this form.

After these preliminaries, we are going to prove the following theorem.

Theorem 1: *Let $(X, \mathfrak{B}, \mu, T)$ be the $(p_0, \ldots, p_{n-1})$-Bernoulli system. Assume that $\underline{C}$ is a partition consisting only of cylinder sets, i.e. $\underline{C}$ is of the form* (4.1), *such that*

$$\max_{\substack{j,l \\ j',l'}} |i_{jl} - i_{j'l'}| \leq d, \qquad d \text{ an integer}. \tag{4.2}$$

This means that the maximal difference between right lower indices occuring on all cylinder sets in $\underline{C}$ is not greater than d. Then

$$G(\underline{C}/T) = G\left(\underline{C}/\bigvee_{i=1}^{d} T^{-i}\underline{C}\right) \tag{4.3}$$

for all functionals of Def. 1(ii).

Example: For T the (p_0, p_1)-Bernoulli shift and $\underline{C} = ({}_1A_0{}^0, {}_2A^{10}_{01}, {}_3A^{11}_{01})$ we find

$$\max_{\substack{j,j' \in \{0,1\} \\ l,l' \in \{1,2,3\}}} |i_{j_l} - i_{j'_{l'}}| = 1. \tag{4.4}$$

So we have

$$G(\underline{C}/T) = G(\underline{C}/T^{-1}C) = F_g(p_0). \tag{4.5}$$

The explicit value of $F_g(p_0)$ can be calculated to be equal to the right hand side of Equ. 3.3 without difficulty.

For the proof of the theorem we need the following lemma.

Lemma 2: *Let $\underline{C}$, $\underline{D}$, $\underline{E}$ be finite partitions such that $\underline{C} \perp^{\underline{E}} \underline{D}$. Then for all generalized relative entropies $G(\cdot/\cdot)$*

$$G(\underline{C}/\underline{E} \vee \underline{D}) = G(\underline{C}/\underline{E}). \tag{4.6}$$

Proof:

$$\underline{C} \perp^{\underline{E}} \underline{D} \quad \text{iff} \quad \frac{\mu(C_i \cap D_j \cap E_l)}{\mu(D_j \cap E_l)} = \frac{\mu(C_i \cap E_l)}{\mu(E_l)} \tag{4.7}$$

for all i, j, l s.t. $\mu(D_j \cap E_l) > 0$. The lemma follows directly from the definition of $G(\cdot/\cdot)$ if we use (4.7) ∎

Proof of the theorem: The only thing to show is

$$\underline{C} \perp^{\underline{D}^d} \underline{D}^n \, (n \geqq d), \quad \text{where} \quad \underline{D}^d = \bigvee_{i=1}^{d} T^{-i}\underline{C} \quad \text{and} \quad \underline{D}^n = \bigvee_{i=d+1}^{n} T^{-i}\underline{C}. \tag{4.8}$$

With Lemma 2 then follows $G(\underline{C}/T) = \lim_n G(\underline{C}/\underline{D}^d \vee \underline{D}^n) = G(\underline{C}/\underline{D}^d)$, but this is the assertion of the theorem.

To show (4.8) we introduce the following notation. Let C_i, D_j^d, D_k^n be elements of the partitions $\underline{C}$, $\underline{D}^d$, $\underline{D}^n$, respectively, such that $\mu(C_i \cap D_j^d \cap D_k^n) > 0$. Of course, the intersection of cylinder sets is a cylinder set too. We denote $p(y_i) := \mu^0(y_i)$, $y_i \in Y$. The right lower indices of C_i (indicating the place where the cylinder C_i is fixed) not occuring on D_j^d are denoted by $l_1, l_2, \ldots$, and analogously, the right lower indices of D_k^n nor occuring on D_j^d are denoted by $m_1, m_2, \ldots$ The corresponding upper indices (which are elements of Y) are denoted by the symbols $y_{l_1}, y_{l_2}, \ldots, y_{m_1}, y_{m_2}, \ldots$

Now because of the cylindrical structure of the sets $C_i \cap D_j^d \cap D_k^n$, $C_i \cap D_j^d$, $D_j^d \cap D_k^n$ and because of the product measure on the cylinder sets, we get

$$\mu(C_i \cap D_j^d) = \mu(D_j^d) \cdot \prod_{l_r} p(y_{l_r}), \tag{4.9}$$

$$\mu(D_j^d \cap D_k^n) = \mu(D_j^d) \cdot \prod_{m_t} p(y_{m_t}), \tag{4.10}$$

$$\mu(C_i \cap D_j^d \cap D_k^n) = \mu(D_j^d) \cdot \prod_{l_r} p(y_{l_r}) \cdot \prod_{m_t} p(y_{m_t}). \tag{4.11}$$

Equ. 4.11 expresses the fact that $\underline{C}$ and $\underline{D}^n$ are independent partitions. This is a consequence of the construction of d. The partition $\underline{C}$ cannot have right lower indices which coincide with right lower indices of some set in $\underline{D}^n = \bigvee_{i=d+1}^{n} T^{-i}\underline{C}$, because the maximal difference of indices of any set C_i is d, and D^n contains only sets of $\underline{C}$ shifted at least $d + 1$ times. The Equations (4.9, 4.10, 4.11) can be combined to

$$\frac{\mu(C_i \cap D_j^d \cap D_k^n)}{\mu(D_j^d)} = \frac{\mu(C_i \cap D_j^d)}{\mu(D_j^d)} \cdot \frac{\mu(D_j^n \cap D_k^d)}{\mu(D_j^d)}. \tag{4.12}$$

In the case that one of the sets involved has zero measure, there is nothing to be shown. Therefore (4.12) proves the theorem ∎

5. A special optimization problem

The example of a process in a Bernoulli system which gives $X_g(s) < G(\underline{C}/T)$ (c.f. Equ. 3.1) can be constructed if we are able to calculate $X_g(s)$ for some concave function g. This is done in this section for the special functions $g_r(x)$ (Equ. 3.2). We consider the problem

$$
\begin{aligned}
&X_r(s) = \sup_{\bar{p} \in \mathscr{S}} G_r(\bar{p}), \qquad G_r: d_+^1 \ni \bar{p} = (p_i) \to \sum g_r(p_i), \\
&\mathscr{S} = \{\bar{p} \in d_+^1 : H(\bar{p}) = s\}, \qquad H: d_+^1 \ni \bar{p} = (p_i) \to \sum h(p_i), \\
&d_+^1 = \{\bar{p} \in d : p_i \geqq 0 \ (i = 1, 2, \ldots), \ \sum p_i = 1\}.
\end{aligned}
\tag{5.1}
$$

Here d denotes the set of all finite sequences. Therefore, d_+^1 is the set of all probability vectors with at most finitely many nonzero components. $\mathscr{S}$ is the set of all finite probability vectors of the given entropy s.

The functional $G_r: d_+^1 \to \mathbf{R}$ is concave. It is not Gateaux-differentiable iff $p_i = r$ for some i. The region $\mathscr{S}$ is not convex. But the special structure of the functionals G_r and H, both being defined as a sum with an underlying concave function, gives the possibility of solving the problem in a somewhat unusual way by using arguments from the order-structure of states.

Solution of (5.1): The solution is performed in three steps.
i) Choose $n \in \mathbf{N}$ s.t. $a_n < r \leqq a_{n-1}$ or $a_n < r \leqq a_n'$. a_n is a solution of the equation

$$nh(a_n) + h(1 - na_n) = s. \tag{5.2}$$

Equ. (5.2) has one (real) solution if $s < \log n$, two solutions if $\log n \leqq s < \log(n + 1)$, and no solution if $s > \log(n + 1)$. If (5.2) has two solutions then we denote the lesser one with a_n and the larger one with a_n'. If $r > a_n'$ for some n then $X_r(s) = 1$.
ii) Calculate b as the lesser solution of the equation

$$(n - 1)\, h(r) + h(b) + h\big(1 - (n - 1)\, r - b\big) = s. \tag{5.3}$$

If $s \neq \log(n + 1)$ we always find two distinct solutions.

iii) $$X_r(s) = \begin{cases} \min\{nr + b, 1\} & \text{if (5.3) has two solutions (a)} \\ 1 & \text{if } r > a_n' \text{ for some } n \quad \text{(b)} \\ 1 & \text{if } s = \log(n+1),\ r = \dfrac{1}{n+1}. \quad \text{(c)} \end{cases}$$

Remarks:
i) At least one of the cases (a, b, c) is fulfilled. In case (b) we cannot perform step (ii) because a_{n-1} cannot be calculated.
ii) If $r < \dfrac{1}{n+1}$ then (a) holds and $nr + b < 1$.
iii) We always mean a real solution when we speak of a solution of an equation.

We show firstly, using the Lemmas 3 and 4, that G_r has a local maximum on $\mathscr{S}$ at the point

$$\bar{r}_n = (\underbrace{r, \ldots, r}_{n-1}, b, 1 - (n - 1)\, r - b, 0, 0, \ldots) \tag{5.4}$$

provided the suppositions of case (a) hold. The lemmas 5, 6 prove that in this case $\bar{r}_n$ is the global maximum point. Because of $X_r(s) \leqq 1$, the other cases are clear. Therefore, we have demonstrated that the construction above leads to the solution of (5.1).

Lemma 3: Let n, a_n, b be as in the solution, i.e. $a_n < r$ and a_n and b are the smallest solutions of Equ. (5.2), (5.3), resp. Then

i) $0 \leqq b < r$ and
ii) $b = 0$ iff $r = a_{n-1}$.

Proof: We see that $1 - na_n \geqq a_n$. Indeed, $a_n \leqq \frac{1}{n+1}$, for the function $nh(x) + h(1 - nx)$ has its only maximum at $x = \frac{1}{n+1}$. Therefore $1 - na_n \geqq \frac{1}{n+1} \geqq a_n$. Suppose now that $1 - (n-1)\, r - b \geqq b \geqq r$. Then

$$\bar{a}_n = (1 - na_n, \underbrace{a_n, \ldots, a_n}_{n}, 0, 0, \ldots) \prec \bar{r}_n$$

according to (2.6). But this leads to $H(\bar{a}_n) \leqq H(\bar{r}_n)$, and equality holds if and only if $\bar{a}_n \simeq \bar{r}_n$. However, mixing equivalence can hold only in the case, $b = r = a_n$. This in turn proves both the assertions, because $1 - (n-1)\, r - b \geqq b$ is supposed in the construction of b ∎

The following definition is needed to make the proofs of the Lemmas 4, 5 and 6 a little more transparent.

Definition 2: Let $r: 0 < r < 1$ and n be a real number and an integer, respectively.

i) We say that a probability vector $\bar{p} \in d_+^1$ is an $r - n$*-typical vector* iff n of its components are equal to r, one component is greater than r, and one of the nonzero components is smaller than r.

ii) A probability vector is said to be *r-typically* iff it is $r - n$-typically for some n.

Lemma 4: Let $\bar{p} \in \mathscr{S}$ be a given probability vector with entropy s. For $\gamma > 0$, $r > 0$ we define some neighbourhoods of $\bar{p}$ by

$$U_\gamma(\bar{p}) = \{\bar{q} \in d_+^1 : \textstyle\sum |q_i - p_i| < \gamma\}, \tag{5.5}$$

$$U_\gamma^r(\bar{p}) = \{\bar{q} \in U_\gamma(\bar{p}) : G_r(\bar{q}) \geqq G_r(\bar{p})\}. \tag{5.6}$$

If $\bar{p}$ is an r-typical vector then there is a $\gamma > 0$ such that $U_\gamma^r(\bar{p}) \cap \mathscr{S} = \bar{p}$.

Remark: The lemma says that $\bar{p}$ is a local maximum point of G_r under the constraints of the problem (5.1).

Proof of the lemma: We assume $\bar{p}$ to be $r - n$-typically and the components of $\bar{p}$ to be rearranged in such a way that $\bar{p} = (c, \underbrace{r, \ldots, r}_{n}, b, 0, 0, \ldots)$, $c > r > b$. This can be done without loss of generality. Then $\bar{q} \in U_\gamma(\bar{p})$ is equivalent to the following assertion: $\bar{q} = \bar{p} + \bar{\varepsilon}$; $\bar{\varepsilon} = (\varepsilon_i) \in d$, d the space of all real vectors with at most finitely many nonzero components; $\sum \varepsilon_i = 0$, $\varepsilon_i \geqq 0 \;\forall\; i > n + 2$, $\sum |\varepsilon_i| < \gamma$. We can rearrange $\bar{\varepsilon}$ so that $\varepsilon_2 \geqq \varepsilon_3 \geqq \cdots \geqq \varepsilon_k > 0 \geqq \varepsilon_{k+1} \geqq \cdots \geqq \varepsilon_{n+1}$ for some $k \geqq 2$. Now suppose that $\bar{q} \in U_\gamma^r(\bar{p})$ and $\gamma < \min\left\{b, \frac{r-b}{2}, \frac{c-r}{2}\right\}$. We get

$$G_r(\bar{q}) = \sum g_r(q_i) = (n+1)\, r + b + \sum_{i=k+1}^{\infty} \varepsilon_i \geqq (n+1)\, r + b = G_r(\bar{p}).$$

Therefore $\sum_{k+1}^{\infty} \varepsilon_i \geqq 0$, but this leads to

$$\sum_{1}^{m} \varepsilon_i \leqq 0 \qquad (m = 1, 2, \ldots). \tag{5.7}$$

This is so because $\varepsilon_2 \geqq \cdots \geqq \varepsilon_k > 0$. The choice of γ and (5.7) guarantee that $\bar{q} = \bar{p} + \bar{\varepsilon} \succ \bar{p}$ and $\bar{q} \simeq \bar{p}$ iff $\bar{\varepsilon} = 0$. The same argument as in the proof of the previous lemma completes the proof ∎

Lemma 5: We assume that there is a r-typical vector $\bar{p} \in \mathscr{S}$ with $r > 0$. Then we have following properties:

i) Any r-typical vector $\bar{q} \in \mathscr{S}$ is equal to $\bar{p}$ up to a rearrangement of the components.

ii) There is no vector $\bar{q} \in \mathscr{S}$ such that $q_i \leqq r \ \forall\ i = 1, 2, \ldots$

iii) There is no vector $\bar{q} \in \mathscr{S}$ such that $\forall\ i$ either $q_i \geqq r$ or $q_i = 0$ and $\sum g_r(q_i) \geqq \sum g_r(p_i)$.

Proof: Let $\bar{p} = (c, \underbrace{r, \ldots, r}_{n}, b, 0, 0, \ldots)$, $c > r > b > 0$.

i) We consider $\bar{q} = (c', \underbrace{r, \ldots, r}_{k}, b', 0, 0, \ldots)$, $c' > r > b' > 0$. Therefore we get $\bar{q} \prec \bar{p}$ in the case $k < n$ because of $c' = 1 - kr - b' \geqq 1 - (k + 1)\, r \geqq 1 - nr > c$. Analogously, $k > n \Rightarrow \bar{q} \succ \bar{p}$. So $k = n$ has to hold. But for a given n the equation $h(x) + h(1 - nr - t) = s$ has at most one solution x such that $x < r$.

ii) $q_i \leqq r \ \forall i \Rightarrow \bar{q} \succ \bar{p}$.

iii) $q_i \geqq r\ (i = 1, 2, \ldots, k)$; $q_i = 0\ (i > k) \Rightarrow \sum g_r(q_i) = kr$. If $kr \geqq \sum g_r(p_i)$ then $kr \geqq (n + 1)\, r + b \Rightarrow k \geqq n + 2$ has to be fulfilled. We can rearrange $\bar{q}$ so that $q_1 \geqq q_2 \geqq \cdots \geqq q_k \geqq r$. $q_1 = 1 - \sum_{2}^{\infty} q_i \leqq 1 - (n + 1)\, r < c$, and therefore $\bar{q} \succ \bar{p}$ ∎

Lemma 6: Assume that there is an r-typically vector $\bar{p} \in \mathscr{S}$. Then the functional G_r has no local maximum in $\mathscr{S}$ at points $\bar{q} = (q_i)$ such that

i) for more than one index i $\ 0 < q_i < r$,

ii) $G_r(\bar{q}) \geqq G_r(\bar{p})$, and for more than one index i $\ q_i > r$.

Proof: i) Suppose that $0 < q_1 < r$, $0 < q_2 < r$. Because of Lemma 5 (ii) we can assume that $q_3 > r$. Now the problem

$$\sum_{1}^{3} g_r(p_i) = \text{Extr.!}, \quad \sum_{1}^{3} p_i = \sum_{1}^{3} q_i = \text{const.}, \quad \sum_{1}^{3} h(p_i) = \sum_{1}^{3} h(q_i)$$

can be solved by application of the Lagrange multiplier rule. Because of the given constraints we get $q_1 = q_2$ as a necessary condition for $\bar{q}$ to be extremally. This however is a local minimum point of the functional $\sum_{1}^{3} g_r(q_i)$. Therefore $\bar{q}$ cannot be a local maximum of G_r.

ii) Analogously. One has to remember that Lemma 5 (iii) allows us to restrict the considerations to the case $0 < q_1 < r$, $q_2 > r$, $q_3 > r$ ∎

With the lemmas proved above we can see that the solution of (5.1) holds. Of course, either we can find n as in step (i) or $r > a_n'$ for some n. In the latter case there is a $\bar{q} \in \mathscr{S}$ such that $q_i \geqq r$ for all i, therefore $\sum g_r(q_i) = 1$. The former case leads to case (a) of the solution iff $s \neq \log (n + 1)$. The calculated b is smaller than r. Therefore either $G_r(\bar{r}_n) = 1$ or $\bar{r}_n$ is $r - (n - 1)$-typically. From Lemma 4 we know that any r-typically vector $\bar{p} \in \mathcal{S}$ is a local maximum of G_r. Lemmas 5 and 6 say that there is no further local maximum of G_r in $\mathcal{S}$ greater than $G_r(\bar{r}_n)$. This means that $\bar{r}_n$ is the global maximum point.

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VERFASSER:

Dr. THOMAS DE PALY
Sektion Physik und Naturwissenschaftlich-Theoretisches Zentrum
der Karl-Marx-Universität
DDR-7010 Leipzig, Karl-Marx-Platz 10

Gegenwärtige Adresse: DDR-8060 Dresden, Bischofsweg 28H