Remarks on the dual least action principle

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Es sei $L$ ein selbstadjungierter Operator mit abgeschlossenem Wertevorrat in einem Hilbert- 
raum $H$, und es sei $\varphi : H \to \mathbb{R}$ eine konvexe Funktion. Unter der Voraussetzung, daß keine 
Resonanz vorhanden ist, wird die Frage behandelt, ob der Wertevorrat von $L + \varphi$ mit 
ganz $H$ zusammenfällt.

Пусть $L$ — самоспрямленный оператор с замкнутой областью значений в гильбертовом 
пространстве $H$, $\psi : H \to \mathbb{R}$ — выпуклая функция. В одном нерезонансном случае 
находят достаточные условия для того, чтобы область значений оператора $L + \varphi$ 
совпадала с пространством $H$.

Let $L$ be a self-adjoint operator with a closed range in a Hilbert space $H$ and let $\varphi$ be a convex 
function on $H$. Under a non resonance assumption the surjectivity of $L + \varphi$ is studied.

Introduction

Let $H$ be a real Hilbert space, let $L : D(L) \subseteq H \to H$ be a self-adjoint operator with 
a closed range and let $\varphi : H \to \mathbb{R}$ be a continuous convex function. The surjeotivity 
of $L + \varphi$ is studied under a non resonance condition due to Dolph [8]. The basic 
tool is the dual least action principle of Clarke and Ekeland. In contrast to the 
previous applications of this principle, we consider the case when the right inverse 
of $L$ is not necessarily a compact operator.

1. The dual least action principle

Let $H$ be a real Hilbert space with inner product $(\cdot, \cdot)$ and corresponding norm $|\cdot|$. 
Let $L : D(L) \subseteq H \to H$ be a self-adjoint operator with a closed range and let $\varphi : H \to \mathbb{R}$ 
be a continuous convex function. Let $\alpha$, $\beta$, $\gamma$ and $c$ be real numbers such that 
$0 < \beta \leq \gamma < \alpha$ and

$$(A_1) \sigma(L) \cap \{0\} = \{\alpha, 0\} = \{\varphi, 0\},$$

where $\sigma(L)$ denotes the spectrum of $L$, $(A_2)$ for every $u \in H$,

$$\beta \frac{|u|^2}{2} - c \leq \psi(u) \leq \gamma \frac{|u|^2}{2} + c.$$ 

Let us write

$K = \{L \mid D(L) \cap R(L)\}^{-1}$,

$$\psi^*(v) = \sup_{u \in H} [(v, u) - \psi(u)], \quad v \in H,$$

and

$$\varphi(v) = \frac{1}{2} (Kv, v) + \psi^*(v), \quad v \in R(L).$$
The function $\psi^*$ is the Fenchel transform of $\psi$. The present formulation of the "dual action" $\varphi$ was introduced in [5] for hyperbolic problems and in [9] for hamiltonian systems. See [7] and [10] for other abstract formulations.

**Lemma 1:** Under assumption $A_2$, if $\varphi$ has a local minimum on $R(L)$, then equation

$$-Lu \in \partial \psi(u)$$

is solvable.

**Proof:** If $\varphi$ has a local minimum at $v \in R(L)$, then for every $h \in R(L)$ sufficiently small and for every $t \in [0, 1[$, we have

$$\psi^*(v) \leq \psi^*(v + th) + t(Kv, h) + \frac{t^2}{2} (Kh, h).$$

Thus

$$-(Kv, h) \leq \frac{\psi^*(v + th) - \psi^*(v)}{t} + \frac{t^2}{2} (Kh, h).$$

If $t \downarrow 0$, we obtain denoting by $\delta^+\psi^*(v, \cdot)$ the right Gateaux variation at $v$

$$-(Kv, h) \leq \delta^+\psi^*(v, h).$$

Since $\delta^+\psi^*(v, \cdot)$ is positively homogeneous and subadditive, the Hahn-Banach theorem insures the existence of $w \in ker L$ such that, for every $h \in H$,

$$(w, h) - (Kv, h) \leq \delta^+\psi^*(v, h).$$

But then

$$(w - Kv, h) \leq \psi^*(v + h) - \psi^*(v),$$

i.e. $w - Kv \in \partial \psi^*(v)$. It follows that $v \in \partial \psi(w - Kv)$. If $u = w - Kv$, $-Lu = v$ and $u$ is a solution of (1) \(\blacksquare\)

The following lemma has been widely used in the study of hamiltonian systems (see [9]).

**Lemma 2:** Under assumptions $A_1$ and $A_2$, $\varphi$ is coercive on $R(L)$, i.e. $\varphi(v) \to \infty$, as $|v| \to \infty$.

**Proof:** It suffices to observe that $A_1$ and $A_2$ imply that

$$\forall v \in R(L) \quad -\frac{1}{\alpha} |v|^2 \leq (Kv, v)$$

and

$$\forall v \in H \quad \frac{1}{\gamma} \frac{|v|^2}{2} - c \leq \psi^*(v) \quad \blacksquare$$
2. Surjectivity theorems

The following result generalizes Theorem 2.1 of [8] and Theorem 3.2 of [6]. It extends Theorem 1 of [7].

**Theorem 1:** Under assumptions $A_1$ and $A_2$, if $\sigma(L) \cap ]-\infty, 0[ \text{ consists of isolated eigenvalues with finite multiplicity, then } L + \partial \varphi \text{ is onto.}$

**Proof:** Since, for any $f \in H$, the function $\varphi(u) - (f, u)$ has the same properties as $\varphi(u)$, it suffices to prove that (1) is solvable. By assumption $\frac{1}{2} (Ku, v)$ is weakly lower semi-continuous (w.l.s.c.). Therefore $\varphi$ itself is w.l.s.c. By Lemma 2, $\varphi$ has a minimum on $R(L)$ and, by Lemma 1, (1) is solvable.

The following result extends Theorem 3.7 of [1].

**Theorem 2:** Under assumptions $A_1$ and $A_2$, if
(a) $\varphi$ is differentiable and $\partial \varphi$ is Lipschitzian with constant $k$
(b) $\sigma(L) \cap ]-k, 0[ \text{ consists of isolated eigenvalues with finite multiplicity, then } L + \partial \varphi \text{ is onto.}$

**Proof:** As in Theorem 1, it suffices to prove that (1) is solvable. If $f_i \in \partial \varphi(v_i)$ we have $v_i = \partial \varphi(f_i)$ ($i = 1, 2$). By Corollary 10 of [3], assumption (a) implies that

$$(v_1 - v_2, f_1 - f_2) \geq \frac{1}{k} |v_1 - v_2|^2.$$

Thus $\varphi(v) = \varphi^*(v) - \frac{1}{k} \frac{|v|^2}{2}$ is convex.

Let $\{P_i : \lambda \in \mathbb{R}\}$ be the spectral resolution of $L$ and let us write

$$Q_1 = \int_{]-\infty, -k]} dP_1, \quad Q_2 = \int_{]-k, 0]} dP_1, \quad u = Q_i u \quad (i = 1, 2; u \in H).$$

It follows from assumption (b) that $\varphi_2(v) = \frac{1}{2} (Kv_2, v_2)$ is w.l.s.c. on $R(L)$. Moreover, for any $v \in R(L)$,

$$(Kv_1, v_1) \geq -\frac{1}{k} |v_1|^2 \geq -\frac{1}{k} |v|^2.$$

Thus $\varphi_2(v) = \frac{1}{2} (Kv_1, v_1) + \frac{1}{k} \frac{|v|^2}{2}$ is convex. Finally $\varphi = \varphi_1 + \varphi_2 + \varphi_2$ is w.l.s.c.

and, by Lemmas 1 and 2, (1) is solvable.

As an obvious consequence of Theorem 2 we obtain:

**Corollary 1:** Under assumptions $A_1$ and $A_2$, if $\varphi$ is differentiable and $\partial \varphi$ is Lipschitzian with constant $\alpha$, then $L + \partial \varphi$ is onto.

**Remark:** Corollary 1 generalizes Theorem 1.2 of [8], Theorem 1 of [11] and extends Theorem I.12 of [4].
3. Periodic solutions of a nonlinear hyperbolic equation

This section is devoted to the existence of $2\pi$-periodic solutions in $t$ and $x$ of the nonlinear hyperbolic equation

$$u_{tt} - u_{xx} + \lambda u + \partial j(u) = f(t, x)$$

where $j: \mathbb{R} \rightarrow \mathbb{R}$ is convex and $f \in H = L^2([0, 2\pi]^2)$. We shall only consider the case when $\lambda = 1$. The other cases are left to the reader. The case when $\lambda = 0$ is treated in [7].

Let $A$ be the linear operator defined by

$$D(A) = \{u \in C^2([0, 2\pi]^2) : u(0, \cdot) - u(2\pi, \cdot) = u_t(0, \cdot) - u_t(2\pi, \cdot) = 0\}$$

$$Au = u_{tt} - u_{xx}.$$ 

Let us write $A = A^*$. Then $A$ is self-adjoint and $\sigma(A)$, which is the set of odd integers and of multiples of 4, consists of eigenvalues which are of finite multiplicity except 0 (see [12]).

Let us define $\psi: H \rightarrow \mathbb{R}$ by

$$\psi(u) = \int_0^{2\pi} \int_0^{2\pi} j(u(t, x)) \, dt \, dx.$$ 

The following theorem extends the results of [13].

**Theorem 3:** Assume that there exists $\beta$, $\gamma$ and $c \in \mathbb{R}$ such that $0 < \beta \leq \gamma < 2$ and, for every $u \in \mathbb{R},$

$$\beta \frac{u^2}{2} - c \leq j(u) \leq \gamma \frac{u^2}{2} + c,$$ 

then equation

$$Au + u + \partial \psi(u) = f$$

is solvable for every $f \in H$.

**Proof:** It suffices to apply Theorem 1 with $L = A + I$ and $\alpha = 2$. 

4. The growth of $\partial \psi$

A sharp estimate of the growth of $\partial \psi$ under assumption $A_2$ is given.

**Proposition 1:** Under assumption $A_2$, there exists $c' \in \mathbb{R}$ such that, for every $f, u \in H,$

$$f \in \partial \psi(u) \Rightarrow |f| \leq 2\gamma |u| + c'.$$ 

**Proof:** Let $u \in H$ and $f \in \partial \psi(u)$. Assuming $f \neq 0$, let us write $g = ||f||$. We have, for every $t \in \mathbb{R},$

$$(f, tg - u) + \psi(u) \leq \psi(tg)$$
or

\[ 0 \leq \varphi(tg) - t \frac{|f|}{2} + (f, u) - \varphi(u). \]

By assumption

\[ 0 \leq t^2 \frac{\gamma}{2} + c - t \frac{|f|}{2} + |u| \frac{|f|}{2} - c'' \]

where \( c'' = \inf_{\omega \in \mathcal{H}} \varphi(u) > -\infty. \) Then

\[ |f|^2 \leq 4 \frac{\gamma}{2} (c - c'' + |u| |f|) \]

and (3) follows easily.

Remark: The argument of the proof is due to Brézis and Nirenberg [4, p. 312].

The following example shows that estimate (3) is sharp. Let \( \gamma \in ]1/2, 1[ \), \( \beta \in ]0, 2\gamma - 1[ \), \( p = \frac{\gamma - \beta}{1 - \gamma} \) and \( q = \frac{p - \beta}{p - 1} \). Let us define inductively \( t_0 = \gamma / \beta \), \( u_n = pt_n \) and \( t_{n+1} = u_n / \beta \). Let us define \( r : \mathbb{R} \to \mathbb{R} \) by

\[
    r(u) = \begin{cases} 
        \gamma u, & u \in ]-\infty, 1] \\
        \gamma, & u \in ]1, t_0] \\
        qu + (1 - q) u_n, & u \in ]t_n, u_n] \\
        u_n, & u \in ]u_n, t_{n+1}]. 
    \end{cases}
\]

If \( \varphi(u) = \int_0^u r(s) \, ds \), for any \( u \in \mathbb{R} \),

\[ \frac{\beta}{2} u^2 \leq \varphi(u) \leq \gamma \frac{u^2}{2}. \]

But, for any \( n \in \mathbb{N} \), \( \partial \varphi(u_n) = u_n \) and \( u_n \to \infty, n \to \infty. \) Moreover, if \( L : \mathbb{R} \to \mathbb{R} \) is defined by \( Lu = -u \), assumptions \( A_1 \) and \( A_2 \) are satisfied with \( \alpha = 1 \), and, for any \( n \in \mathbb{N} \), \( Lu_n + \partial \varphi(u_n) = 0. \) Thus conditions \( A_1 \) and \( A_2 \) doesn’t imply any a priori bound for the solutions of (1) (see also [8] and [2]).

REFERENCES


Manuskripteingang: 08. 05. 1981

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