Energy estimates for a class of semilinear elliptic equations on half Euclidean balls

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Abstract. For a class of semi-linear elliptic equations with critical Sobolev exponents and boundary conditions, we prove pointwise estimates for blowup solutions and energy estimates. A special case of this class of equations is a locally defined prescribing scalar curvature and mean curvature type equation.

1. Introduction

In this article we consider the problem

\begin{align*}
-\Delta u &= g(u), \quad \text{in } B^n_3^+, \\
\frac{\partial u}{\partial x_n} &= h(u), \quad \text{on } \partial B^n_3^+ \cap \partial \mathbb{R}_n^+,
\end{align*}

where \( u > 0 \) is a positive continuous solution, \( B^n_3^+ \) is the upper half ball centered at the origin with radius 3, \( g \) is a continuous function on \((0, \infty)\) and \( h \) is locally Hölder continuous on \((0, \infty)\).

If \( g(s) = s^{(n+2)/(n-2)} \) and \( h(s) = cs^{n/(n-2)} \), the equation (1.1) is a typical curvature equation. If we use \( \delta \) to represent the Euclidean metric, then \( u^{4/(n-2)} \delta \) is conformal to \( \delta \). Equation (1.1) in this special case means that the scalar curvature under the new metric is \( 4(n-1)/(n-2) \), and that the boundary mean curvature under the new metric is \( -\frac{2}{n-2} c \). Equation (1.1) is very closely related to the well-known Yamabe problem and to the boundary Yamabe problem. For \( g \) and \( h \) we assume:

\( GH_0 : g \) is a continuous function on \((0, \infty)\), \( h \) is Hölder continuous on \((0, \infty)\), and

\( GH_1 : \begin{cases} g(s) s^{-(n+2)/(n-2)} \text{ is non-increasing,} \\
\lim_{s \to \infty} g(s) s^{-(n+2)/(n-2)} \in (0, \infty), \\
\lim_{s \to \infty} s^{n/(n-2)} h(s) \text{ is non-decreasing and} \\
\lim_{s \to \infty} s^{n/(n-2)} h(s) < \infty.
\end{cases} \)

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The main result of this article is concerned with the case if $c \neq 0$ our assumption on $g$, $h$ is

$$c := \lim_{s \to \infty} s^{-n/(n-2)} h(s).$$

Then if $c > 0$ we assume

$$GH_2 : \sup_{0 < s \leq 1} \frac{g(s)}{s} < \infty, \quad \text{and} \quad \sup_{0 < s \leq 1} \frac{|h(s)|}{s} < \infty.$$ 

If $c \leq 0$ our assumption on $g$, $h$ is

$$GH_3 : \sup_{0 < s \leq 1} g(s) < \infty, \quad \text{and} \quad \sup_{0 < s \leq 1} |h(s)| < \infty.$$ 

The main result of this article is concerned with the case $c > 0$:

**Theorem 1.1.** Let $u > 0$ be a solution of (1.1) where $g$ and $h$ satisfy $GH_0$ and $GH_1$. Suppose $c > 0$ and $GH_2$ also holds. Then,

$$\int_{B^+_R} |\nabla u|^2 + u^{2n/(n-2)} \leq C,$$

for some $C > 0$ that depends only on $g$, $h$ and $n$.

Obviously, if

$$g(s) = c_1 s^{(n+2)/(n-2)}, \quad c_1 > 0 \quad \text{and} \quad h(s) = c_h s^{n/(n-2)}, \quad c_h > 0,$$

then $g$ and $h$ satisfy the assumptions in Theorem 1.1. The energy estimate (1.3) for this special case has been proved by Li–Zhang [9]. It is easy to see that the assumptions on $g$ and $h$ in Theorem 1.1 include a much larger class of functions. For example, for any non-increasing function $c_1(s)$ satisfying $\lim_{s \to \infty} c_1(s) > 0$ and $\lim_{s \to 0^+} c_1(s) s^{4/(n-2)} < \infty$, $g(s) = c_1(s) s^{(n+2)/(n-2)}$ satisfies the assumptions of $g$. Similarly $h(s) = c_2(s) s^{n/(n-2)}$ for a nondecreasing function $c_2(s)$ with $\lim_{s \to \infty} c_2(s) = c_h$ and $\lim_{s \to 0^+} |c_2(s)| s^{2/(n-2)} < \infty$, satisfies the requirement of $h$ in Theorem 1.1.

For the case $c_h \leq 0$ we have:

**Theorem 1.2.** Let $u > 0$ be a solution of (1.1) where $g$ and $h$ satisfy $GH_0$ and $GH_1$. Suppose $c_h \leq 0$ and $g$ and $h$ satisfy $GH_3$. Then the energy estimate (1.3) holds for $C$ depending only on $g$, $h$ and $n$.

If we allow $\lim_{s \to \infty} s^{-(n+2)/(n-2)} g(s) = 0$, then the energy estimate (1.3) may not hold. For example, let $g(s) = \frac{1}{4} (s+1)^{-3}$; then $g$ satisfies the assumption in Theorem 1.2 except that $\lim_{s \to \infty} s^{-(n+2)/(n-2)} g(s) = 0$. Let $u_j(x) = \sqrt{x_1 + j - 1}$. It is easy to verify that $u_j$ satisfies

$$\begin{cases}
-\Delta u_j = g(u_j) & \text{in } B^+_3, \\
\frac{\partial u_j}{\partial x_n} = 0, & \text{on } \partial B^+_3 \cap \partial \mathbb{R}^n_+.
\end{cases}$$

Note that $h = 0$ in this case. Then clearly (1.3) does not hold for $u_j$. 

Let

$$u := \lim_{j \to \infty} u_j,$$
The energy estimate (1.3) is closely related to the following Harnack type inequality:

\[(\max_{B_{1}^{+}} u)(\min_{B_{2}^{-}} u) \leq C,\]

which was proved by Li–Zhang [9] for the special case (1.4). Li–Zhang [9] also proved (1.3) for equation (1.4) using (1.5) in their argument in a nontrivial way.

In the past two decades Harnack type inequalities similar to (1.5) have played an important role in blowup analysis for semilinear elliptic equations with critical Sobolev exponents. Pioneer works in this respect can be found in Schoen [13], Schoen–Zhang [14], Chen–Lin [3] and Li [8], etc. Further results can be found in [2], [4], [5], [7], [9], [10], [12], [14], [16] and the references therein. Usually for a semi-linear equation without boundary condition, for example the conformal scalar curvature equation

\[\Delta u + K(x) u^{(n+2)/(n-2)} = 0, \quad \text{in } B_{3},\]
a Harnack inequality of the type

\[(\max_{B_{1}} u)(\min_{B_{2}} u) \leq C\]

immediately leads to the energy estimate

\[\int_{B_{1}} |\nabla u|^{2} + u^{2n/(n-2)} \leq C\]

by Green’s representation theorem and integration by parts (see [3] for a proof). However, when a boundary condition as in (1.1) intervenes, using the Harnack inequality (1.5) to derive (1.3) is much more involved. In order to derive energy estimate (1.3) and pointwise estimates for blow up solutions, Li and Zhang prove the following results in [9]:

**Theorem A** (Li–Zhang, [9]). Let \(u > 0\) be a solution of (1.1), where \(g\) and \(h\) satisfy \(GH_{0}, GH_{1}\) and \(GH_{3}\). Then

\[(\max_{B_{1}^{+}} u)(\min_{B_{2}^{-}} u) \leq C.\]

Here we note that in Theorem A no sign of \(c_{h}\) is specified. One would expect the energy estimate (1.3) to follow directly from Li–Zhang’s theorem. This is indeed the case if \(c_{h} \leq 0\). However for \(c_{h} > 0\) substantially more estimates are needed in order to establish a precise pointwise estimate for blowup solutions. As a matter of fact we need to assume \((GH_{2})\) instead of \((GH_{3})\) in order to obtain (1.3).

The organization of this article is as follows. In Section 2 we prove Theorem 1.1. The idea of the proof is as follows. First we use a selection process to locate regions in which the bubbling solutions look like global solutions. Then we consider the interaction of the bubbling regions. Using delicate blowup analysis and Pohozaev identity we prove that bubbling regions must be a positive distance apart. In Section 3 we prove Theorem 1.2 using Theorem A and integration by parts.
The following notations will be used throughout the paper:
\[ B(x, \sigma) \] is the ball centered at \( x \) with radius \( \sigma \),
\[ B^+(x, \sigma) := B(x, \sigma) \cap \mathbb{R}^n_+ \], \( B_\sigma = B(0, \sigma) \), \( B^+_\sigma = B_\sigma \cap \mathbb{R}^n_+ \),
\[ B_T(x, \sigma) := B(x, \sigma) \cap \{ y_n > T \}, \quad B^+_T = B_T(0, T) \],
\[ \partial' B^T(x, \sigma) := \partial B_T(x, \sigma) \cap \{ y_n = T \}, \quad \partial'' B^T(x, \sigma) \cap \{ y_n > T \} \].

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2. Proof of Theorem 1.1

First we recall that Theorem 1.1 deals with the \( c_h > 0 \) case, which is substantially harder than the other case. The proof of Theorem 1.1 is by contradiction. Suppose there is no energy bound; then there exists a sequence \( u_k \) such that
\[ \int_{B_1} |\nabla u_k|^2 + u_k^{2n/(n-2)} \to \infty. \]

We claim that \( \max_{B^{+}_{3/2}} u_k \to \infty \). Indeed, if this is not the case, which means that there is a uniform bound for \( u_k \) on \( B^{+}_{3/2} \), we just take a cut-off function \( \eta \in C^\infty \) such that \( \eta \equiv 1 \) on \( B^+_1 \) and \( \eta \equiv 0 \) on \( B^+_2 \setminus B^+_1 \) and \( |\nabla \eta| \leq C \). Multiplying the equation (1.1) by \( u_k \eta^2 \), using integration by parts and Cauchy’s inequality we obtain a uniform bound of \( \int_{B_1} |\nabla u_k|^2 \), a contradiction to (2.1).

Since the remaining part of the proof is technical in nature, it may be helpful to explain the outline of the approach. First we use a selection process to determine a bubbling area which consists of disjoint balls. Each ball is shrinking to a point as \( k \) tends to infinity and the profile of bubbling solutions in each ball is very similar to that of a globally defined solution. In the second step we focus on the interaction of the bubbling balls, which is the most essential part of the proof. We shall employ the standard moving sphere method to obtain an upper bound of solutions not only within each bubbling ball, but also on the region outside the bubbling balls. Then by comparing the lower and upper bound of solutions we prove that all the bubbles are of comparable magnitude. Then we use the Pohozaev identity, a balancing condition, around a bubbling ball to prove that all bubbling balls have to be a positive distance apart and the distance is independent of \( k \). Finally in step three the conclusion of Theorem 1.1 follows from standard elliptic estimates.

2.1. Step one: locating bubbling balls

Proposition 2.1. Let \( u \) be a solution of (1.1). Then, for any \( \epsilon \in (0, 1) \) and \( R > 1 \), there exist positive constants \( C_0(n, \epsilon, R, g, h) \) and \( C_1(n, \epsilon, R, g, h) \) such that if \( \max_{B^+_1} u > C_0 \), there exists a set \( Z = \{ q_1, \ldots, q_L \} \subset B^+_2 \) of local maximum points of \( u \) such that
\[ \left\| \frac{u(q_j)^{-1} u(q_j)^{-2/(n-2)} y + q_j}{y^2} \right\|^2_{C^2(B^+_{3/4})} < \epsilon, \]
Proof of Lemma 2.1 requires the following lemma.

Lemma 2.2. Given any $R > 1$, $\epsilon \in (0, 1)$, there exists $C_2(R, \epsilon, g, h) > 1$ such that for any compact $K \subset \bar{B}_1^+$ and any $u$ of (1.1) with

$$\max_{q \in \bar{B}_2^+ \setminus K} \text{dist} (q, K)^{2/(n-2)} u(q) \geq C_2,$$

there exists $q_0 \in \bar{B}_{n/2}^+ \setminus K$, which is a local maximum of $u$, and

$$u^{-1}(q_0) u\left(u(q_0)^{2/(n-2)} y + q_0\right) - \left(1 + \frac{A}{n(n-2)} |y|^2\right)^{-(n-2)/2} \|_{C^2(B_{n/2}^+)} < \epsilon,$$

where $T = u(q_0)^{2/(n-2)} q_{0n}$.

Proof of Lemma 2.2. We prove it by contradiction. Suppose no such $C_2$ exists for some $\epsilon$ and $R$. Then there exist compact subsets $K_k \subset \bar{B}_1^+$ and a sequence of solutions $u_k$ such that $\max_{B_2^+ \setminus K_k} \text{dist}(x, K_k)^{2/(n-2)} u_k(x) > k$ and no $q_0$ as in (2.3) exists. Let $x_k$ satisfy

$$u_k(x_k) \text{dist} (x_k, K_k)^{(n-2)/2} > k, \quad d_k = \text{dist} (x_k, K_k),$$

and

$$S_k(y) = u_k(y) (d_k - |y - x_k|)^{(n-2)/2}, \quad \forall y \in B_{n/2}^+.$$

Suppose that $S_k$ reaches its maximum in $B^+(x_k, d_k)$ at $\hat{x}_k$. Then

$$S_k(\hat{x}_k) \geq S_k(x_k) = u_k(x_k) d_k^{(n-2)/2} > k.$$

Let $\sigma_k = \frac{1}{2} (d_k - |x_k - \hat{x}_k|)$. Then clearly (2.4) can be written as

$$u_k(\hat{x}_k) 2^{(n-2)/2} \sigma_k^{(n-2)/2} \geq u_k(x_k) d_k^{(n-2)/2} \longrightarrow \infty \quad \text{as} \quad k \longrightarrow \infty.$$

For all $x \in B^+(\hat{x}_k, \sigma_k)$, since

$$u_k(x) (d_k - |x - x_k|)^{(n-2)/2} \leq u_k(\hat{x}_k) (d_k - |x_k - \hat{x}_k|)^{(n-2)/2},$$
we have
\[ u_k(x) \leq u_k(\hat{x}_k) \left( \frac{d_k - |x_k - \hat{x}_k|}{d_k - |x - \hat{x}_k|} \right)^{(n-2)/2}. \]

Using \(|x - \hat{x}_k| \leq \sigma_k\), and
\[ d_k - |x - x_k| \geq d_k - |x_k - \hat{x}_k| - |x - \hat{x}_k| \geq \sigma_k, \]
we obtain
\[ u_k(x) \leq 2^{(n-2)/2} u_k(\hat{x}_k), \quad \text{for all } x \in B^+(\hat{x}_k, \sigma_k). \]

Let \( v_k(y) = M_k^{-1} u_k(M_k^{-(n-2)/2} y + \hat{x}_k) \), \( M_k^{-(n-2)/2} y + \hat{x}_k \in B^+_1 \).

Direct computation shows
\[ \Delta v_k(y) + (M_k v_k(y))^{-(n+2)/(n-2)} g\{M_k v_k(y)\} \cdot v_k(y)^{(n+2)/(n-2)} = 0. \]

By (2.6) we have
\[ 0 \leq v_k(y) \leq 2^{(n-2)/2} \quad \forall y \in B(0, M_k^{2/(n-2)} \sigma_k) \cap \{ y_n \geq -M_k^{2/(n-2)} \hat{x}_kn \}. \]

We consider two cases.

**Case one in Lemma 2.2:** along a subsequence \( \lim_{k \to \infty} M_k^{2/(n-2)} \hat{x}_kn = \infty. \)

Throughout this article we do not distinguish sequences and their subsequences. Since we always consider subsequence instead of the whole sequence with no difference on their notation, this process will not be repeatedly stated in the remaining part of the article.

Since \( M_k^{2/(n-2)} \sigma_k \) and \( M_k^{2/(n-2)} \hat{x}nk \) both tend to infinity, (2.7) is defined on \(|y| \leq l_k\) for some \( l_k \to \infty\). By (2.8) \( v_k \) is bounded above in \( B_{l_k} \). We claim that along a subsequence \( v_k \to V \) uniformly over all compact subsets of \( \mathbb{R}^n \), where \( V \) satisfies
\[ \Delta V + AV^{(n+2)/(n-2)} = 0, \quad \mathbb{R}^n, \quad V > 0 \text{ in } \mathbb{R}^n. \]

with \( A = \lim_{s \to \infty} s^{-(n+2)/(n-2)} g(s) \). To prove the claim we shall show that for any \( R > 1 \),
\[ v_k(y) \geq C(R) > 0, \quad |y| \leq R. \]

Once (2.10) is established, we have \( M_k v_k \to \infty \) over all \( B_R \), thus
\[ M_k^{-(n+2)/(n-2)} g(M_k v_k) = (M_k v_k)^{-(n+2)/(n-2)} g(M_k v_k) v_k^{(n+2)/(n-2)} \to AV^{(n+2)/(n-2)} \]
over all compact subsets of \( \mathbb{R}^n \). Then it is easy to see that \( V \) solves (2.9).
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Therefore we only need to establish (2.10) for fixed $R > 1$. Let

$$\Omega_{R,k} := \{ y \in B_R : v_k(y) \leq 3M_k^{-1} \}$$

and

$$a_k(y) = M_k^{-(n+2)/(n-2)} g(M_k v_k)/v_k.$$ 

It follows from $(GH_1)$ that in $B_R \setminus \Omega_{R,k}$

$$a_k(y) \leq g(3) v_k^{4/(n-2)} \leq 4 g(3).$$

For $y \in \Omega_{R,k}$ we use $(GH_2)$ to obtain

$$a_k(y) \leq C M_k^{-4/(n-2)}, \quad y \in \Omega_{R,k}.$$ 

In either case $a_k(y)$ is a bounded function. From

$$\Delta v_k(y) + a_k(y) v_k(y) = 0 \quad \text{in} \ B_R$$

and the standard Harnack inequality we have

$$1 = v_k(0) \leq \max_{B_{R/2}} v_k \leq C(R) \min_{B_{R/2}} v_k.$$ 

Thus (2.10) is established.

By the classification theorem of Caffarelli–Gidas–Spruck [1], $V$ is comparable to $O(|y|^{2-n})$ at infinity and there is only one maximum point, $\bar{y}$, in $\mathbb{R}^n$. Correspondingly there exists a sequence of local maximum points of $u_k$, denoted $\bar{x}_k$, that tends to $\bar{y}$ after scaling. Thus if the scaling is centered at $\bar{x}_k$ in the first place, the limit function would be a solution to (2.9) with $V(0) = 1 = \max_{\mathbb{R}^n} V$. By the classification theorem of Caffarelli–Gidas–Spruck,

$$V(y) = \left(1 + \frac{A}{n(n-2)} |y|^2 \right)^{-(n-2)/2}.$$ 

Thus (2.3) holds for all large $k$. Consequently this case is ruled out and we only need to consider:

**Case two in Lemma 2.2:** $\lim_{k \to \infty} M_k^{2/(n-2)} \bar{x}_kn < \infty$.

It is easy to verify that $v_k$ satisfies

$$\begin{cases}
\Delta v_k + (M_k v_k)^{-\frac{n+2}{n-2}} g(M_k v_k) v_k^{\frac{n+2}{n-2}} = 0, & \text{in } \{ y; M_k^{2/(n-2)} y + \bar{x}_k \in B^+_3 \}, \\
\frac{\partial v_k}{\partial y_n} = (M_k v_k)^{-n/(n-2)} h(M_k v_k) v_k^{2/(n-2)} v_k, & \text{on } \{ y_n = -M_k^{2/(n-2)} \bar{x}_kn \}.
\end{cases}$$

We claim that for any $R > 1$, there exists $C(R) > 0$ such that

$$v_k(y) \geq C(R) \quad \text{in } B_R \cap \{ y_n \geq -M_k^{2/(n-2)} \bar{x}_kn \}.$$ 

(2.12)
The proof of (2.12) is similar to the interior case. Let $T_k = M_k^{2/(n-2)}x_kn$ and $p_k = (0', -T_k)$. On $B(p_k, R) \cap \{y_n \geq -T_k\}$ we write the equation for $v_k$ as
\begin{equation}
\begin{cases}
\Delta v_k + a_k v_k = 0, & \text{in } B(p_k, R) \cap \{y_n > -T_k\}, \\
\partial_n v_k + b_k v_k = 0, & \text{on } B(p_k, R) \cap \{y_n = -T_k\}.
\end{cases}
\end{equation}
where it is easy to use $GH_k$ to prove that $|a_k| + |b_k| \leq C$ for some $C$ independent of $k$ and $R$. By a classical Harnack inequality with boundary terms (see, for example, Lemma 6.2 of [15], or Han–Li [6]), we have
\[1 = v_k(0) \leq \max_{B(p_k, R/2)\cap\{y_n \geq -T_k\}} v_k \leq C(R) \min_{B(p_k, R/2)\cap\{y_n \geq -T_k\}} v_k.
\]
Therefore $v_k$ is bounded below by positive constants over all compact subsets. Thus the limit function $V_1$ solves
\begin{equation}
\begin{cases}
\Delta V_1 + AV_1^{1/(n+2)/(n-2)} = 0, & \text{in } \mathbb{R}^n \cap \{y_n > -T\}, \\
\partial_n V_1 = c_h V_1^{n/(n-2)}, & \text{on } \{y_n = -T\},
\end{cases}
\end{equation}
where $T = \lim_{k \to \infty} T_k$. Note that $T > 0$ because $c_h > 0$. By Li–Zhu’s classification theorem, $V_1$ is just the restriction of a solution to (2.9) to $\{y_n > -T\}$. Thus there is a global maximum of $V_1$ in the interior of $\{y_n > -T\}$. Correspondingly there is a sequence of local maximum points $x_k$ of $u_k$ tending to that point after scaling. If the scaling is centered at $x_k$ in the first place, the limit function $V_1$ is just $V$ as in (2.11), and (2.3) holds for all large $k$ in this case as well. Lemma 2.2 is established.

Proof of Proposition 2.1. First we apply Lemma 2.2 by letting $K = \emptyset$ (which implies $d(q, K) = 1$). From Lemma 2.2 we obtain $q_1$. Then we let $K = B_+(q_1, r_1)$, where $r_1 = Rn^{-2/(n-2)}(q_1)$. If
\[\max_{q \in B_+^+ \setminus K} \dist(q, K)^{2/(n-2)} u(q) \leq C_0,
\]
we stop. Otherwise, there is $q_2$ that satisfies $B(q_2, r_2) \cap B(q_1, r_1) = \emptyset$, where $r_2 = u(q_2)^{-2/(n-2)}R$. We continue this process by adding $B_+(q_2, r_2)$ to $K$. This process stops in a finite number of steps, since each selection process implies $\int_{B_+(q, r_1)} |\nabla u|^2 \geq C(n)$ because of the profile of the standard bubbles. Then it is easy to conclude that Proposition 2.1 holds.

2.2. Step two: all bubbles are far apart

The following theorem plays an important role in the proof of our main theorem, Theorem 1.1.

Theorem 2.3. Let $u$ be a solution to (1.1) and $Z$ be the set of maximum points determined in Proposition 2.1. Then for suitably large $R$ (that only depends on $n, g, h$) and $\epsilon \in (0, e^{-R})$, there exists $d_0(R, \epsilon) > 0$ such that if $\max_{B_+^+} u \geq C_0(n, R, \epsilon, g, h),
\min \{\dist(q, q_j), \forall q_i, q_j \in Z \cap B_+^+, q_i \neq q_j\} \geq d_0$.
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Proof of Theorem 2.3. By the way of contradiction, we assume that there exists a sequence of solutions \( \{u_k\} \) such that \( \max_{B^+_1} u_k \to \infty \) and

\[
\min \{ \text{dist}(q_{a_k}^k, q_{b_k}^k); \ 1 \leq a, b \leq N_k, \ a \neq b \} \to 0 \quad \text{as} \quad k \to \infty,
\]

where \( q_1^k, \ldots, q_{N_k}^k \) are the points determined by Proposition 2.1 for \( u = u_k \).

Let \( Z_k \) be the set of local maximum points of \( u_k \) determined in Proposition 2.1. Let \( q_k \in Z_k \) and suppose \( \sigma_k = \text{dist}(q_k, Z_k \setminus \{q_k\}) \) and we let

\[
\tilde{u}_k(y) = \sigma_k^{(n-2)/2} u_k(q_k + \sigma_k y), \quad \text{in} \ \Omega_k
\]

where \( \Omega_k := \{ y : q_k + \sigma_k y \in B_{1/3}^+ \} \). By the selection process we have

\[
\tilde{u}_k(y) \leq C |y|^{-(n-2)/2}, \quad |y| \leq 3/4, \ y \in \Omega_k
\]

and

\[
\tilde{u}_k(0) \to \infty.
\]

We prove in the following proposition that \( \tilde{u}_k \) decays like a harmonic function.

**Proposition 2.4.** There exists \( C > 0 \) independent of \( k \) such that along a subsequence

\[
\tilde{u}_k(0) \tilde{u}_k(y) |y|^{n-2} \leq C, \quad \text{for} \ y \in B_{2/3} \cap \Omega_k.
\]

**Proof of Proposition 2.4.** Direct computation shows that \( \tilde{u}_k \) satisfies

\[
\begin{aligned}
\Delta \tilde{u}_k(y) + \sigma_k^{(n+2)/2} g(\sigma_k^{(n-2)/2} \tilde{u}_k) &= 0, \quad \text{in} \ \Omega_k, \\
\partial_n \tilde{u}_k(y) + \sigma_k^{n/2} h(\sigma_k^{(n-2)/2} \tilde{u}_k) &= 0, \quad \text{on} \ \partial \Omega_k \cap \{ y_n = -\sigma_k^{-1} q_{kn} \},
\end{aligned}
\]

Let \( \tilde{M}_k = \tilde{u}_k(0) \). By (2.16) \( \tilde{M}_k \to \infty \). Set

\[
v_k(z) = \tilde{M}_k^{-1} \tilde{u}_k(\tilde{M}_k^{2/(n-2)} z), \quad \text{for} \ z \in \tilde{\Omega}_k,
\]

where

\[
\tilde{\Omega}_k := \{ z : |z| \leq \tilde{M}_k^{2/(n-2)}, \ \tilde{M}_k^{2/(n-2)} z \in \Omega_k \}
\]

Note that \( v_k \) is defined on a bigger set, but for the proof of Proposition 2.4 we only need to consider the part in \( \tilde{\Omega}_k \).

Direct computation gives

\[
\begin{aligned}
\Delta v_k(z) + l_k^{-(n+2)/(n-2)} g(l_k v_k) &= 0, \quad z \in \tilde{\Omega}_k, \\
\partial_n v_k(z) &= l_k^{-n/(n-2)} h(l_k v_k), \quad \{ z_n = -T_k \} \cap \partial \tilde{\Omega}_k,
\end{aligned}
\]

where \( l_k = \sigma_k^{-(n-2)/2} \tilde{M}_k \) and \( T_k = l_k^{2/(n-2)} q_{kn} \). We consider two cases.
Case one in Proposition 2.4: \( T_k \rightarrow \infty \).

As in the proof of Proposition 2.1, there exist \( R_k \rightarrow \infty \) such that 
\[
\|v_k(z) - \left(1 + \frac{A}{n(n-2)|z|^2}\right)^{-(n-2)/2}\|_{C^{1,\alpha}(B_{R_k})} \leq CR_k^{-1}.
\]

Clearly (2.17) holds for \(|z| \leq M_k^{2/(n-2)}R_k\), so we just need to prove (2.17) for the case \(|z| > M_k^{2/(n-2)}R_k\).

Lemma 2.5. There exists \( k_0 > 1 \) such that for all \( k \geq k_0 \) and \( r \in (R_k, \tilde{M}_k^{2/(n-2)})\),
\[
\min_{\partial B_r \cap \tilde{\Omega}_k} v_k \leq 2 \left(\frac{n(n-2)}{A}\right)^{(n-2)/2} r^{2-n}.
\]

Proof of Lemma 2.5. Suppose (2.20) does not hold. Then there exist \( r_k \) such that
\[
v_k(z) \geq 2 \left(\frac{n(n-2)}{A}\right)^{(n-2)/2} r_k^{2-n}, \quad |z| = r_k, \ z \in \tilde{\Omega}_k.
\]

Clearly \( r_k \geq R_k \). Let
\[
v_k^\lambda(z) = \left(\frac{\lambda}{|z|}\right)^{n-2} v_k(z^\lambda), \quad z^\lambda = \frac{\lambda^2 z}{|z|^2}.
\]

One checks that \( v_k^\lambda \) satisfies
\[
\Delta v_k^\lambda(z) + \left(\frac{\lambda}{|z|}\right)^{n+2} \left|\frac{\partial}{\partial \lambda}\right| v_k^\lambda(z) = 0, \quad \text{in } \Sigma^\lambda
\]
where
\[
\Sigma^\lambda := \{ z \in \tilde{\Omega}_k; \ |\lambda| < |z| < r_k \}.
\]

Clearly \( v_k^\lambda \rightarrow V^\lambda \) in \( C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n) \) for fixed \( \lambda > 0 \). By direct computation,
\[
V(z) > V^\lambda(z), \text{ for } \lambda \in \left(0, \left(\frac{n(n-2)}{A}\right)^{1/2}\right), \quad |z| > \lambda
\]
\[
V(z) < V^\lambda(z), \text{ for } \lambda > \left(\frac{n(n-2)}{A}\right)^{1/2}, \quad |z| > \lambda.
\]

We shall apply the method of moving spheres for \( \lambda \in \left(\frac{1}{2}\left(\frac{n(n-2)}{A}\right)^{1/2}, 2\left(\frac{n(n-2)}{A}\right)^{1/2}\right) \).

First we prove that, for
\[
\lambda_0 = \frac{1}{2} \left(\frac{n(n-2)}{A}\right)^{1/2},
\]
we have
\[
v_k(z) > v_k^{\lambda_0}(z), \quad z \in \Sigma_{\lambda_0}.
\]
To prove (2.25) we first observe that $v_k > v_k^{\lambda_0}$ in $B_R \setminus B_{\lambda}$ for any fixed $R$ large. Indeed, $v_k = v_k^{\lambda_0}$ on $\partial B_{\lambda_0}$. On $\partial B_{\lambda_0}$, we have $\partial_v V > \partial_v V^{\lambda_0}$. Thus the $C^{1,\alpha}$ convergence of $v_k$ to $V$ gives that $v_k > v_k^{\lambda_0}$ near $\partial B_{\lambda_0}$. Then by the uniform convergence we further know that (2.25) holds on $B_R \setminus B_{\lambda_0}$. On $\partial B_R$, we have

\begin{equation}
(2.26) \quad v_k(z) \geq \left( \left( \frac{(n-2)}{A} \right)^{(n-2)/2} - \varepsilon \right) |z|^{2-n}, \quad |z| = R
\end{equation}

and

\begin{equation}
(2.27) \quad v_k^\alpha(z) \leq \left( \left( \frac{(n-2)}{A} \right)^{(n-2)/2} - 2\varepsilon \right) |z|^{2-n}, \quad |z| \geq R
\end{equation}

for some $\varepsilon > 0$ independent of $k$. Next we shall use the maximum principle to prove that

\begin{equation}
(2.28) \quad v_k(z) > \left( \left( \frac{(n-2)}{A} \right)^{(n-2)/2} - 2\varepsilon \right) |z|^{2-n} > v_k^\alpha(z), \quad z \in \Sigma_{\lambda_0} \setminus B_R.
\end{equation}

The proof of (2.28) is by contradiction. We shall compare $v_k$ and

\[ f_k := \left( \left( \frac{(n-2)}{A} \right)^{(n-2)/2} - 2\varepsilon \right) |z|^{2-n}. \]

Clearly $v_k - f_k$ is super harmonic in $\Sigma_{\lambda_0} = B_R$ and, by (2.26), (2.27) and (2.21), $v_k - f_k > 0$ on $\partial B_R$ and $\partial \Sigma_{\lambda_0} \cap \{ z_n = -T_k \}$ and

\[ 0 > v_k(z_0) - f_k(z_0) = \min_{\Sigma_{\lambda_0} \setminus B_R} v_k - f_k \]

we would have

\begin{equation}
(2.29) \quad 0 < \partial_n (v_k - f_k)(z_0) = l_k^{-n/(n-2)} h(l_k v_k(z_0)) - \partial_n f_k(z_0).
\end{equation}

It is easy to verify that $\partial_n f_k(z_0) > N_k f_k(z_0)^{n/(n-2)}$ for some $N_k \to \infty$. However, by $GH_1,$

\[ l_k^{-n/(n-2)} h(l_k v_k(z_0)) \leq C v_k(z_0)^{n/(n-2)}. \]

Thus it is impossible to have $v_k(z_0) < f_k(z_0)$ and (2.29). Therefore, (2.28) is established.

Before we employ the method of moving spheres, we set

\[ O_\lambda := \left\{ z \in \Sigma_\lambda : v_k(z) < \min \left( \left( \frac{|z|}{\lambda} \right)^{n-2}, 2 \right) v_k^\alpha(z) \right\}. \]

Clearly $O_\lambda$ contains a neighborhood of $\partial B_\lambda$ in $\Sigma_\lambda$. Later we shall consider the equation of $v_k - v_k^{\lambda}$ in $O_\lambda$ only, since outside this region $v_k$ is already much greater than $v_k^{\lambda}$, there is no need to apply maximum principles.
In order to apply the maximum principle in $O_\lambda$ we first estimate the second term in (2.22): by $GH_1$,

$$
\left( \frac{\lambda}{|z|} \right)^{n+2} l_k^{(n+2)/(n-2)} g \left( l_k \left( \frac{|z|}{\lambda} \right)^{n-2} v_k^\lambda \right) 
= \left( \left( \frac{|z|}{\lambda} \right)^{n-2} l_k^\lambda v_k^\lambda \right)^{-(n+2)/(n-2)} g \left( \left( \frac{|z|}{\lambda} \right)^{n-2} l_k^\lambda v_k^\lambda \right) (v_k^\lambda)^{(n+2)/(n-2)} 
\leq (l_k v_k)^{-(n+2)/(n-2)} g(l_k v_k) (v_k^\lambda)^{(n+2)/(n-2)}, \quad \text{in } O_\lambda.
$$

Therefore, we have

$$
\Delta v_k^\lambda + (l_k v_k)^{-(n+2)/(n-2)} g(l_k v_k) (v_k^\lambda)^{(n+2)/(n-2)} \geq 0, \quad \text{in } O_\lambda.
$$

Then we write (2.19) as

$$
\Delta v_k + (l_k v_k)^{-(n+2)/(n-2)} g(l_k v_k) v_k^{(n+2)/(n-2)} = 0.
$$

Let $w_{\lambda,k} = v_k - v_k^\lambda$. We have, from (2.30) and (2.31),

$$
\Delta w_{\lambda,k} + n(n-2) (l_k v_k)^{-(n+2)/(n-2)} g(l_k v_k) \xi_k^{4/(n-2)} w_{\lambda,k} \leq 0, \quad \text{in } O_\lambda,
$$

where $\xi_k$ is obtained from the mean value theorem.

Now we apply the method of moving spheres to $w_{\lambda,k}$. Let

$$
\bar{\lambda}_k = \sup \left\{ \lambda \in [\lambda_0, \lambda_1] : v_k > v_k^\mu \text{ in } \Sigma_\mu, \forall \mu \in (0, \lambda) \right\}
$$

where $\lambda_0$ is defined in (2.24), $\lambda_1$ is given by

$$
\lambda_1 = \left( \frac{n(n-2)}{A} \right)^{1/2} + \epsilon_0,
$$

$\epsilon_0 > 0$ is chosen to be independent of $k$, and

$$
(2.33) \quad v_k(z) > v^\lambda(z), \quad \text{on } \partial B(0, r_k) \cap \hat{\Gamma}_k, \quad \forall \lambda \in [\lambda_0, \lambda_1].
$$

Here we recall that $r_k$ is defined in (2.21). From (2.21) we see that $\epsilon_0$ can be chosen easily. By (2.25), $\bar{\lambda}_k > \lambda_0$. We claim that $\bar{\lambda}_k = \lambda_1$. Suppose that this is not the case and that we have $\bar{\lambda}_k < \lambda_1$. By continuity, $w_{\lambda_k,k} \geq 0$, and by (2.21), $w_{\lambda_k,k} > 0$ on the outside boundary: $\partial \Sigma_{\lambda_k} \setminus (\partial B_{\lambda_k} \cup \{z_n = -T_k\})$. By (2.32), if $\min_{\Sigma_{\lambda_k}} w_{\lambda_k,k} = 0$, the minimum has to appear on $\partial \Sigma_{\lambda_k}$. From (2.33) we see that the minimum does not appear on $\partial \Sigma_{\lambda_k} \setminus (\partial B_{\lambda_k} \cup \{z_n = -T_k\})$. If there exists a minimum $x_0 \in \partial \Sigma_{\lambda_k} \cap \{z_n = -T_k\}$, we have

$$
\partial_n w_{\lambda_k,k}(x_0) > 0.
$$

Note that we have strict inequality because of Hopf’s lemma. On the other hand,
using $T_k \to \infty$ we have

$$\frac{\partial v_k^{\lambda_k}}{\partial z_n} = \frac{\partial}{\partial z_n} \left( \frac{\lambda_k}{|z|} \right) ^{n-2} v_k(z^{\lambda_k}), \quad \text{where} \quad z^{\lambda_k} := \frac{\lambda_k^2 z}{|z|^2},$$

$$= (n-2)\frac{\lambda_k}{|z|} T_k v_k(z^{\lambda_k}) - 2 \sum_{j=1}^{n-1} \frac{\partial v_k(z^{\lambda_k})}{|z|^2} \partial_j \left( \frac{\lambda_k^2 z}{|z|^2} \right),$$

$$\geq \frac{n-2}{2} \frac{\lambda_k}{|z|} T_k v_k(z^{\lambda_k}) > N_k (v_k^{\lambda_k})^{n/(n-2)}, \quad \text{in} \quad O_{\lambda_k} \cap \{ z_n = -T_k \},$$

for some $N_k \to \infty$.

For $v_k$, $GH_1$ implies

$$\frac{\partial z_n}{v_k} \leq c_h v_k^{n/(n-2)}, \quad \text{in} \quad O_{\lambda_k} \cap \{ z_n = -T_k \}$$

where $c_h = \lim_{s \to \infty} s^{-n/(n-2)} h(s)$. It is easy to see that $w_{\lambda_k} > 0$ on $\{ z_n = -T_k \}$.

Finally, an application of Hopf's lemma on $\partial B_{\lambda_k}$ gives that $\partial_z w_{\lambda_k} > 0$ on $\partial B_{\lambda_k}$. Then it is easy to see that one can move the spheres a little further than $\lambda_k$, a contradiction of the definition of $\bar{\lambda}_k$. Thus we have proved $\lambda_k = \lambda_1$. However by (2.23) it is impossible to have $\lim k \to \infty \lambda_k > (n(n-2)/A)^{1/2}$. This contradiction proves (2.20) under Case one. Lemma 2.5 is established.

From Lemma 2.5 we further prove the spherical Harnack inequality for $v_k$. For fixed $k$, consider $2R_k \leq r \leq \frac{1}{2} M_k ^{2/(n-2)}$ and let

$$\tilde{v}_k(z) = r^{(n-2)/2} v_k(rz).$$

By (2.15), $\tilde{v}_k(z) \leq C$. Direct computation yields

$$\begin{cases}
\Delta \tilde{v}_k(z) + r^{(n+2)/2} \frac{r^{-n/2}(n-2)}{l_k} g(l_k r^{-n/2} \tilde{v}_k) = 0, \quad 1/2 < |z| < 2, \quad r z \in \tilde{\Omega}_k, \\
\partial_n \tilde{v}_k = r^{n/2} l_k ^{-n/(n-2)} h(r^{-n/2} l_k \tilde{v}_k), \quad \partial\tilde{v}_k.
\end{cases}$$

Let

$$a_k = r^{n/2} l_k ^{-n/(n-2)} h(r^{-n/2} l_k \tilde{v}_k) / \tilde{v}_k$$

$$b_k = r^{n/2} l_k ^{-n/(n-2)} h(r^{-n/2} l_k \tilde{v}_k) / \tilde{v}_k$$

By the definition of $l_k$ and $r$, we see that $r = o(1) l_k ^{2/(n-2)}$ (recall that $l_k = \sigma_k ^{-(n-2)/2} M_k$). Using the assumptions on $g, h$ we have

$$a_k(z) \leq \begin{cases}
g(1) \tilde{v}_k^{4/(n-2)} \leq C, & \text{if} \quad l_k r^{-n/2} \tilde{v}_k(z) \geq 1, \\
C r^2 l_k ^{4/(n-2)} = o(1), & \text{if} \quad l_k r^{-n/2} \tilde{v}_k(z) \leq 1,
\end{cases}$$

and

$$\int_{\partial B_{\lambda_k}} \frac{\partial v_k(z^{\lambda_k})}{|z|^2} \partial_j \left( \frac{\lambda_k^2 z}{|z|^2} \right) - \frac{\partial z_n}{v_k} v_k \leq c_h v_k^{n/(n-2)}, \quad \text{in} \quad O_{\lambda_k} \cap \{ z_n = -T_k \}.$$
and

\[ |b_k(z)| \leq \begin{cases} 
  c_k \frac{2}{(n-2)} \tilde{v}_k(z)^{-2} \leq C, & \text{if } l_k r^{-(n-2)/2} \tilde{v}_k(z) \geq 1, \\
  C r l_k^{2/(n-2)} = o(1), & \text{if } l_k r^{-(n-2)/2} \tilde{v}_k(z) \leq 1. 
\end{cases} \]

Hence \( a_k \) and \( b_k \) are both bounded functions.

Consequently, the equation for \( \tilde{v}_k \) can be written as

\[
\Delta \tilde{v}_k(z) + a_k \tilde{v}_k = 0, \quad \frac{1}{2} < |z| < 2, rz \in \tilde{\Omega}_k, \\
\partial_n \tilde{v}_k = b_k \tilde{v}_k, \quad \partial \tilde{\Omega}_k \cap \{ z_n = -T_k/r \}.
\]

We apply the classical Harnack inequality for two cases: either \( T_k/r > 1 \) or \( T_k/r \leq 1 \). In the first case we have

\[
\max_{|z|=3/4} \tilde{v}_k(z) \leq C \min_{|z|=3/4} \tilde{v}_k.
\]

In the second case we have

\[
\max_{|z|=1, z_n \geq -T_k/r} \tilde{v}_k(z) \leq C \min_{|z|=1, z_n \geq -T_k/r} \tilde{v}_k.
\]

Now (2.17) follows from (2.20) and the spherical Harnack inequality above. Proposition 2.4 is established for Case one.

**Case two in Proposition 2.4:** \( \lim_{k \to \infty} T_k = T \).

Recall that \( v_k \) satisfies (2.19). As in Case one there exists \( R_k \to \infty \) such that

\[
\| v_k(y) - \left( 1 + \frac{A}{n(n-2)} |y|^2 \right)^{-\frac{(n-2)/2}{4}} \|_{C^{1/2}(B_r^{-T_k})} \leq CR_k^{-1}. \]

Clearly (2.17) holds for \( |y| \leq \tilde{M}_k^{2/(n-2)} R_k \cap \{ y_n \geq -T_k \} \), so we just need to prove (2.17) for \( \{ |y| > \tilde{M}_k^{2/(n-2)} R_k \} \cap \{ y_n \geq -T_k \} \).

**Lemma 2.6.** There exists \( k_0 > 1 \) such that, for all \( k \geq k_0 \) and \( r \in (R_k, \tilde{M}_k^{2/(n-2)}) \), the estimate (2.20) still holds.

**Remark 2.7.** Even though (2.20) also holds for case two, the domain for case two is different.

**Proof of Lemma 2.6.** Just like in the interior case, suppose there exist \( r_k \geq R_k \) such that

\[
(2.34) \quad \min_{\partial B_{r_k} \cap \Omega_k} v_k > 2 \left( \frac{n(n-2)}{A} \right)^{(n-2)/2} r_k^{2-n}. \]

Let

\[
\tilde{v}_k(z) = v_k(z - T_k e_n), \quad \tilde{v}_k(z) = \left( \frac{\lambda}{|z|} \right)^{n-2} \tilde{v}_k \left( \frac{\lambda^2 z}{|z|^2} \right)
\]
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and

\[ D_k := \{ z; \quad \tilde{M}_k e_n - \frac{2}{n-2}(z - T_k e_n) \in \Omega_k \cap B_{r_k} \} \]

be the domain of \( \tilde{v}_k \). Then \( D_k \subset \mathbb{R}^n_+ \). Set

\[ \Sigma_\lambda := \{ z \in D_k; \quad |z| > \lambda \}. \]

Let \( \tilde{V} \) be the limit of \( \tilde{v}_k \) in \( C^2_{\text{loc}}(\mathbb{R}^n_+) \):

\[ \tilde{V}(z) = \left( 1 + \frac{A}{n(n-2)} |z-Te_n|^2 \right)^{-(n-2)/2}. \]

Then there exist \( \lambda_2 \) and \( \lambda_3 \) (\( \lambda_2 < \lambda_3 \)), depending only on \( n, A \) and \( T \), such that

\[ \tilde{V} > \tilde{V}^{\lambda_2} \quad \text{in} \quad \mathbb{R}^n_+ \setminus B_{\lambda_2} \]

and

(2.35) \[ \tilde{V} < \tilde{V}^{\lambda_3} \quad \text{in} \quad \mathbb{R}^n_+ \setminus B_{\lambda_3}. \]

We shall employ the method of moving spheres to compare \( \tilde{v}_k \) and \( \tilde{v}_k^{\lambda} \) on \( \Sigma_\lambda \) for \( \lambda \in [\lambda_2, \lambda_3] \).

We use the uniform convergence of \( \tilde{v}_k \) to \( \tilde{V} \) to assert that, for any fixed \( R > 1 \),

(2.36) \[ \tilde{v}_k(y) > \tilde{v}_k^{\lambda_2}(y), \quad y \in \Sigma_{\lambda_2} \cap B_R. \]

For \( R \) large we have, with \( a_1 = (n(n-2)/A)^{(n-2)/2} \),

\[ \tilde{v}_k(y) \geq (a_1 - \epsilon/5) |y|^{2-n} \quad \text{on} \quad \partial B_R \cap \mathbb{R}^n_+ \]

and

\[ \tilde{v}_k^{\lambda_2}(y) \leq (a_1 - 2\epsilon/5) |y|^{2-n}, \quad |y| > \lambda_2. \]

To prove \( \tilde{v}_k > \tilde{v}_k^{\lambda_2} \) in \( \Sigma_{\lambda_2} \setminus B_R \), we compare \( \tilde{v}_k \) with

\[ w = (a_1 - 3\epsilon/10) |y - A_1 e_n|^{2-n} \]

where \( A_1 = \frac{1}{n-2} c_h a_1^{2/(n-2)} \). For \( R \) chosen sufficiently large we have

\[ w \geq \tilde{v}_k^{\lambda_2} \quad \text{in} \quad \Sigma_{\lambda_2} \setminus B_R, \quad \text{and} \quad \tilde{v}_k > w \quad \text{on} \quad \partial B_R \cap \Sigma_{\lambda_2}. \]

To compare \( \tilde{v}_k \) and \( w \) over \( \Sigma_{\lambda_2} \setminus B_R \), it is easy to see that \( \tilde{v}_k > w \) on \( \partial B_R \cap \Sigma_{\lambda_2} \) and \( \partial \Sigma_{\lambda_2} \setminus (B_R \cup \{ z_n \geq 0 \}) \). Since \( \tilde{v}_k - w \) is super-harmonic, the only thing we need to prove is, on \( \partial B_R \setminus \Sigma_{\lambda_2} \),

(2.37) \[ \partial_n (\tilde{v}_k - w) < \xi_k (\tilde{v}_k - w), \quad \text{on} \quad \{ z_n = 0 \} \setminus B_R. \]

for some positive function \( \xi_k \). Then standard maximum principle can be used to conclude that \( \tilde{v}_k > w_k \) on \( \Sigma_{\lambda_0} \setminus B_R \).
To obtain (2.37) first for \( \tilde{v}_k \) we use \( GH_2 \) to have
\[
\partial_n \tilde{v}_k \leq c_h \tilde{v}_k^{n/(n-2)}, \quad \{z_n = 0\}.
\]
On the other hand, by the choice of \( A_1 \) we verify easily that
\[
\partial_n w > c_h w^{n/(n-2)}, \quad \text{on} \quad \{z_n = 0\}.
\]
Thus (2.37) holds from mean value theorem. We have proved that the moving sphere process can start at \( \lambda = \lambda_2 \):
\[
\tilde{v}_k > \tilde{v}_k^{\lambda_2} \quad \text{in} \quad \Sigma_{\lambda_2}.
\]
Let \( \tilde{\lambda} \) be the critical moving sphere position:
\[
\tilde{\lambda} := \sup \{ \lambda \in [\lambda_2, \lambda_3] : \tilde{v}_k > \tilde{v}_k^{\lambda} \quad \text{in} \quad \Sigma_{\mu}, \quad \forall \mu \in (0, \lambda) \}.
\]
As in Case one we shall prove that \( \tilde{\lambda} = \lambda_3 \), thus getting a contradiction to (2.35).
To this end we let
\[
w_{\lambda,k} = \tilde{v}_k - \tilde{v}_k^{\lambda}.
\]
To derive the equation for \( w_{\lambda,k} \), we first recall from (2.19) and the definition of \( \tilde{v}_k \) that
\[
\begin{align*}
\Delta \tilde{v}_k(z) + l_k^{-(n+2)/(n-2)} g(l_k \tilde{v}_k) &= 0, \quad z \in \tilde{\Omega}_k, \\
\frac{\partial \tilde{v}_k}{\partial z_n} &= l_k^{-n/(n-2)} h(l_k \tilde{v}_k), \quad \{z_n = 0\} \cap \partial \tilde{\Omega}_k.
\end{align*}
\]
where \( l_k = \sigma_k^{-(n-2)/2} M_k \). Correspondingly \( \tilde{v}_k^{\lambda} \) satisfies
\[
\begin{align*}
\Delta \tilde{v}_k^{\lambda}(z) + \left( \frac{\lambda}{|z|} \right)^{n+2} l_k^{-(n+2)/(n-2)} g \left( l_k \left( \frac{|z|}{\lambda} \right)^{n-2} \tilde{v}_k^{\lambda}(z) \right) &= 0, \quad \text{in} \quad \tilde{\Sigma}_\lambda, \\
\frac{\partial \tilde{v}_k^{\lambda}}{\partial z_n} &= \left( \frac{\lambda}{|z|} \right)^{n-n/(n-2)} h \left( l_k \left( \frac{|z|}{\lambda} \right)^{n-2} \tilde{v}_k^{\lambda}(z) \right), \quad \text{on} \quad \partial \Sigma_{\lambda} \cap \{z_n = 0\}.
\end{align*}
\]
Let \( O_{\lambda} \) be defined as before. Then in \( O_{\lambda} \) we have, by \( GH_1 \),
\[
\left( \frac{\lambda}{|z|} \right)^{n+2} l_k^{-(n+2)/(n-2)} g \left( l_k \left( \frac{|z|}{\lambda} \right)^{n-2} \tilde{v}_k^{\lambda}(z) \right)
\leq (v_k l_k)^{-(n+2)/(n-2)} g(l_k v_k) (v_k^{\lambda})^{(n+2)/(n-2)},
\]
and on \( \partial O_{\lambda} \cap \{z_n = 0\} \),
\[
\left( \frac{\lambda}{|z|} \right)^{n-n/(n-2)} h \left( l_k \left( \frac{|z|}{\lambda} \right)^{n-2} \tilde{v}_k^{\lambda}(z) \right)
\geq (l_k v_k) - \frac{\lambda}{|z|} h(l_k v_k) (v_k^{\lambda})^{-1/2},
\]
The inequalities above yield
\[
\begin{align*}
\Delta w_{\lambda,k} + \xi_{1,k} w_{\lambda,k} &\leq 0, \quad \text{in} \quad O_{\lambda}, \\
\partial_n w_{\lambda,k} &\leq \xi_{2,K} w_{\lambda,k}, \quad \text{on} \quad \partial O_{\lambda} \cap \{z_n = 0\},
\end{align*}
\]
Thus standard maximum principle implies that $\bar{\lambda}$ is always comparable to $|z|^{2-n}$. Re-scaling $v_k$ as $v_{k}(r z)$ we see the spherical Harnack inequality holds by the $GH_2$ and $GH_3$. Thus Proposition 2.4 is established. \hfill $\square$

**Lemma 2.8.** Let $\{u_k\}$ be a sequence of solutions of (1.1) and $q_k \to q \in B_1^+$ be a sequence of points in $Z_k$. Then there exist $C > 0$, $r_2 > 0$ independent of $k$ and \( R_k \to \infty \) such that

$$u_k(q_k)u_k(x) \geq C|x - q_k|^{2-n} \quad \text{in} \quad u_k(q_k)^{-2/(n-2)}R_k \leq |x - q_k| \leq r_2, \quad x \in B_3^+.$$ 

**Proof.** We consider two cases:

**Case one:** $u_k(q_k)^{2/(n-2)}q_k \to \infty$.

Let $M_k = u_k(q_k)$ and

$$v_k(y) = M_k^{-1}u_k(M_k^{-2/(n-2)}y + q_k),$$

for $y \in \Omega_k := \{y : M_k^{-2/(n-2)}y + q_k \in B_1^+\}$. In this case, $v_k$ converges uniformly to

$$V(y) = \left(1 + \frac{A}{n(n-2)}|y|\right)^{-(n-2)/2}$$

over all compact subsets of $\mathbb{R}^n$. For $\epsilon > 0$ small we let

$$\phi = \left(\frac{n(n-2)}{A} - \epsilon\right)^{(n-2)/2} (|y|^{2-n} - M_k^{-2}), \quad |y| \leq M_k^{2/(n-2)}$$

on $|y| \geq R$, where $R > 1$ is chosen so that $v_k \geq \phi$ on $\partial B_R$. By direct computation we have

$$\frac{\partial \phi}{\partial y_n} > N_k \phi^{n/(n-2)} \quad \text{on} \quad \{y_n = -q_k \}$$

for some $N_k \to \infty$. It is easy to see that $v_k \geq \phi$ on $\partial \Omega_k \setminus \{y_n = -M_k^{2/(n-2)}q_k\}$. On $\{y_n = -M_k^{2/(n-2)}q_k\}$ we have

$$\partial_{y_n}(v_k - \phi) \leq c_k(v_k - \phi).$$

Thus standard maximum principle implies $v_k \geq \phi$ on $\Omega_k$. Lemma 2.8 is established in this case.

**Case two:** $M_k^{2/(n-2)}q_k \to C$.

Let $v_k$ be defined as in (2.40). In this case the boundary condition is written as

$$\partial_{y_n}v_k = \left(M_k^{-2/(n-2)}v_k\right)^{-n/(n-2)}h(M_k^{-2/(n-2)}v_k)^{n/(n-2)},$$

on $\{y_n = -M_k^{2/(n-2)}q_k\}$. Where $h$ is a function depending on $\Omega_k$. The proof of Lemma 2.8 is completed. \hfill $\square$
The function $v_k$ converges to $V$ of (2.41) over all compact subsets of \( \{ y_n \geq -T \} \), where
\[
T = \lim_{k \to \infty} M_k^{-2/(n-2)} q_{kn}.
\]
For $R$ large and $\epsilon > 0$ small, both independent of $k$, we have
\[
v_k(y) \geq \left( \frac{n(n-2)}{A} - \epsilon \right)^{(n-2)/2} |y|^{2-n}, \quad |y| = R.
\]
In $B_R^T$ we have the uniform convergence of $v_k$ to $V_1$. Our goal is to prove that $v_k$ is bounded below by $O(1)$ outside $B_R$. To this end let
\[
w(y) = \left( \frac{n(n-2)}{A} - 2\epsilon \right)^{(n-2)/2} |y|^{2-n} |y - A_1 c_n|^{2-n}
\]
where
\[
A_1 = ch \left( \frac{n(n-2)}{A} \right) - T.
\]
Then it is easy to check that
\[
\frac{\partial w}{\partial y_n} > ch w(y)^{n/(n-2)}, \quad \text{on} \quad \{ y_n = -M_k^{-2/(n-2)} q_{kn} \}.
\]
By choosing $R$ larger if needed we have
\[
v_k(y) > \left( \frac{n(n-2)}{A} - \epsilon \right)^{(n-2)/2} |y|^{2-n} w(y), \quad |y| = R, \quad y \in \mathbb{R}^n.
\]
Then it is easy to apply maximum principle to prove $v_k > w$ in $\Omega_k \setminus B_R$. Lemma 2.8 is established.

Let $q_k^1 \in Z_k$ and $q_k^2$ be its nearest or almost nearest sequence in $Z_k$:
\[
|q_k^2 - q_k^1| = (1 + o(1)) d(q_k^1, Z_k \setminus \{ q_k^1 \}).
\]

**Lemma 2.9.** There exists $C > 0$ independent of $k$ such that
\[
\frac{1}{C} u_k(q_k^1) \leq u_k(q_k^2) \leq C u_k(q_k^1).
\]

**Proof.** Let $\sigma_k = d(q_k^1, Z_k \setminus \{ q_k^1 \})$ and
\[
\bar{u}_k(y) = \sigma_k^{(n-2)/2} u_k(q_k^1 + \sigma_k y).
\]
We use $c_k$ to denote the image of $q_k^1$ after scaling (so $|c_k| \to 1$). Then in $B_1$, $\bar{u}_k(x) \sim \bar{u}_k(0)^{-1} |x|^{2-n}$ for $|x| \sim 1/2$. On one hand, for $|x| = 1/2$ we have, by Lemma 2.8 applied to $c_k$,
\[
\bar{u}_k(0)^{-1} \left( \frac{1}{2} \right)^{2-n} \geq C \bar{u}_k(c_k)^{-1}
\]
which is just $u_k(q_k^1) \leq C u_k(q_k^2)$. 

\[\square\]
On the other hand, the same moving sphere argument can be applied to $u_k$ near $q_2^k$ with no difference. The Harnack type inequality gives
\[
\max_{B(q_2^k,1/4)\cap B^+_k} u_k \geq \min_{B(q_2^k,1/2)\cap B^+_k} u_k \leq C.
\]
Using
\[
\max_{B(q_2^k,1/4)\cap B^+_k} u_k \geq u_k(q_2^k), \quad \text{and} \quad \min_{B(q_2^k,1/2)\cap B^+_k} u_k \geq \min_{B(q_1^k,\sigma_k)\cap B^+_k} u_k,
\]
we have
\[
(2.42) \quad \tilde{u}_k(e_k) \tilde{u}_k(0)^{-1} \leq C.
\]
Thus (2.42) gives $u_k(q_2^k) \leq C u_k(q_1^k)$. Lemma 2.9 is established. \(\square\)

**Remark 2.10.** Proposition 2.4 is not needed in the proof of Lemma 2.9.

The following lemma is concerned with Pohozaev identity that can be verified by direct computation.

**Lemma 2.11.** Let $u$ solve
\[
\begin{cases}
\Delta u + g(u) = 0, & \text{in } B^+_\sigma, \\
\frac{\partial u}{\partial \nu} = h(u) & \text{on } \partial B^+_\sigma.
\end{cases}
\]
Then
\[
(2.43) \quad \int_{\partial^+ B^+_\sigma} h(u) \left( \sum_{i=1}^{n-1} x_i \partial_i u + \frac{n-2}{2} u \right) + \int_{B^+_\sigma} \left( n G(u) - \frac{n-2}{2} g(u) u \right) = \int_{\partial^+ B^+_\sigma} \left( \sigma (G(u) - \frac{1}{2} \vert \nabla u \vert^2 + (\partial_n u)^2) + \frac{n-2}{2} u \partial_n u \right)
\]
where $G(s) = \int_0^s g(t)dt$, $\nu$ stands for the outer normal vector of the domain.

Now we finish the proof of Theorem 2.3.

Recall that $\sigma_k = (1 + o(1))(q_1^k - q_2^k)$. We prove by contradiction. Suppose $\sigma_k \to 0$. Let $\tilde{M}_k = \tilde{u}_k(0)$. We claim that
\[
(2.44) \quad \tilde{M}_k \tilde{u}_k(y) \to a |y|^{2-n} + b(y) \quad \text{in } C^2_{\text{loc}}(B_{3/4} \cap \tilde{\Omega}_k \setminus \{0\}), \quad \text{with } a > 0, \ b(0) > 0
\]
where $\tilde{\Omega}_k = \{y; \sigma_k y + q_k^i \in B^+_1\}$.

**Proof of (2.44).** As usual we consider the following two cases:

**Case one in (2.44):** $\lim_{k \to \infty} q_k^{1/n} \tilde{M}_k^{2/(n-2)} \to \infty$.

Let
\[
T_k = \tilde{M}_k^{2/(n-2)} q_k^{1/n}.
\]
Recall the equation for $\tilde{u}_k$ is (2.18). Multiplying $\tilde{M}_k$ on both sides and letting $k \to \infty$ we see from the assumptions of $g$ and $h$ that $\tilde{M}_k \tilde{u}_k \to h$ in $C^2_{\text{loc}}(B_1 \setminus \{0\})$ where $h$ is a harmonic function defined in $B_1 \setminus \{0\}$. 

Thus,
\[ h(y) = a|y|^{2-n} + b(y) \]
for some harmonic function \( b(y) \) in \( B_1 \). From the pointwise estimate in Lemma 2.8 we see that \( a > 0 \). Given any \( \epsilon > 0 \), we compare \( \tilde{u}_k \) and
\[ w_k := (a - \epsilon) \left( |y|^{2-n} - R_k^{2-n} \right) \]
on \(|y| \leq R_k\). Here \( R_k \to \infty \) is less than \( T_k \). Observe that \( \tilde{u}_k > w_k \) on \( \partial B_{R_k} \) and \(|y| = \epsilon_1 \) for \( \epsilon_1 \) sufficiently small. Thus \( \tilde{u}_k > w_k \) by the maximum principle. Letting \( k \to \infty \), we have, in \( B_1 \),
\[ a|y|^{2-n} + b(y) \geq (a - \epsilon)|y|^{2-n}, \quad B_1 \setminus B_{\epsilon_1}. \]
Then let \( \epsilon \to 0 \), which implies \( \epsilon_1 \to 0 \) we have \( b(y) \geq 0 \) in \( B_1 \). Next we claim that \( b(0) > 0 \) because by Lemma 2.8 and Lemma 2.9 we have
\[ a|y|^{2-n} + b(y) \geq a_1|y - \epsilon|^{2-n} \quad \text{in } B_1 \]
for some \( a_1 > 0 \), where \( \epsilon = \lim_{k \to \infty} \epsilon_k \). Thus \( b(y) > 0 \) when \( y \) is close to \( \epsilon \), which leads to \( b(0) > 0 \). (2.44) is established in Case one.

Case two in (2.44): \( \lim_{k \to \infty} q^2_{1n} M_k^{2/(n-2)} \to T < \infty \).
Again we first have \( M_k \tilde{u}_k \to h \) in \( C^2_{\text{loc}}(B_1^{-T} \setminus \{0\}) \) and \( h \) is of the form
\[ h(y) = a|y|^{2-n} + b(y), \quad y_n \geq -T. \]
To prove \( b(y) \geq 0 \) we compare, for fixed \( \epsilon > 0 \), \( M_k \tilde{u}_k \) with
\[ w_k(y) = (a - \epsilon) \left( |y| - b_k \epsilon !2-n - (R_k - 1)^{2-n} \right) \]
where \( b_k \to 0 \) and \( R_k \to \infty \) are chosen to satisfy
\[ (n - 2)b_k R_k^{-2} > c_0 \sigma_k, \quad c_0 = \sup_{0 < s \leq 1} s|h(s)| \]
and
\[ (n - 2)b_k = c_k M_k^{-2/(n-2)} a^{2/(n-2)}. \]
It is easy to see that such \( b_k \) and \( R_k \) can be found easily. Let \( h_k = M_k \tilde{u}_k \), and \( \partial \Omega_k = \partial \Omega_k \cap \{y_n = -T_k\} \). We divide \( \partial \Omega_k \) into two parts:
\[ E_1 = \{ z \in \partial \Omega; \tilde{u}_k(z) \sigma_k^{-2(n-2)/2} \geq 1 \}, \quad E_2 = \partial \Omega_k \setminus E_1. \]
Then, by the assumptions on \( h \),
\[ \partial_n \tilde{h}_k \leq \begin{cases} c_0 \sigma_k h_k, & x \in E_2, \\ c_k M_k^{-2/(n-2)} h_k^{n/(n-2)}, & x \in E_1, \end{cases} \]
With the choice of $b_k$ and $R_k$ it is easy to verify that
\[ \partial_h w_k \geq \max \left\{ c_0 \sigma_k w_k, \, c_b M_k^{-2/(n-2)} w_k^{n/(n-2)} \right\} \text{ on } \partial \Omega_k \cap B_{R_k}. \]

Thus standard maximum principle can be applied to prove $h_k \geq w_k$ on $\Omega_k \cap B_{R_k}$. Letting $k \to \infty$ first and $\epsilon \to 0$ next we have $b(y) \geq 0$ in $B_1 \cap \{ y_n \geq -T \}$. Then by Proposition 2.9 we see that $b(y) > 0$ when $y$ is close to $e$, the limit of $e_k$. The fact $\partial_b b = 0$ at $0$ implies $b(0) > 0$. Claim (2.44) is proved in both cases.

Finally to finish the proof of Theorem 2.3 we derive a contradiction from each of the following two cases:

**Case one:** $\lim_{k \to \infty} M_k^{2/(n-2)} q_{1n}^k > 0$.

We use the following Pohozaev identity on $B_\sigma$ for $\sigma < \lim_{k \to \infty} M_k^{2/(n-2)} q_{1n}^k$:

\[
\int_{B_\sigma} \left( n G_k(\tilde{u}_k) - \frac{n-2}{2} \tilde{u}_k g_k(\tilde{u}_k) \right) = \int_{\partial B_\sigma} \left( \sigma (G_k(\tilde{u}_k) - \frac{1}{2} \nabla \tilde{u}_k|^2 + |\partial_{n} \tilde{u}_k|^2) + \frac{n-2}{2} \tilde{u}_k \partial_n \tilde{u}_k \right),
\]

where

\[ g_k(s) = \sigma_k^{(n+2)/2} g(\sigma_k^{-1/2} s), \quad G(t) = \int_0^t g(s) \, ds, \quad G_k(s) = \sigma_k^n G(\sigma_k^{-1/2} s). \]

First we claim that for $s > 0$,

\[
G_k(s) \geq \frac{n-2}{2n} s g_k(s).
\]

Indeed, writing $g(t) = c(t) t^{(n+2)/(n-2)}$, we see from $GH_1$ that $c(t)$ is a non-increasing function, thus

\[
G_k(s) = \sigma_k^n G(\sigma_k^{-1/2} s) = \sigma_k^n \int_0^{\sigma_k^{-1/2} s} c(t) t^{(n+2)/(n-2)} \, dt \\
\geq \sigma_k^n c(\sigma_k^{-1/2} s) \int_0^{\sigma_k^{-1/2} s} t^{(n+2)/(n-2)} \, dt \\
= \frac{n-2}{2n} \sigma_k c(\sigma_k^{-1/2} s) s^{2n/(n-2)} = \frac{n-2}{2n} s g_k(s).
\]

Replacing $s$ by $\tilde{u}_k$ we see that the left hand side of (2.45) is non-negative. Next we prove that

\[
\lim_{k \to \infty} \tilde{M}_k^2 \int_{\partial B_\sigma} \left( \sigma (G_k(\tilde{u}_k) - \frac{1}{2} \nabla \tilde{u}_k|^2 + |\partial_{n} \tilde{u}_k|^2) + \frac{n-2}{2} \tilde{u}_k \partial_n \tilde{u}_k \right) < 0
\]

for $\sigma > 0$ small. Clearly after (2.47) is established we obtain a contradiction to (2.45). To this end first we prove that

\[
\tilde{M}_k^2 G_k(\tilde{u}_k) = o(1).
\]
Indeed, by GH$_1$ and GH$_3$, 

\[ G_k(\tilde{u}_k) = \sigma_k^n \int_0^{\sigma_k^{-(n-2)/2} \tilde{u}_k} g(t) \, dt \]

\[
\leq \begin{cases} 
\sigma_k^n \int_0^{\sigma_k^{-(n-2)/2} \tilde{u}_k} c \, dt \, dt, & \text{if } \sigma_k^{-(n-2)/2} \tilde{M}_k^{-1} \leq 1, \\
\sigma_k^n \left( \int_0^1 c \, dt + \int_1^{\sigma_k^{-(n-2)/2} \tilde{u}_k} c t^{(n+2)/(n-2)} \right), & \text{if } \sigma_k^{-(n-2)/2} \tilde{M}_k^{-1} > 1.
\end{cases}
\]

Therefore,

\[ G_k(\tilde{u}_k) \leq \begin{cases} 
C \sigma_k^n \tilde{M}_k^{-2}, & \text{if } \sigma_k^{-(n-2)/2} \tilde{M}_k^{-1} \leq 1, \\
C \sigma_k^n + C \tilde{M}_k^{-2n/(n-2)}, & \text{if } \sigma_k^{-(n-2)/2} \tilde{M}_k^{-1} > 1.
\end{cases}
\]

Clearly (2.48) holds in either case. Consequently we write the left hand side of (2.47) as

\[ \int_{\partial B_\sigma} \left( -\frac{1}{2} \sigma |\nabla h|^2 + \sigma |\partial_x h|^2 + \frac{n-2}{2} h \partial_x h \right) + o(1), \]

where

\[ h(y) = a |y|^{2-n} + b(y), \quad b(0) > 0, \quad a > 0. \]

By direct computation we have

\[ \int_{\partial B_\sigma} \left( -\frac{1}{2} \sigma |\nabla h|^2 + \sigma |\partial_x h|^2 + \frac{n-2}{2} h \partial_x h \right) = \int_{\partial B_\sigma} \left( -\frac{(n-2)^2}{a} b(0) \cdot \sigma^{1-n} + O(\sigma^{2-n}) \right) \, dS. \]

Thus (2.47) is verified when \( \sigma > 0 \) is small.

**Case two:** \( \lim_{k \to \infty} \tilde{M}_k^{2/(n-2)} \tilde{q}_\tau = 0 \).

In this case we use the following Pohozaev identity on \( B^+_{\sigma} \): let

\[ h_k(s) = \sigma_k^{n/2} h \left( \sigma_k^{-(n-2)/2} s \right). \]

Then we have

\[ \int_{\partial B^+_{\sigma} \cap \mathbb{R}^n_+} h_k(\tilde{u}_k) \left( \sum_{i=1}^{n-1} x_i \partial_i \tilde{u}_k + \frac{n-2}{2} \tilde{u}_k \right) + \int_{B^+_{\sigma}} \left( n G_k(\tilde{u}_k) - \frac{n-2}{2} g_k(\tilde{u}_k) \tilde{u}_k \right) = \int_{\partial B^+_{\sigma} \cap \mathbb{R}^n_+} \left( \sigma \left( G_k(\tilde{u}_k) - \frac{1}{2} |\nabla \tilde{u}_k|^2 + (\partial_x \tilde{u}_k)^2 \right) + \frac{n-2}{2} \tilde{u}_k \partial_x \tilde{u}_k \right). \]

Multiplying \( \tilde{M}_k^2 \) on both sides and letting \( k \to \infty \) we see by the same estimate as in **Case one** that the second term on the left hand side is non-negative, the right hand side is strictly negative.
The only term we need to consider is
\[
\lim_{k \to \infty} \tilde{M}_k^2 \int_{\partial B^+_\xi \cap \partial \mathbb{R}^n_+} h_k(\tilde{u}_k) \left( \sum_{i=1}^{n-1} x_i \partial_i \tilde{u}_k + \frac{n-2}{2} \tilde{u}_k \right).
\]

Let \( H(s) = \int_0^s h(t) \, dt \), then from integration by parts we have
\[
(2.50) \quad \int_{\partial B^+_\xi \cap \partial \mathbb{R}^n_+} h_k(\tilde{u}_k) \left( \sum_{i=1}^{n-1} x_i \partial_i \tilde{u}_k + \frac{n-2}{2} \tilde{u}_k \right) = \int_{\partial B^+_\xi \cap \partial \mathbb{R}^n_+} \sigma_k^{n-1} H(\sigma_k^{-(n-2)/2} \tilde{u}_k) \sigma_k^{(n-2)/2} \tilde{u}_k + \frac{n-2}{2} \tilde{u}_k h_k(\tilde{u}_k) \, dx'.
\]

For the first term on the right hand side of (2.50) we claim that
\[
(2.51) \quad \tilde{M}_k^2 \sigma_k^{n-1} H(\sigma_k^{-(n-2)/2} \tilde{u}_k) = o(1) \quad \text{on} \quad \partial B_\xi.
\]

Indeed, by \( GH_1 \) and \( GH_2 \),
\[
|H(\sigma_k^{-(n-2)/2} \tilde{u}_k)| \leq \left\{ \begin{array}{ll}
O(\sigma_k), & \text{if } \sigma_k^{-(n-2)/2} \tilde{u}_k \leq 1, \\
O(\sigma_k) + O(\tilde{M}_k^{-(n-2)/2}), & \text{if } \sigma_k^{-(n-2)/2} \tilde{u}_k > 1.
\end{array} \right.
\]

Using \( \tilde{u}_k = O(1/\tilde{M}_k) \) on \( \partial B_\xi \) we then have
\[
\tilde{M}_k^2 \sigma_k^{n-1} |H(\sigma_k^{-(n-2)/2} \tilde{u}_k)| \leq \left\{ \begin{array}{ll}
O(\sigma_k), & \text{if } \sigma_k^{-(n-2)/2} \tilde{u}_k \leq 1, \\
O(\sigma_k) + O(\tilde{M}_k^{-(n-2)/2}), & \text{if } \sigma_k^{-(n-2)/2} \tilde{u}_k > 1.
\end{array} \right.
\]

Thus (2.51) is verified and the first term on the right hand side of (2.50) is o(1).

Therefore we only need to estimate the last term of (2.50), which we claim is non-negative. Indeed, for \( t > 0 \), we write \( h(t) = b(t) t^{n/(n-2)} \) for some non-decreasing function \( b \). Then we have
\[
\sigma_k^{n-1} H(\sigma_k^{-(n-2)/2} s) = \sigma_k^{n-1} \int_0^{\sigma_k^{-(n-2)/2} s} h(t) \, dt = \sigma_k^{n-1} \int_0^{\sigma_k^{-(n-2)/2} s} b(t) t^{n/(n-2)} \, dt \\
\leq \sigma_k^{n-1} b(\sigma_k^{-(n-2)/2} s) \int_0^{\sigma_k^{-(n-2)/2} s} t^{n/(n-2)} \, dt \\
= \frac{n-2}{2n-2} b(\sigma_k^{-(n-2)/2} s) s^{(2n-2)/(n-2)} = \frac{n-2}{2n-2} b_k(s) s.
\]

Replacing \( s \) by \( \tilde{u}_k \) in the above we see that the last term of (2.50) is non-negative. Thus there is a contradiction in (2.49) in Case two as well. Theorem 2.3 is established. \( \Box \)
2.3. Step three: the completion of the proof of Theorem 1.1

Let $u_k$ be a sequence of blowup solutions of (1.1) that satisfies $\max_{B^+_{3/2}} u_k \to \infty$ and (2.1) (see the beginning part of the proof before Proposition 2.1). By Proposition 2.3 there are finite local maximum points $q_i^k \in B^+_{3/2}$ for $i = 1, \ldots, L$, where $L$ is independent of $k$ and $\min\{q_i^k, q_j^k\} \geq 2\delta_0$ for all $q_i^k \neq q_j^k$ and some $\delta_0 > 0$ independent of $k$. In each $B^+(q_i^k, \delta_0)$ if we define

$$v_i^k(y) = u_k(q_i^k)^{-1}u_k\left(u_k^{-2/(n-2)}(q_i^k) y + q_i^k\right)$$

by the proof of Theorem 2.3 we have

$$v_i^k(y) \leq C(1 + |y|)^{2-n}, \quad \text{for } |y| \leq \delta_0 u_k(q_i^k)^{2/(n-2)}.$$  \hfill (2.52)

The corresponding estimate for $u_k$ is

$$u_k(x) \leq C u_k(q_i^k)(1 + u_k(q_i^k)^{2/(n-2)}|x - q_i^k|)^{2-n}, \quad \text{in } B^+(q_i^k, \delta_0).$$

Then direct computation shows that

$$\int_{B^+(q_i^k, \delta_0)} u_k^{2n/(n-2)} \leq C$$

for some $C > 0$ independent of $k$. By Proposition 2.1 we also see that $u_k \leq C$ in $B^+_{3/2} \setminus (\cup_i B(q_i^k, \delta_0))$. Thus we have

$$\int_{B^+_{3/2}} u_k^{2n/(n-2)} \leq C. \hfill (2.53)$$

Let $\phi$ be a radial and smooth function such that $\phi \equiv 1$ in $B_1$ and $\phi \equiv 0$ near $\partial B_{3/2}$. Moreover, $\phi \geq 0$. Multiplying $u_k \phi^2$ to both sides of (1.1), we have

$$\int_{\partial B^+_{3/2}} h(u_k) u_k \phi^2 + \int_{B^+_{3/2}} \nabla u_k \nabla (u_k \phi^2) = \int_{B^+_{3/2}} g(u_k) u_k \phi^2.$$  \hfill (3.1)

Using Cauchy’s inequality we obtain

$$\frac{1}{2} \int_{B^+_{3/2}} |\nabla u_k|^2 \phi^2 \leq 2 \int_{B^+_{3/2}} u_k^2 |\nabla \phi|^2 + \int_{B^+_{3/2}} g(u_k) u_k \phi^2 - \int_{\partial B^+_{3/2}} h(u_k) u_k \phi^2.$$  \hfill (3.2)

By (2.52) and the assumptions on $g, h$ it is easy to see that the right hand side of the above is bounded by $C$ independent of $k$. Therefore $\int_{B^+_{3/2}} |\nabla u_k|^2 \leq C$ for some $C > 0$ independent of $k$. A contradiction to (2.1). Theorem 1.1 is established. \hfill \Box

**Remark 2.12.** Finally we summarize some technical points in the proof of Theorem 1.1. In the proof of Proposition 2.1 the assumptions on $g, h$ play a central role. The proof of Proposition 2.4 relies on the classification theorems of Caffarelli–Gidas–Spruck [1] and Li–Zhu [11] in an essential way. The main reason is equation (1.1) is not scaling invariant so we have to prove the decay rate of bubbling solutions in each blowup disk (Lemma 2.5 and Lemma 2.6). At the first glance the reader may feel that the proof of Theorem 2.3 is similar to corresponding theorems in [8] or [6]. Actually this is not the case. The main difference lies on the fact that (1.1) is a locally defined equation, while the equations in [8] or [6] are globally
defined. This difference is particularly subtle when the interaction of bubbles is concerned. In [8] and [6] it is possible to find two bubbles closest to each other because the equations are globally defined. This is not possible for the local equation (1.1) and the approach in this article is significantly different. We mainly follow the line of proof in [9] for this part.

3. Proof of Theorem 1.2

If \( h \) is non-positive, the energy estimate follows from the Harnack inequality in a straightforward way. Indeed, let \( G(x, y) \) be a Green’s function on \( B_3^+ \) such that \( G(x, y) = 0 \) if \( x \in B_3^+, y \in \partial B_3^+ \cap \mathbb{R}_+^{n} \) and \( \partial_y G(x, y) = 0 \) for \( x \in B_3^+, y \in \partial B_3^+ \cap \partial \mathbb{R}_+^{n} \). It is easy to see that \( G \) can be constructed by adding the standard Green’s function on \( B_3 \) its reflection over \( \partial \mathbb{R}_+^{n} \). It is also immediate to observe that

\[
G(x, y) \geq C_n|x - y|^{2-n}, \quad x \in B_3^+, \quad y \in B_3^+.
\]

Multiplying \( G \) on both sides of (1.1) and integrating by parts, we have

\[
u(x) + \int_{\partial B_3^+ \cap \partial \mathbb{R}_+^{n}} h(u(y)) G(x, y) \, dS_y + \int_{\partial B_3^+ \cap \mathbb{R}_+^{n}} u(y) \frac{\partial G(x, y)}{\partial \nu} \, dS_y = \int_{B_3^+} g(u(y)) G(x, y) \, dy.
\]

Here \( \nu \) represents the outer normal vector of the domain. Using \( h \leq 0 \) and \( \partial \nu G \leq 0 \), we have

\[
u(x) \geq \int_{B_3^+} g(u(y)) G(x, y) \, dy, \quad x \in B_3^+.
\]

In particular, let \( u(x_0) = \min_{\partial B_2^+} u \). Then \( |x_0| = 2 \), thus

\[
C \geq \max_{B_3^+} u \cdot \min_{B_2^+} u \geq \int_{B_{3/2}^+} g(u(y)) u(y) G(x_0, y) \, dy \geq C \int_{B_{3/2}^+} g(u) u \, dy.
\]

Therefore we have obtained the bound on \( \int_{B_{3/2}^+} g(u) u \, dy \). To obtain the bound on \( \int_{B_{3/2}^+} |\nabla u|^2 \), we use a cut-off function \( \eta \) which is 1 on \( B_3^+ \) and is 0 on \( B_2^+ \setminus B_{3/2}^+ \) and \( |\nabla \eta| \leq C \). Multiplying \( u \eta^2 \) to both sides of (1.1) and using integration by parts and Cauchy inequality we obtain the desired bound on \( \int_{B_3^+} |\nabla u|^2 \). Theorem 1.2 is established.

References


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