Brownian motion on treebolic space:
escape to infinity

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Abstract. Treebolic space is an analog of the Sol geometry, namely, it
is the horocyclic product of the hyperbolic upper half plane \( \mathbb{H} \) and the
homogeneous tree \( T = T_p \) with degree \( p + 1 \geq 3 \), the latter seen as a
one-complex. Let \( h \) be the Busemann function of \( T \) with respect to a fixed
boundary point. Then for real \( q > 1 \) and integer \( p \geq 2 \), treebolic space
\( HT(q, p) \) consists of all pairs \( (z = x + iy, w) \in \mathbb{H} \times T \) with
\( h(w) = \log_q y \).

It can also be obtained by gluing together horizontal strips of \( \mathbb{H} \) in a
tree-like fashion. We explain the geometry and metric of \( HT \) and exhibit a
locally compact group of isometries (a horocyclic product of affine groups)
that acts with compact quotient. When \( q = p \), that group contains the
amenable Baumslag–Solitar group \( BS(p) \) as a cocompact lattice, while
when \( q \neq p \), it is amenable, but non-unimodular. \( HT(q, p) \) is a key example
of a strip complex in the sense of [4].

Relying on the analysis of strip complexes developed by the same au-
thors in [4], we consider a family of natural Laplacians with “vertical drift”
and describe the associated Brownian motion. The main difficulties come
from the singularities which treebolic space (as any strip complex) has
along its bifurcation lines. In this first part, we obtain the rate of escape
and a central limit theorem, and describe how Brownian motion converges
to the natural geometric boundary at infinity. Forthcoming work will be
dedicated to positive harmonic functions.

1. Introduction

Let \( H = \{ x + iy : x \in \mathbb{R} \, , \, y > 0 \} \) be hyperbolic upper half space, and \( T = T_p \) be the
homogeneous tree, drawn in such a way that every vertex of \( T \) has one predecessor
and \( p \) successors. Treebolic space is a Riemannian 2-complex, a horocyclic product
of \( H \) and \( T \). Let us start with a picture and an informal description.

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cape, central limit theorem, boundary convergence.
Let $1 < q \in \mathbb{R}$. For $k \in \mathbb{Z}$, subdivide $\mathbb{H}$ into the strips $S_k = \{ x + i y : x \in \mathbb{R}, q^{k-1} \leq y \leq q^k \}$ (see Figure 3 further below). Each strip is bounded by two horizontal lines of the form $L_k = \{ x + i q^k : x \in \mathbb{R} \}$, which in hyperbolic geometry are horocycles with respect to the “upper” boundary point $\infty$ (or rather $i \infty$).

In treebolic space $\mathbb{H}T(q, p)$, infinitely many copies of those strips are glued together in a tree-like fashion: for each $k \in \mathbb{Z}$, the bottom lines of $p$ copies of $S_k$ are identified with each other and with the top line of a copy of $S_{k-1}$. Thus, every copy of any of the $L_k$ becomes a bifurcation line whose “side view” is a vertex $v$ of $T$ that can be used to identify the line as $L_v$ (instead of $L_k$). In the same way, we write $S_v$ for the strip sitting below $L_v$ in our picture. Each strip is equipped with the standard hyperbolic length element, and combining this with the tree metric, one obtains a natural metric on $\mathbb{H}T(q, p)$. A more formal description will be given in §2.

![Figure 1](image1.png)

**Figure 1.** A finite section of treebolic space, with $p = 2$.

Why is this space interesting? First of all, it is a key example of a *strip complex* in the sense of [4]. Strip complexes are a class of Riemannian complexes. Laplacians and the associated potential theory on Riemannian complexes appear in the book of Eells and Fuglede [18]. A study of Brownian motion and harmonic functions on Euclidean complexes was undertaken by Brin and Kifer [9]. In [4], the theory of Laplacians and diffusion on strip complexes, properties of the heat kernel, etc., were studied in a careful and rigorous way. In this spirit, the present paper is the first detailed case study of what can be achieved on the basis of that theory, which provides a highly non-trivial extension of the very popular subject of analysis and probability on “quantum graphs” (metric graphs) to what one might also call “quantum complexes”.

Second, treebolic space is a horosphere in the product space $\mathbb{H} \times T$, where the tree $T$ is viewed as a one-dimensional complex in which each edge is a copy of a suitable compact interval. In other words, it is the *horocyclic product* of $\mathbb{H}$ and $T$. A first appearance of such a horocyclic product was that of two trees with (integer) branching numbers $p$ and $q \geq 2$, respectively. This is the Diestel–Leader graph $DL(p, q)$, which for $p \neq q$ was proposed by Diestel and Leader [17] as a candidate example to answer the following question of Woess [30]: is there a vertex-transitive graph which is not quasi-isometric with a Cayley graph? It was only

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1This figure also appears in [4].
quite recently that Eskin, Fisher and Whyte finally showed, as part of impressive work \[19\], \[20\], \[21\], that DL(p, q) is indeed such an example. On the other hand, in case of equal branching numbers of the two trees, DL(p, p) is a Cayley graph of the lamplighter group \(\mathbb{Z}(p) \wr \mathbb{Z}\). This geometric realisation of the latter lead to a good understanding of random walks, spectra and boundary theory of those groups, the DL-graphs, and of horocyclic products of more than 2 trees, see the work of Bertacchi, Bartholdi, Brofferio, Neuhauser, Woess \[6\], \[33\], \[1\], \[12\], \[13\], \[2\].

Besides DL(p, q), treebolic space has another, more classical sister structure. This is Sol(p, q), the horocyclic product of two hyperbolic planes with curvatures \(-p^2\) and \(-q^2\), respectively, where \(p, q > 0\). Besides being a 3-dimensional Riemannian manifold, Sol(p, q) can be seen as a Lie group, which is the semidirect product of \(\mathbb{R}\) with \(\mathbb{R}^2\) induced by the action \((x, y) \mapsto (e^{p^2 x}, e^{-q^2 y})\), \(z \in \mathbb{R}\). Sol(1, 1) is one of Thurston’s eight model geometries in dimension 3. The Brownian motion generated by the Laplace–Beltrami operator on Sol(p, q) is studied in detail in a sister paper to the present one, by Brofferio, Salvatori and Woess \[11\].

The analogy between DL(p, q) and Sol(p, q) becomes also apparent in \[19\], \[20\], and \[21\], where the quasi-isometry classes of these graphs, resp. manifolds are determined. Coming back to treebolic space, we mention that like the above sister structures, it is neither Gromovhyperbolic nor Cat(0). We shall explain below that the amenable Baumslag–Solitar group BS(p) = \(\langle a, b \mid ab = bpa \rangle\) acts on HT(p, p) by isometries and with compact quotient. This fact has been exploited by Farb and Mosher \[22\] (without describing the space as a horocyclic product) in order to determine the quasi-isometry types of the Baumslag–Solitar groups. On the other hand, we shall see that for \(p \neq q\), no discrete group can act in such a way on our space.

In the present paper, in \$2\ we first exhibit more details about the geometry of treebolic space and its metric \(d(\cdot, \cdot) = d_{HT}(\cdot, \cdot)\) and explain its isometry group, which is (up to the obvious reflections with respect to vertical hyperplanes) obtained as a “horocyclic” product of the group Aff(H, q) of all affine mappings \(z \mapsto q^k z + b\) \((z \in \mathbb{H}, k \in \mathbb{Z}, b \in \mathbb{R})\) and the affine group of the tree \(T_p\), that is, the group of all automorphisms (self-isometries) of the tree that fix a given boundary point.

We next, in \$3\, turn our attention to the Laplace operator on HT, whose rigorous construction as an essentially self-adjoint diffusion operator bears a serious challenge in view of the singularities which our structure has along the bifurcation lines. This challenge was faced in the general setting of strip complexes in \[4\]. As a matter of fact, we consider a family of Laplacians \(\Delta_{\alpha, \beta}\) with two “vertical drift” parameters \(\alpha\) and \(\beta\). When looking at Brownian motion (BM), that is, the diffusion \((X_t)_{t \geq 0}\) on HT whose infinitesimal generator is \(\Delta_{\alpha, \beta}\), it is hyperbolic BM with linear drift parameter \(\alpha\) in the interior of each strip. On the other hand, \(\beta\) is responsible for the random choice of the strip into which BM should make its next infinitesimal step when the current position is on one of the bifurcation lines. The overall drift relies on both in terms of the number \(a = \beta p q^{\alpha-1}\). The drift is 0 if and only if \(a = 1\), while BM has an “upwards” (resp. “downwards”) drift when \(a > 1\) (resp. \(< 1\). The Laplacian and Brownian motion admit natural projections
on $\mathbb{T}$ and $\mathbb{H}$, as well as on $\mathbb{R}$. The projection onto $\mathbb{R}$ associates with each point its height: it is the Busemann function with respect to the boundary point at infinity of $\mathbb{H}$ (as well as of the tree). The projected Brownian motion $(Z_t)_{t \geq 0}$ on $\mathbb{H}$ is in general not ordinary hyperbolic BM with drift parameter $\alpha$, except when $\beta = 1/p$. That is, it evolves like hyperbolic BM with drift in the interior of each of the strips $S_k$ into which $\mathbb{H}$ has been “sliced”, while it receives an additional vertical “kick” (absent only when $\beta = 1/p$) on each of the lines $L_k$.

The projection $(W_t)_{t \geq 0}$ on $\mathbb{T}$ is a typical example of BM on a metric graph (the tree). The study of the corresponding Laplace operators is by now well established, and more straightforward than the higher dimensional version on strip complexes that we are dealing with here. See e.g. Cattaneo [14], Keller and Lenz [29], Bendikov and Saloff-Coste [3]. Analogously, the projection $(Y_t)_{t \geq 0}$ on $\mathbb{R}$ evolves like ordinary BM with drift as long as it does not visit any integer. When it visits an integer, BM receives an additional random “kick” in the positive or negative direction.

The main goal of this paper is to describe how Brownian motion on $\mathbb{H} \times \mathbb{T}$ evolves spatially. Main tool for this study is the sequence $(\tau(n))$ of the stopping times of the successive visits of $(X_t)$ in the bifurcation lines $L_v$, $v \in V(\mathbb{T})$ (the vertex set of the tree). The increments $\tau(n) - \tau(n - 1)$ are i.i.d. for $n \geq 2$, have exponential moments and an explicitly computable Laplace transform, see §4. That section contains further basic preliminary results. In particular, we study the distribution of the location of the process at time $\tau(1)$, which is the law governing the process $(X_{\tau(n)})$. This is quite subtle, because the singularities of our structure require care when trying to implement methods that appear to be “obvious” in the classical smooth setting.

The state space of the induced Markov process $(X_{\tau(n)})_{n \geq 0}$ is the disjoint union of all bifurcation lines. The projection $(Z_{\tau(n)})_{n \geq 0}$ of that process on $\mathbb{H}$ can be interpreted as a random walk on the group $\text{Aff}(\mathbb{H}, q)$. It can be treated via the methods of the work of Grincevicius [24], [25]. At the same time, the projection $(W_{\tau(n)})_{n \geq 0}$ is a nearest neighbour random walk on the (vertex set of the) tree whose transition probabilities are invariant under the action of the affine group of $\mathbb{T}$. It can also be considered as a random walk on that group. Random walks of this type were studied in detail by Cartwright, Kaimanovich and Woess [15]. The synthesis of those results on the two affine groups of $\mathbb{H}$ and of $\mathbb{T}$ is crucial for our study.

In §5, we consider the natural geometric boundary at infinity of $\mathbb{H} \times \mathbb{T}$. Since $\mathbb{H} \times \mathbb{T}$ is naturally embedded in the direct product $\mathbb{H} \times \mathbb{T}$, its natural compactification is its closure in $\mathbb{H} \times \mathbb{T}$. The boundary of $\mathbb{H} \times \mathbb{T}$ is the set of points added in this way. Here, $\hat{\mathbb{T}}$ is the well-known end compactification of the tree, while $\hat{\mathbb{H}}$ is the classical compactification of hyperbolic plane (the closed unit disk in the disk model of $\mathbb{H}$, or equivalently – in the upper half plane situation – the upper half plane together with its bottom line $\mathbb{R}$ and the “upper” boundary point at infinity). We show that in the topology of that compactification, Brownian motion on $\mathbb{H} \times \mathbb{T}$ converges almost surely to a limit random variable that lives on the boundary. In general, we can get quite good information about the law of that limit random variable, but it can
be given explicitly only in special cases regarding the choice of the parameters $\alpha$ and $\beta$. Convergence to the boundary goes hand in hand with computation of the linear rate of escape $\ell(\alpha, \beta)$, that is,

$$
\frac{d_{HT}(X_t, X_0)}{t} \to \ell(\alpha, \beta) \quad \text{almost surely, as } t \to \infty.
$$

It is the same as the rate of escape of $(Y_t)$ on $\mathbb{R}$. A basic tool for boundary convergence and rate of escape is the notion of regular sequences of Kaimanovich [28].

Next, in §6, we derive a central limit theorem, concerning convergence in law of

$$
\left(\frac{d_{HT}(X_t, X_0) - t \ell(\alpha, \beta)}{\sqrt{t}}\right).
$$

When $\ell(\alpha, \beta) > 0$, the limit law is centred normal distribution, and we also explain how to compute its variance $\sigma^2(\alpha, \beta) > 0$. When $\ell(\alpha, \beta) = 0$, the result as well as the limit distribution are somewhat more complicated.

The interplay of BM with the boundary provides the bridge to the potential theoretic part of our work, that will be laid out in forthcoming work [5].

In concluding the introduction, we want to underline how similar the geometric features as well as the properties of Brownian motion (resp. random walks) and the associated harmonic functions are on DL-graphs and lamplighter groups, the Sol-manifold (resp. -group) and treebolic space. In spite of the different techniques needed for each of the three, the realisation of those analogies, as well as the detailed study undertaken here, have become possible via the geometric interpretation of those structures as horocyclic products.

On the other hand, as already indicated, the elaboration and use of the analytic and probabilistic tools for this study are quite subtle in view of the singularities of HT at the bifurcation lines, thus providing a first concrete implementation of the analysis on strip complexes developed in [4].

2. Geometry and isometries of treebolic space

We start by describing the relevant features of the homogeneous tree $\mathbb{T} = \mathbb{T}_n$. Here, we consider $\mathbb{T}$ as a one-complex, where each edge is a copy of the unit interval $[0, 1]$. The discrete graph metric $d_{\mathbb{T}}(v_1, v_2)$ on the vertex set (0-skeleton) $V(\mathbb{T})$ of $\mathbb{T}$ is the length (number of edges) on the shortest path between $v_1$ and $v_2$. This metric has an obvious “linear” extension to the one-skeleton.

We partition the vertex set into countably many sets $H_k$, $k \in \mathbb{Z}$, such that each $H_k$ is countably infinite, and every vertex $v \in H_k$ has precisely one neighbour $v^-$ (the predecessor of $v$) in $H_{k-1}$ and $p$ neighbours in $H_{k+1}$ (the successors of $v$), each of which has $v$ as its predecessor. See Figure 2. The sets $H_k$ are called horocycles. For $v \in H_k$, we define $h(v) = k$. There is also a horocycle $H_t$ for any real $t$: if $k = \lceil t \rceil$ and $v \in V(\mathbb{T})$ with $h(v) = k$, then the metric edge $[v^-, v]$ meets $H_t$ precisely in the point $w$ which is at distance $k - t$ from $v$, and we set $h(w) = t$.

In addition to this basic description, we shall need further details. A geodesic path, resp. geodesic ray, resp. infinite geodesic in $\mathbb{T}$ is the image of an isometric
embedding \( t \mapsto w_t \in T \) of a finite interval \([a, b]\), resp. one-sided infinite interval \([a, \infty)\), resp. \( \mathbb{R} \), that is, \( d(w_s, w_t) = |t - s| \) for all \( s, t \). An end of \( T \) is an equivalence class of rays, where two rays \((w_t)\) and \((\bar{w}_t)\) are equivalent if they coincide up to finite initial pieces, i.e., there are \( s, t_0 \in \mathbb{R} \) such that \( \bar{w}_t = w_{s+t} \) for all \( t \geq t_0 \). We write \( \partial T \) for the space of ends, and \( \overline{T} = T \cup \partial T \). For all \( \eta, \zeta \in \overline{T} \) there is a unique geodesic \( \eta \zeta \) that connects the two. In particular, if \( w \in T \) and \( \xi \in \partial T \) then \( w \xi \) is the ray that starts at \( w \) and represents \( \xi \). Furthermore, if \( \xi, \eta \in \partial T \) (\( \xi \neq \eta \)) then \( \eta \xi \) is the infinite geodesic whose two halves (split at any of its points) are rays that represent \( \eta \) and \( \xi \), respectively. For \( v, w \in T \), \( v \neq w \), we define the cone \( \hat{T}(v, w) = \{ \zeta \in \overline{T} : w \in v \zeta \} \). For \( \xi \in \partial T \), the collection of all cones containing \( \xi \) is a neighbourhood basis of \( \xi \), while a neighbourhood basis of \( w \in T \) is given by all open balls in the tree metric. Thus, we obtain a topology which makes \( \overline{T} \) a compact Hausdorff space with the vertex set \( V(T) \) as a discrete subset and \( \partial T \).

![Figure 2](image.png)

**Figure 2.** The “upper half plane” drawing of \( T_2 \) (top down). \(^2\)

We choose and fix a reference vertex (root) \( o \in H_0 \). The geodesic ray whose vertices consist of the root and all its ancestors (= iterated predecessors) defines a reference end \( \varpi \in \partial T \), the lower boundary point in Figure 2. We set \( \partial^* T = \partial T \setminus \{\varpi\} \), the upper boundary in Figure 2. For \( w_1, w_2 \in \hat{T} \setminus \{\varpi\} \), their confluent (or maximal common ancestor) \( b = w_1 \wedge w_2 \) with respect to \( \varpi \) is defined by \( w_1 \varpi \cap w_2 \varpi = b \varpi \).

The function \( h : T \to \mathbb{R} \) defined above is the Busemann function of \( T \) with respect

\(^2\)This figure also appears in [4].
to \( \pi \), which can be written as

\[
(2.1) \quad \mathfrak{h}(w) = d(w, w \wedge o) - d(o, w \wedge o).
\]

In addition, we note that

\[
(2.2) \quad d_T(v, w) = d_T(v, v \wedge w) + d_T(v \wedge w, w) = \mathfrak{h}(v) + \mathfrak{h}(w) - 2\mathfrak{h}(v \wedge w).
\]

There is a natural Lebesgue measure \( dv \) on \( T \), which on each edge is a copy of standard Lebesgue measure on the unit interval.

The natural compactification \( \hat{H} \) of the hyperbolic plane \( H \) is the closed unit disk, when we use the Poincaré disk model. In our upper half plane model, \( \hat{H} \) is the closed upper half plane together with the the point at infinity \( \infty \). The corresponding boundary \( \partial \hat{H} \) consists of \( \infty \) together with the lower boundary line \( \partial^* \hat{H} = \mathbb{R} \). The Busemann function on \( H \) with respect to \( \infty \) is \( z \mapsto \log(\text{Im} z) \), where \( \text{Im} z \) is the imaginary part of \( z \). For \( z, z' \in \hat{H} \setminus \{\infty\} \), we can define the hyperbolic analogue \( z \wedge z' \) of the confluent: \( z \wedge z = z \), and when \( z \neq z' \), then \( z \wedge z' \) is the point on the (hyperbolic) geodesic \( z z' \) with maximal imaginary part. Recall that \( z z' \) is part of a circle centred on \( \mathbb{R} \) which is orthogonal to that boundary line. The function \( (z, z') \mapsto z \wedge z' \) is continuous from \((\hat{H} \setminus \{\infty\}) \times (\hat{H} \setminus \{\infty\})\) to \( \hat{H} \setminus \{\infty\} \). (The analogous property holds for the tree.) Similarly to (2.2), we have for \( z, z' \in \hat{H} \)

\[
(2.3) \quad d_{\hat{H}}(z, z') = d_{\hat{H}}(z, z \wedge z') + d_{\hat{H}}(z \wedge z', z') \quad \text{and} \quad |d_{\hat{H}}(z, z') - (2\log(\text{Im} z \wedge z') - \log(\text{Im} z) - \log(\text{Im} z'))| \leq \log 4.
\]

We are not sure whether the last inequality appears in the literature very often; its proof is an amusing exercise of handling the hyperbolic metric.

In the same way as \( T \) is subdivided horizontally by the horocycles \( H_k, k \in \mathbb{Z} \), we subdivide \( \hat{H} \) into the horizontal strips \( S_k \) delimited by the lines \( L_k \) consisting of all \( x + iy \in \hat{H} \) with \( y = q^k, k \in \mathbb{Z} \), see Figure 3. Note that all \( S_k \) are hyperbolically isometric.

As outlined in introduction and abstract, treebolic space with parameters \( q \) and \( p \) is

\[
(2.4) \quad HT(q, p) = \{J = (z, w) \in H \times T_p : \mathfrak{h}(w) = \log_q(\text{Im} z)\}.
\]

Thus, Figures 2 and 3 are the “side” and “front” views of \( HT \), that is, the images of \( HT \) under the projections \( \pi_H : (z, w) \mapsto z \) and \( \pi_T : (z, w) \mapsto w \), respectively. For a vertex \( v \in V(T) \), let

\[
(2.5) \quad L_v = \{J = (z, v) \in HT : \text{Im} z = q^{h(v)}\} = L_{h(v)} \times \{v\}.
\]

Then \( L_v \) and \( L_v^- \) are the upper and lower lines (respectively) in \( HT \) that delimit the strip

\[
(2.6) \quad S_v = \{J = (z, w) \in HT : w \in [v^-, v]\}.
\]
Here, \( w \in [v^-, v] \) is an element of the edge \([v^-, v]\) of \( T \), which is (recall) a copy of the unit interval. For \( \mathfrak{z} = (z, w) \in HT \), we shall sometimes write \( \text{Re}\mathfrak{z} = \text{Re} \, z \) for the real part of \( z \).

For each end \( \xi \in \partial^* T \), treebolic space contains the isometric copy
\[
\mathbb{H}_\xi = \{ \mathfrak{z} = (z, w) \in HT : w \in \xi \} \cap \mathbb{H},
\]
of \( \mathbb{H} \), and if \( \xi, \eta \in \partial^* T \) are distinct and \( v = \xi \wedge \eta \) (a vertex), then \( \mathbb{H}_\xi \) and \( \mathbb{H}_\eta \) ramify along the line \( L_v \), that is, \( \mathbb{H}_\xi \cap \mathbb{H}_\eta = \{ (z, w) \in HT : w \in \nu \Delta \} \).

The metric of \( HT \) is given by the hyperbolic length element in the interior of each strip. Its natural geodesic continuation is given as follows: consider two points \( \mathfrak{z}_1 = (z_1, w_1), \mathfrak{z}_2 = (z_2, w_2) \in HT \). Let \( d_H(z_1, z_2) \) by the hyperbolic distance between \( z_1 \) and \( z_2 \), and let \( v = w_1 \wedge w_2 \). Then
\[
(2.7) \quad d_{HT}(\mathfrak{z}_1, \mathfrak{z}_2) = \begin{cases} \, d_H(z_1, z_2), & \text{if } v \in \{w_1, w_2\}, \\ \min \{d_H(z_1, z) + d_H(z, z_2) : z \in L_v \}, & \text{if } v \notin \{w_1, w_2\}. \end{cases}
\]

Indeed, in the first case, \( \mathfrak{z}_1 \) and \( \mathfrak{z}_2 \) belong to a common copy \( \mathbb{H}_\xi \) of \( \mathbb{H} \). In the second case, \( v \in V(T) \), and there are \( \xi_1, \xi_2 \in \partial^* T \) such that \( \xi_1 \wedge \xi_2 = v \) and \( \mathfrak{z}_1 \in \mathbb{H}_\xi \), lie above the line \( L_v \), so that it is necessary to pass through some point \( \mathfrak{z} = (z, v) \in L_v \) on the way from \( \mathfrak{z}_1 \) to \( \mathfrak{z}_2 \). See Figure 4.

**Proposition 2.8.** For \( \mathfrak{z}_1 = (z_1, w_1), \mathfrak{z}_2 = (z_2, w_2) \in HT \), with \( \delta = \log(1 + \sqrt{2}) \),
\[
d_{HT}(\mathfrak{z}_1, \mathfrak{z}_2) \leq d_H(z_1, z_2) + (\log q) \, d_T(w_1, w_2) \\
- (\log q) \, |h(w_1) - h(w_2)| \leq d_{HT}(\mathfrak{z}_1, \mathfrak{z}_2) + 2\delta.
\]

\[\text{This figure also appears in [4].}\]
Proof. In the first case of (2.7), we have $d_{HT}(z_1, z_2) = d_{H}(z_1, z_2)$ and $d_T(w_1, w_2) = |h(w_1) - h(w_2)|$. Therefore, the left-hand side inequality of the proposition is indeed an equality.

We consider the second case of (2.7). We suppose without loss of generality that $\text{Im}z_1 \leq \text{Im}z_2$ and $\text{Re}z_1 \leq \text{Re}z_2$. Set $k = h(v)$, and let $z \in L_k$ be a point that realizes the minimum in (2.7), corresponding to $\xi = (z, v)$ in Figure 4. On the vertical ray in $\mathbb{H}$ going upwards from $z$ to $\infty$, let $z_1'$ be the point with $\text{Im}z_1' = \text{Im}z_1$. Also, we let $z'$ be the point on $L_k$ with $\text{Re}z' = \text{Re}z_1$. See Figure 5, showing the respective points and geodesic arcs in $\mathbb{H}$.

We start with the left-hand one of the two proposed inequalities, and use the minimising property of $z$: the distance $d_{HT}(z_1, z_2)$ is bounded above by the length of any path in $\mathbb{H}$ that starts at $z_1$, ends at $z_2$ and visits $L_k$ in between. We choose the following path: we first move vertically from $z_1$ down to $z_1'$, then back up to $z_1$, and then along the geodesic arc from $z_1$ to $z_2$. The length of this path is

$$2(\log \text{Im}z_1 - \log \text{Im}z') + d_{\mathbb{H}}(z_1, z_2),$$

which coincides with the middle term of our double inequality.

We now consider the right-hand one of the two proposed inequalities. By (2.7), $d_{HT}(z_1, z_2) = d_{\mathbb{H}}(z_1, z) + d_{\mathbb{H}}(z_2, z)$. On the other hand,

$$(\log q) \, d_T(w_1, w_2) = (\log \text{Im}z_1 - \log \text{Im}z) + (\log \text{Im}z_2 - \log \text{Im}z) \quad \text{and} \quad (\log q) \, |h(w_1) - h(w_2)| = \log \text{Im}z_2 - \log \text{Im}z_1,$$
because the last term was assumed to be $\geq 0$. Thus,

$$(\log q) d_T(w_1, w_2) - (\log q) |h(w_1) - h(w_2)| = 2d_{\mathbb{H}}(z'_1, z).$$

Thus, the claim of the proposition is equivalent with

$$d_{\mathbb{H}}(z'_1, z) - 2\delta \leq \frac{1}{2} (d_{\mathbb{H}}(z_1, z) + d_{\mathbb{H}}(z_2, z) - d_{\mathbb{H}}(z_1, z_2)) \leq d_{\mathbb{H}}(z'_1, z),$$

and we still need to prove the left-hand inequality. Now recall that $\mathbb{H}$ is the classical model of a geodesic metric space which is Gromov-hyperbolic: for the given value of $\delta \geq 0$, every geodesic triangle is $\delta$-thin. (That is, for any point on one of the three sides, there is a point at distance at most $\delta$ on the union of the other two sides.) We refer to Gromov [26], Coornaert, Delzant and Papadopoulos [16] and/or Ghys and de la Harpe [23] for all details regarding hyperbolic metrics and spaces. Now, $\frac{1}{2}(d(z_1, z) + d(z_2, z) - d(z_1, z_2)) = (z_1|z_2)_z$ is just the so-called Gromov product of $z_1$ and $z_2$ with respect to the reference point $z$. It is well known that the Gromov product on any geodesic hyperbolic metric space satisfies

$$d(z, z_1 z_2) - 2\delta \leq (z_1|z_2)_z \leq d(z, z_1 z_2),$$

where $z_1 z_2$ is of course the geodesic arc between $z_1$ and $z_2$. In our situation, we have $d_{\mathbb{H}}(z, z_1 z_2) \geq d_{\mathbb{H}}(z'_1, z)$, and the desired inequality follows. \hfill \Box

Proposition 2.8 should be compared with the formula of Proposition 3.1 in [6] for the graph metric of the DL-graphs, which is of the same form (without the $\delta$).

Note that the width of a strip $S_v$ of $\mathbb{H}$ is $d_{\mathbb{H}}(L_{v^-}, L_v) = \log q$, while its image under the projection $p_\mathbb{H}$ is the (metric) edge $[v^-, v]$, which has length 1. That is, in the construction of HT from $\mathbb{T}$ and $\mathbb{H}$, the tree is stretched by a factor of $\log q$.

Also note that the coordinates $(z, w)$ of HT used in (2.4) are useful in order to see the nature of HT as a horocyclic product and for deducing algebraic–geometric properties. However, by their nature, these coordinates are not independent. The resulting redundancy can be avoided by yet another description, more suitable for analytic purposes; see [4], §2.B. It is not used here in order to avoid abundance of multiple notation.

The area element of HT is $d_3 = y^{-2} dx\, dy$ for $\mathbb{H} = \{z, w\}$ in the interior of every $S_v$, where $z = x + iy$ and $dx$, $dy$ are Lebesgue measure: this is (a copy of) the standard hyperbolic upper half plane area element. The area of the lines $L_v$ is of course 0.

**Definition 2.9.** For a real function $f$ on HT, we write $f_v$ for its restriction to the closed strip $S_v$, where $v \in V(T)$. For its values, we write $f_v(z) = f(z, w)$, where $w \in T$ is the unique element on the edge $[v^-, v]$ such that $(z, w) \in HT$ (that is, $h(w) = \log q(w)$).

Analogous notation is used for the restriction of a function $f$ defined on $\Omega \subset HT$ to $\Omega \cap S_v$. 
While we think of \( f_v \) as a function on \( S_v \), it is formally a function defined for complex \( z = x + iy \in S_{h(v)} \subset \mathbb{H} \). The integral of \( f \) with respect to the area element is given by

(2.10) \[
\int_{HT} f(z) \, d\mu = \sum_{v \in V(T)} \int_{S_{h(v)}} f_v(x + iy) \, y^{-2} \, dx \, dy,
\]
whenever this is well defined in the sense of Lebesgue integration.

Next, we determine the isometry group of \( HT(q, p) \) and its modular function. Recall here that the modular function \( \delta \) of an arbitrary locally compact group \( G \) is the continuous homomorphism from the group into the multiplicative group \( \mathbb{R}^+ \) with the property that for left Haar measure \( dg \) on \( G \), one has

(2.11) \[
\int_G f(gg_0) \, dg = \delta(g_0)^{-1} \int_G f(g) \, dg
\]
for every integrable function \( f \) on \( G \).

Consider the action on \( \mathbb{H} \) of the group of affine transformations

(2.12) \[
\text{Aff}(\mathbb{H}, q) = \left\{ g = \begin{pmatrix} a \, b \\ 0 \, 1 \end{pmatrix} : n \in \mathbb{Z}, \ b \in \mathbb{R} \right\}, \text{ where } gz = q^n z + b, \ z \in \mathbb{H}.
\]

Thus, \( g_1 g_2 = \begin{pmatrix} a_1^{n_1} + b_1 & a_1 b_1 \\ 0 & 1 \end{pmatrix} \) for \( g_i = \begin{pmatrix} a_i \, b_i \\ 0 \, 1 \end{pmatrix}, \ i = 1, 2 \). This group acts by isometries on \( \mathbb{H} \) and leaves the set of lines \( y = q^k, \ k \in \mathbb{Z}, \) invariant. The full group of isometries of \( \mathbb{H} \) with the latter property is generated by \( \text{Aff}(\mathbb{H}, q) \) and the reflection along the \( y \)-axis. Our group is locally compact, and left Haar measure \( dg \) and its modular function \( \delta_{\mathbb{H}} = \delta_{\mathbb{H}, q} \) are given by

(2.13) \[
dg = q^{-n} \, dn \, db \quad \text{and} \quad \delta_{\mathbb{H}}(g) = q^{-n}, \ \text{if} \ g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.
\]

Here, \( dn \) is counting measure on \( \mathbb{Z} \) and \( db \) is Lebesgue measure on \( \mathbb{R} \).

Regarding the tree, first note that every isometry is the natural linear extension of an automorphism, that is, a neighbourhood preserving permutation of the vertex set \( V(T) \). Also, note that the action of each isometry extends continuously to \( \hat{T} \), since isometries send geodesic rays to geodesic rays and preserve their equivalence. Let \( \text{Aut}(T_p) \) denote the full isometry group of \( T_p \). Following Cartwright, Kaimanovich and Woess [15], the affine group of \( T_p \) is

(2.13) \[
\text{Aff}(T_p) = \left\{ \gamma \in \text{Aut}(T_p) : \gamma \varpi = \varpi \right\}.
\]

This is a locally compact, totally disconnected and compactly generated group with respect to the topology of pointwise convergence, and it acts transitively on \( V(T) \). The name is chosen (1) because of the analogy with the classical affine group which is just the group of (orientation preserving) isometries of \( \mathbb{H} \) that fix the boundary point \( \infty \), and (2) because the affine group over any local field whose residual field has order \( p \) embeds naturally into \( \text{Aff}(T_p) \), see [15] and below. The elements \( \gamma \) of \( \text{Aff}(T) \) are also characterized by the property \( \gamma(v^-) = (\gamma v)^- \) for every \( v \in V(T) \),
or equivalently, by $\gamma(w_1 \wedge w_2) = (\gamma w_1) \wedge (\gamma w_2)$ for all $w_i \in \mathbb{T}$. Consequently, the mapping $\Phi : \text{Aff}(\mathbb{T}) \to \mathbb{Z}$ defined by $\gamma \mapsto h(\gamma w) - h(w)$ is independent of $w \in \mathbb{T}$ and a homomorphism. Thus $\gamma(H_t) = H_{t+k}$ if $h(\gamma w) - h(w) = k$. As a matter of fact, this mapping appears in the modular function $\delta_T = \delta_{p_0}$ of $\text{Aff}(\mathbb{T}_p)$, see [15]:

\[(2.14) \quad \delta_T(\gamma) = p^{\Phi(\gamma)} \quad \text{where} \quad \Phi(\gamma) = h(\gamma w) - h(w), \quad \text{if} \quad \gamma \in \text{Aff}(\mathbb{T}_p), \quad w \in \mathbb{T}.
\]

In the following theorem, we collect several rather straightforward properties of the isometry group of $\text{HT}(q, p)$.

**Theorem 2.15.** The group

$$\mathcal{A} = \mathcal{A}(q, p) = \{(g, \gamma) \in \text{Aff}(\mathbb{H}, q) \times \text{Aff}(\mathbb{T}_p) : \log_q \delta_{\mathbb{H}}(g) + \log_p \delta_T(\gamma) = 0\}$$

acts on $\text{HT}(q, p)$ by isometries $(g, \gamma)(z, w) = (gz, \gamma w)$. It is the semidirect product

$$\mathcal{A} = \mathbb{R} \times \text{Aff}(\mathbb{T})$$

with respect to the action $b \mapsto q^{\Phi(\gamma)}b$, $\gamma \in \text{Aff}(\mathbb{T}), \ b \in \mathbb{R}$.

The full group of isometries of $\text{HT}(q, p)$ is generated by $\mathcal{A}(q, p)$ and the reflection

$$s(x + iy, w) = (-x + iy, w).$$

It acts on $\text{HT}(q, p)$ with compact quotient isomorphic with the circle of length $\log q$, and it leaves the area element of $\text{HT}$ invariant.

As a closed subgroup of $\text{Aff}(\mathbb{H}, q) \times \text{Aff}(\mathbb{T}_p)$, the group $\mathcal{A}$ is locally compact, compactly generated and amenable, and its modular function is given by

$$\delta_\mathcal{A}(g, \gamma) = (p/q)^{\Phi(\gamma)}.$$

**Proof.** (1) Let $(g, \gamma) \in \mathcal{A}$ and $(z, w) \in \text{HT}$, with $g = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ and $z = x + iy$. Then $gz = (b + q^n x) + i(q^n y)$, and $h(\gamma w) = h(w) + n$. Thus, $h(w) = \log_p(\text{Im} g z)$, whence $(gz, \gamma w) \in \text{HT}$. From (2.7), one sees that this is an isometry. Indeed, let $(z_i, w_i) \in \text{HT}$ ($i = 1, 2$) and $v = w_1 \wedge w_2$. Then $\gamma v = \gamma w_1 \wedge \gamma w_2$. So, if $v \in \{w_1, w_2\}$ then $\gamma v \in \{\gamma w_1, \gamma w_2\}$ and

$$d((gz_1, \gamma w_1), (gz_2, \gamma w_2)) = d_{\mathbb{H}}(gz_1, gz_2) = d_{\mathbb{H}}(z_1, z_2) = d((z_1, w_1), (z_2, w_2)).$$

If $v \notin \{w_1, w_2\}$ then $v \in V(\mathbb{T})$ and $\gamma v = \gamma v$. If $z_0$ minimizes $d_{\mathbb{H}}(z_1, z) + d_{\mathbb{H}}(z, z_2)$ among all $z \in L_{\gamma}(v)$, then $g z_0$ minimizes $d_{\mathbb{H}}(gz_1, \tilde{z}) + d_{\mathbb{H}}(\tilde{z}, z_2)$ among all $\tilde{z} \in L_{\gamma}(v)$. Thus, $d((gz_1, \gamma w_1), (gz_2, \gamma w_2)) = d((z_1, w_1), (z_2, w_2))$ in this case as well. Thus, $\mathcal{A}$ acts by isometries.

(2) We can identify each element $g = (g, \gamma) \in \mathcal{A}$ with the pair $[b, \gamma] \in \mathbb{R} \times \text{Aff}(\mathbb{T})$, where $g = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ as an affine mapping. It is immediate that with this identification, $\mathcal{A} = \mathbb{R} \times \text{Aff}(\mathbb{T})$ with the proposed action of $\text{Aff}(\mathbb{T})$ on $\mathbb{R}$, namely, the group operation is $[b_1, \gamma_1][b_2, \gamma_1] = [b_1 + q^{\Phi(\gamma_1)}b_2, \gamma_1 \gamma_2]$.

(3) Let $g$ be an isometry of $\text{HT}$. Then it is clear that $g$ sends each line $L_v$ to some other line $L_{\gamma v}$; compare with [22]. Thus, there is some $\gamma \in \text{Aut}(\mathbb{T})$ such that $g_{L_v} = L_{\gamma v}$ for every $v \in V(\mathbb{T})$. We claim that $\gamma \in \text{Aff}(\mathbb{T})$, that is, $\gamma v^{-} = (\gamma v)^{-}$ for all $v \in V(\mathbb{T})$. 

For \( v \in V(T) \), let \( \tilde{v} = \gamma v \). Suppose that \( \gamma v^- \neq \tilde{v}^- \). Then \( \tilde{u}_1 = \gamma v^- \) must be a successor of \( \tilde{v} \), that is, \( \tilde{u}_1^- = \tilde{v} \). Also, since \( p \geq 2 \), there must be some successor \( u \) of \( v \) \((u^- = v)\) such that \( \tilde{u}_2 = \gamma u \) is a successor of \( \tilde{v} \). Then \( g \) maps \( S_u \cup S_v \) isometrically to \( S_{u_1} \cup S_{u_2} \). Now, writing \( k = h(v) \) and \( \tilde{k} = h(\gamma v) \), we have that \( S_u \cup S_v \) is an isometric copy of \( S_{k-1} \cup S_k \subset \mathbb{H} \), while \( S_{u_1} \cup S_{u_2} \) consists of two copies of \( S_{k-1} \) glued together along their bottom line. With the metric (2.7), these two pieces are not isometric. Thus, it must be \( \gamma v^- = (\gamma v)^- \), and \( \gamma \in \text{Aff}(T) \). We now also see that \( gS_u = S_{\gamma u} \) for all \( v \in V(T) \).

Now consider an end \( \xi \in \partial^* \mathbb{T} \). It follows from the above that \( gH_\xi = H_{\gamma \xi} \). Thus, there must be an isometry \( g_\xi \) of \( \mathbb{H} \) such that for \((z, w) \in H_\xi \), \( g(z, w) = (g_\xi z, \gamma w) \). Then \( g_\xi \) must be either a Möbius transformation or a Möbius transformation followed by reflection along the \( y \)-axis. Since \( gL_v = L_{\gamma v} \) for each \( v \in V(T) \cap \xi \), in both of the last cases, that Möbius transformation is in \( \text{Aff}(\mathbb{H}, q) \). Now let \( \eta \in \partial^* \mathbb{T} \setminus \{\xi\} \) and set \( v = \xi \wedge \eta \). Then \( \mathbb{H}_\xi \) and \( \mathbb{H}_\eta \) coincide below (and including) the line \( L_\xi \subset HT \), whence \( g_\xi \) and \( g_\eta \) coincide below the line \( L_{h(v)} \). But this implies that \( g_\xi = g_\eta = g \) for all \( \xi, \eta \in \partial^* \mathbb{T} \). Every \((z, w) \in HT \) lies in \( \mathbb{H}_\xi \) for some \( \xi \in \partial^* \mathbb{T} \). Therefore \( g(z, w) = (gz, \gamma gw) \) for all \((z, w) \in HT \). This means that either \( g \in A \) (when \( g \) itself is a Möbius transformation), or \( sg \in A \) (when \( g \) is a Möbius transformation followed by reflection along the \( y \)-axis).

The statement about the co-compact action and factor space is obvious, and it is straightforward that the action of \( A \), as well as \( s \), preserve the area element of \( HT \).

(4) We compute the modular function of \( A \). Let \( db \) be Lebesgue measure on \( \text{Aff}(T) \). It will be useful to normalise \( d\gamma \) such that

\[
\int_{\text{Aff}(T)} 1_{\text{Stab}(o)}(\gamma) \, d\gamma = 1, \quad \text{where} \quad \text{Stab}(x) = \{ \gamma \in \text{Aff}(T) : \gamma o = o \}
\]

is the stabiliser of \( o \). It is an open-compact subgroup of \( \text{Aff}(\mathbb{T}) \). It is a straightforward exercise that in the \([b, \gamma]\)-coordinates of the semidirect product, left Haar measure on \( A \) is given by

\[
d\gamma = q^{-\Phi(\gamma)} \, db \, d\gamma.
\]

Now let \( g_0 = [b_0, \gamma_0] \), and let \( f \in \mathcal{C}_c(HT) \), the space of continuous, compactly supported functions. Then, using (2.14)

\[
\int_A f(gg_0) \, dg = \int_{\text{Aff}(T)} \int_R q^{-\Phi(\gamma)} f[b + q^\Phi(\gamma)b_0, \gamma \gamma_0] \, db \, d\gamma
\]

\[
= q^{\Phi(\gamma_0)} \int_R \int_{\text{Aff}(T)} q^{-\Phi(\gamma \gamma_0)} f[b, \gamma \gamma_0] \, d\gamma \, db
\]

\[
= q^{\Phi(\gamma_0)} \int_R \delta_\gamma(\gamma_0)^{-1} \int_{\text{Aff}(T)} q^{-\Phi(\gamma)} f[b, \gamma] \, d\gamma \, db = (q/p)^{\Phi(\gamma_0)} \int_A f(g) \, dg.
\]

This yields \( \delta_A(\gamma_0) = (p/q)^{\Phi(\gamma_0)} \), as proposed. \( \square \)

**Corollary 2.18.** When \( p \neq q \), there is no discrete group that acts on \( HT(q, p) \) with compact quotient.
Indeed, such a group would be a co-compact lattice in the isometry group of $\text{HT}(q, p)$, which cannot exist, since the latter group is non-unimodular. When $p = q$, the situation is different.

**Proposition 2.19.** The Baumslag–Solitar group $\text{BS}(p) = \langle a, b : ab = b^p a \rangle$ embeds as a co-compact, discrete subgroup into $\mathcal{A}(p, p)$.

**Proof.** It is well known that

\begin{equation}
\text{BS}(p) = \left\{ \begin{pmatrix} p^n & k/p^l \\ 0 & 1 \end{pmatrix} : k, l, n \in \mathbb{Z} \right\}.
\end{equation}

In this representation, $a = \begin{pmatrix} p^0 & 0 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Thus, it is immediate that $\text{BS}(p)$ is a (non-discrete) subgroup of $\text{Aff}(\mathbb{H}, p)$.

We now explain how our group acts on $T = T_p$, compare e.g. with § 4.A in [15].

The ring $\mathbb{Q}_p$ of $p$-adic numbers consists of all Laurent series in powers of $p$ of the form

\begin{equation}
u = \sum_{k=m}^{\infty} a_k p^k, \quad m \in \mathbb{Z}, \quad a_k \in \{0, \ldots, p-1\}.
\end{equation}

If $a_m \neq 0$ in (2.21), then we set $|u|_p = p^{-m}$, the $p$-adic norm of $u$. In addition, we get the neutral element 0 of $\mathbb{Q}_p$ when $a_k = 0$ for all $k$ in (2.21). Of course, $|0|_p = 0$. Addition and multiplication in $\mathbb{Q}_p$ extend the respective operations on those elements (2.21) which are finite sums (i.e., $a_k = 0$ for all but finitely many $k$), performed within the rational numbers. That is, carries to higher positions of coefficients that exceed $p-1$ have to be taken care of.

We have the following for all $u, v \in \mathbb{Q}_p$ and $m \in \mathbb{Z}$:

\begin{equation}
\begin{aligned}
(i) & \quad |u|_p = 0 \iff u = 0, \\
(ii) & \quad |u + v|_p \leq \max\{|u|_p, |v|_p\}, \\
(iii) & \quad |uv|_p \leq |u|_p |v|_p, \\
(iv) & \quad |p^m v|_p = p^{-m} |v|_p.
\end{aligned}
\end{equation}

If $p$ is prime, then we always have equality in (iii), and $\mathbb{Q}_p$ is a field. Otherwise, it is only a ring. By (ii), the norm induces an ultrametric. Any metric ball in $\mathbb{Q}_p$ is open and compact, and $\mathbb{Q}_p$ is totally disconnected (a Cantor set). Let $\overline{B}(u, p^{-k})$ be the closed ball with radius $p^{-k}$ and centre $u$. Each of its points is a centre for that ball. It is the disjoint union of $p$ closed balls with radius $p^{-k-1}$. Now consider

\[ H_m = \{ v = \overline{B}(u, p^{-m}) : u \in \mathbb{Q}_p \}. \]

This is going to be the horocycle at level $m$ of our tree, and for $v = \overline{B}(u, p^{-m})$ as a vertex of $T = T_p$, its predecessor is $v^- = \overline{B}(u, p^{-m+1})$. This gives us the tree structure. We find that $\partial^* \mathbb{T} = \mathbb{Q}_p$. 

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We see that via the matrix representation (2.20), $BS(p)$ acts on $\partial^* T$ by affine transformations of the ring $\mathbb{Q}_p$. This action extends to the tree: if $\gamma = \left( \begin{smallmatrix} p^n & k/p^l \\ 1 & 1 \end{smallmatrix} \right)$ and $v = \overline{B}(u, p^{-m})$ then

$$\gamma v = \{ p^m v + k/p^l : v \in \overline{B}(u, p^{m-n}) \} = \overline{B}(p^m u + k/p^l, p^{m-n})$$

is a closed ball with radius $p^{m-n}$, so that it is another vertex of our tree, which lies in $H_{m+n}$. (iv) of (2.22) is crucial here. In this way, $\gamma$ defines an element of $Aff(T_p)$.

We can now take the diagonal embedding $\gamma \mapsto (\gamma, \gamma)$ of $BS(p)$ into $Aff(H, p) \times Aff(T_p)$. This embedding is compatible with the level structure of both $H$ and the tree, so that $BS(p)$ is embedded into $A(p, p)$. It is easily seen to be discrete. It is co-compact because the factor space is compact. Indeed, a fundamental domain $\{ F \}$ satisfies, for all $\gamma$ of $Lebesgue$ measure on $H$, the reader is invited to elaborate these last details as an exercise; compare once more with [22].

**Remark 2.23.** In the analogy between tree and hyperbolic upper half plane, $\partial^* T$ corresponds to the lower boundary line $R$ of $H$. In this spirit, the natural analogue of Lebesgue measure on $R$ is the measure $\lambda^*$ on $\partial T$ which corresponds to (suitably normalised) Haar measure on the Abelian group $\mathbb{Q}_p$, under the identification of $\partial^* T$ with $\mathbb{Q}_p$. The basic open-closed sets in $\partial T$ (the ultrametric balls) and their measures are

$$\partial^* \alpha \partial^* \beta = \{ \xi \in \partial \alpha \partial^* \beta : \xi \in \overline{\alpha \partial^* \beta} \} \quad \text{and} \quad \lambda^*(\partial^* \alpha \partial^* \beta) = p^{-\alpha \beta}.$$

### 3. Laplacians with drift

We now explain our family of natural Laplace operators $\Delta^{HT} = \Delta^{HT}_{\alpha, \beta}$ on $HT(q, p)$ with “vertical drift” parameters $\alpha \in \mathbb{R}$ and $\beta > 0$. Their rigorous construction is carried out in detail in [4]. Here, we reproduce the basic facts.

**Definition 3.1.** We let $C^\infty(HT)$ be the set of those continuous functions $f$ on $HT$ such that, for each $v \in V(T)$, the restriction $f_\alpha$ of $f$ to the strip $S_\alpha$ (as in Definition 2.9) has continuous derivatives $\partial^m \partial^\nu f_\alpha(z)$ of all orders in the interior $S^\alpha_\nu$ which satisfy, for all $R > 0$,

$$\sup \{ |\partial^m \partial^\nu f_\alpha(z) : z = x + iy \in S^\alpha_\nu, \ |Rez| \leq R \} < \infty.$$

Thus, on each strip $S_\alpha$, each partial derivative has a continuous extension $\partial^m \partial^\nu f_\alpha(z)$ to the strip’s boundary. Note that when $w^- = v$, it is in general not true that $\partial^m \partial^\nu f_\alpha = \partial^m \partial^\nu f_\alpha$ on $L_v = S_\alpha \cap S_\omega$, unless $m = n = 0$. We have the (hyperbolic) gradient $\nabla f$ given by

$$\nabla f(z) = (y^2 \partial_x f(z), y^2 \partial_y f(z))$$
which is defined without ambiguity in the interior of each strip. However, on any bifurcation line \( L_v \), we have to distinguish between all the one-sided limits of the gradient, obtaining the family \( \nabla f_v(z) \) and \( \nabla f_w(z) \) for all \( w \in V(\mathbb{T}) \) with \( w^- = v \), \( (z, v) \in L_v \).

Let \( C^\infty(\Omega) \) be the space of those functions in \( C^\infty(HT) \) that have compact support contained in \( \Omega \). We shall write

\[
\text{LT} = \bigcup_{v \in V(\mathbb{T})} L_v \quad \text{and} \quad \text{HT}^\circ = \bigcup_{v \in V(\mathbb{T})} S_v^\circ = HT \setminus \text{LT}.
\]

For \( \alpha \in \mathbb{R}, \beta > 0 \), we define the measure \( \mathbf{m}_{\alpha,\beta} \) on \( HT \) by

\[
d\mathbf{m}_{\alpha,\beta}(\mathbf{z}) = \phi_{\alpha,\beta}(\mathbf{z}) \, d\mathbf{z}
\]

with \( \phi_{\alpha,\beta}(\mathbf{z}) = \beta h(v) y^\alpha \) for \( \mathbf{z} = (x + iy, w) \in S_v \setminus L_v^- \), where \( v \in V(\mathbb{T}) \), that is, \( w \in (v^-, v] \) and \( \log q_y = h(w) \).

**Definition 3.4.** For \( f \in C^\infty(HT) \) and \( \mathbf{z} = (x + iy, w) \in \text{HT}^\circ \), we set

\[
\Delta_{\alpha,\beta} f(\mathbf{z}) = y^2 (\partial^2_x + \partial^2_y) f(\mathbf{z}) + \alpha y \partial_y f(\mathbf{z}).
\]

Let \( D^\infty_{\alpha,\beta,c} \) be the space of all functions \( f \in C^\infty_c(HT) \) with the following properties.

(i) For any \( k \), the \( k \)-th iterate \( \Delta_{\alpha,\beta}^k f \), originally defined on \( \text{HT}^\circ \), admits a continuous extension to all of \( HT \) (which then belongs to \( C^\infty_c(HT) \) and is also denoted \( \Delta_{\alpha,\beta}^k f \)).

(ii) The function \( f \), as well as each of its iterates \( \Delta_{\alpha,\beta}^k f \), satisfies the bifurcation conditions

\[
\partial_y f_v = \beta \sum_{w : w^- = v} \partial_y f_w \quad \text{on } L_v \quad \text{for each } v \in V(\mathbb{T}).
\]

The Laplacian \( \Delta_{\alpha,\beta} \) as a differential operator on \( \text{HT}^\circ \) apparently depends only on \( \alpha \). Dependence on \( \beta \) is through the domain of functions on which the Laplacian acts, which have to satisfy (3.5). In the following propositions, we present some of the essential properties proved in [4], where additional details can be found.

**Proposition 3.6.** The space \( D^\infty_{\alpha,\beta,c} \) is dense in the Hilbert space \( L^2(HT, \mathbf{m}_{\alpha,\beta}) \).

The operator \( (\Delta_{\alpha,\beta}, D^\infty_{\alpha,\beta,c}) \) is essentially self-adjoint in \( L^2(HT, \mathbf{m}_{\alpha,\beta}) \).

With a small abuse of notation, we write \( (\Delta_{\alpha,\beta}, \text{Dom}(\Delta_{\alpha,\beta})) \) for its unique self-adjoint extension. Indeed, at a higher level of rigour in Definition 2.16 of [4], the differential operator of Definition 3.4 (i), defined on \( D^\infty_{\alpha,\beta,c} \), is denoted \( \mathfrak{A}_\alpha \), and the notation \( \Delta_{\alpha,\beta} \) is reserved for the extension. The detailed construction of the latter in [4] is carried out via Dirichlet form theory.
Proposition 3.7. (a) The heat semigroup \(e^{t\Delta_{\alpha,\beta}}\), \(t > 0\), acting on \(L^2(HT, m_{\alpha,\beta})\) admits a continuous positive symmetric transition kernel \((0, \infty) \times HT \times HT \ni (t, \mathbf{w}, \mathbf{z}) \mapsto h_{\alpha,\beta}(t, \mathbf{w}, \mathbf{z})\) such that for all \(f \in C_c(HT)\),
\[
e^{t\Delta_{\alpha,\beta}} f(\mathbf{z}) = \int_{HT} h_{\alpha,\beta}(t, \mathbf{w}, \mathbf{z}) f(\mathbf{z}) \, dm_{\alpha,\beta}(\mathbf{z}).
\]

(b) For each fixed \((t, \mathbf{w})\), the function \(\mathbf{z} \mapsto h_{\alpha,\beta}(t, \mathbf{w}, \mathbf{z})\) is in \(C^\infty(HT)\) and satisfies (3.5).

(c) The heat semigroup is conservative, that is, \(\int_{HT} h_{\alpha,\beta}(t, \mathbf{w}, \cdot) \, dm_{\alpha,\beta} = 1\).

(d) It sends \(L^\infty(HT)\) into \(C^\infty(HT)\cap L^\infty(HT)\) and \(C_0(HT)\), the space of continuous functions vanishing at infinity, into itself.

The general theory of Markov processes tells us that \(\Delta_{\alpha,\beta}\) is the infinitesimal generator of a Hunt process \((X_t)_{t \geq 0}\). This is our Brownian motion on HT. It is defined for every starting point \(\mathbf{w} \in HT\), has infinite life time and continuous sample paths. Its family of distributions \((\mathbb{P}_{\mathbf{w},\beta}^t)_{t \in HT}\) on \(\Omega = C([0,\infty] \to HT)\) is determined by the one-dimensional distributions
\[
\mathbb{P}_{\mathbf{w},\beta}^t[X_t \in U] = \int_U h_{\alpha,\beta}(t, \mathbf{w}, \mathbf{z}) \, dm_{\alpha,\beta}(\mathbf{z}) = \int_U p_{\alpha,\beta}(t, \mathbf{w}, \mathbf{z}) \, d\mathbf{z},
\]
where \(U\) is any Borel subset of HT and
\[
p_{\alpha,\beta}(t, \mathbf{w}, \mathbf{z}) = h_{\alpha,\beta}(t, \mathbf{w}, \mathbf{z}) \phi_{\alpha,\beta}(\mathbf{z})
\]
with the function \(\phi_{\alpha,\beta}\) as in (3.3). We note that this transition density with respect to \(d\mathbf{z}\) is invariant under the action of the group \(A\) of Theorem 2.15:
\[
(3.8) \quad p_{\alpha,\beta}(t, g\mathbf{w}, g\mathbf{z}) = p_{\alpha,\beta}(t, \mathbf{w}, \mathbf{z}) \quad \text{for all} \quad t > 0, \mathbf{w}, \mathbf{z} \in HT \text{ and } g \in A.
\]

We next say a few words about the natural projections of HT. We have
\[
\pi^H : HT \to \mathbb{H}, \quad \mathbf{z} = (z, w) \mapsto z, \quad \pi^T : HT \to \mathbb{T}, \quad \mathbf{z} = (z, w) \mapsto w, \quad \text{and} \quad \pi^R : HT \to \mathbb{R}, \quad \mathbf{z} = (z, w) \mapsto \log \mathbf{g} \Im(z).
\]
We also interpret \(\pi^R\) as a projection \(H \to \mathbb{R}\), where \(z \mapsto \log \mathbf{g} \Im(z)\), and as a projection \(\mathbb{T} \to \mathbb{R}\), where \(w \mapsto b(w)\). Thus, the following diagram commutes.
The “sliced” hyperbolic plane as in Figure 3 can be interpreted as \( HT(q, 1) \), that is, the tree is \( \mathbb{Z} \), the bi-infinite line graph. Everything that has been said above also applies here, so that we have the operator \( \Delta_{\alpha, \beta}^\mathbb{H} \) on \( \mathbb{H} \).

Analogously, we have a Laplacian \( \Delta_{\alpha, \beta}^T \) on the metric tree, introduced in the same way as above. However, we should take care of the slightly different parametrisation, that is, the stretching factor \( \log q \) in the construction of \( HT \), while in \( T \), each edge \([v^-, v]\) corresponds to the real interval \([h(v) - 1, h(v)]\). The functions that we consider now depend on one real variable in each open edge. We write \( f_v \) for the restriction of \( f : T \to \mathbb{R} \) to \([v^-, v]\). We have to redefine the analogue of the measure of (3.3):

\[
\begin{align*}
\text{Let } & f_v \text{ for the restriction of } f : T \to \mathbb{R} \text{ to } [v^-, v]. \\
& \text{We have to redefine the analogue of the measure of (3.3):}
\end{align*}
\]

\[
\begin{align*}
(3.9) \quad dm_{\alpha, \beta}^T(w) = \phi_{\alpha, \beta}^T(w) dw \quad \text{with} \quad \\
\phi_{\alpha, \beta}^T(w) = \beta^{h(v)} q^{(\alpha-1)h(w)} \log q \quad \text{for } w \in (v^-, v),
\end{align*}
\]

where \( v \in V(T) \) and (recall) \( dw \) is the standard Lebesgue measure in each edge. The space \( C^\infty(T) \) is defined as in Definition 3.1, considering the edges of \( T \) as the strips. The analogues of the crucial Definition 3.4 plus the bifurcation condition (3.5) now become the following: every \( f \in \text{Dom}(\Delta_{\alpha, \beta}^T) \cap C^\infty(T) \) must satisfy for every \( v \in V(T) \)

\[
\begin{align*}
(3.10) \quad f_v'(v) = \beta \sum_{w : v^- \rightarrow v} f_w'(v) \\
\Delta_{\alpha, \beta}^T f = \frac{1}{(\log q)^2} f'' + \frac{\alpha - 1}{\log q} f' \quad \text{in the open edge } (v^-, v).
\end{align*}
\]

Finally, the analogue on the real line is comprised in the above by identifying \( \mathbb{R} \) with the tree with branching number 1 (degree 2). In this case, the vertices are the integers, the edges are the intervals \([k-1, k]\), where \( k \in \mathbb{Z} \), and the Laplacian becomes \( \Delta_{\alpha, \beta}^\mathbb{R} \). Its definition as a differential operator in each open interval \([k-1, k]\) is the same as in (3.10), while the bifurcation condition becomes \( f'(k-) = \beta f'(k+) \) for all \( k \in \mathbb{Z} \).

With these modifications, propositions 3.6 and 3.7 apply to all those Laplacians. We write \( h_{\alpha, \beta}^\mathbb{H}, h_{\alpha, \beta}^T \) and \( h_{\alpha, \beta}^\mathbb{R} \) for the respective associated transition kernels.

**Proposition 3.11.** Let \((X_t)\) be the process on \( HT(q, p) \) whose infinitesimal generator is \( \Delta_{\alpha, \beta} \). Set

\[
Z_t = \pi^\mathbb{H}(X_t), \quad W_t = \pi^T(X_t), \quad \text{and} \quad Y_t = \pi^\mathbb{R}(X_t), \quad t \geq 0.
\]

(a) The process \((Z_t)\) is a Markov process on \( \mathbb{H} \) whose infinitesimal generator is \( \Delta_{\alpha, \beta}^\mathbb{H} \). Its transition kernel with respect to the measure \( m_{\alpha, \beta}^\mathbb{H} \) is \( h_{\alpha, \beta}^\mathbb{H} \).

(b) The process \((W_t)\) is a Markov process on \( T \) whose infinitesimal generator is \( \Delta_{\alpha, \beta}^T \). Its transition kernel with respect to the measure \( m_{\alpha, \beta}^T \) is \( h_{\alpha, \beta}^T \).

(c) The process \((Y_t)\) is a Markov process on \( \mathbb{R} \) whose infinitesimal generator is \( \Delta_{\alpha, \beta}^\mathbb{R} \). Its transition kernel with respect to the measure \( m_{\alpha, \beta}^\mathbb{R} \) is \( h_{\alpha, \beta}^\mathbb{R} \).
Definition 3.12. For any open domain $\Omega \subset HT$, we let $\tau^{\Omega} = \inf\{t > 0 : X_t \in HT \setminus \Omega\}$ be the first exit time of $(X_t)$ from $\Omega$, and if $\tau = \tau^{\Omega} < \infty$ almost surely for the starting point $X_0 = w \in \Omega$, then we write $\mu_w^{\Omega}$ for the distribution of $X_{\tau}$.

The probability measure $\mu_w^{\Omega}$ is usually supported by $\partial \Omega$ (we do not specify the meaning of “usually”; for the sets that we are going to consider, this will be true). We shall use analogous notation on $H$, $T$ and $R$. We note that

\begin{equation}
\mu_w^{\Omega}(B) = \mu_{gw}^{\Omega}(gB) \quad \text{for every } g \in A \text{ and Borel set } B \subset HT.
\end{equation}

Definition 3.14. Let $\Omega \subset HT$ be open. A continuous function $f : \Omega \to \mathbb{R}$ is called harmonic on $\Omega$ if for every open, relatively compact set $U$ with $U \subset \Omega$,

\begin{equation}
\int f(x) \mu_{\pi}^U(d\pi) \quad \text{for all } x \in U.
\end{equation}

From the classical analytic viewpoint, this definition may be unsatisfactory; “harmonic” should mean “annihilated by the Laplacian” (as a differential operator). However, for general open domains in $HT$, the correct formulation in these terms is quite subtle in view of the relative location of the bifurcations.

More details will be stated and used in [5].

4. Brownian motion and the induced random walks

Our basic approach is to study BM on $HT$ via the random walk resulting from observing the processes during its successive visits in the set $LT$ of all bifurcation lines.

Thus, we define the stopping times $\tau(n)$, $n \in \mathbb{N}_0$,

\begin{equation}
\tau(0) = 0, \quad \tau(n + 1) = \inf\{t > \tau(n) : Y_t \in \mathbb{Z} \setminus \{Y_{\tau(n)}\}\}.
\end{equation}

They are not only the times of the successive visits of $(Y_t)$ in $\mathbb{Z}$: by Proposition 3.11, if $X_0$ lies in some open strip $S^v_0$, then $\tau(1)$ is the exit time from that open strip, that is, the instant when $X_t$ first meets a point on $L_v \cup L_{w^v}$. If $X_{\tau(n)} \in L_v$ for some $v \in V(T)$ (which holds for all $n \geq 1$, and possibly also for $n = 0$), then $\tau(n + 1)$ is the first instant $t > \tau(n)$ when $X_t$ meets one of the bifurcation lines $L_{v^+}$ or $L_{w^w}$ with $w^w = v$. The $\tau(n)$ are also the times of the successive visits of $(Z_t)$ in the union of all the lines $L_v$ that subdivide $H$, as well as the times of the successive visits of $(W_t)$ in the vertex set $V(T)$ of $T$. Later on, we shall also need the integer random variables $n_t$, defined by

\begin{equation}
\tau(n_t) \leq t < \tau(n_t + 1), \quad \text{where } t \geq 0,
\end{equation}

as well as the stopping time

\begin{equation}
\sigma = \inf\{t > 0 : Y_t \in \{-1, 1\}\}, \quad \text{where } Y_0 = y_0 \in [-1, 1].
\end{equation}

This is the exit time from $[-1, 1]$. Note that $\sigma = \tau(1)$ when $y_0 = 0$, but not when $0 < |y_0| < 1$. 
Lemma 4.4. For any starting point in $\mathbb{R}$, resp. $[-1, 1]$, the stopping times $\tau(1)$ and $\sigma$ are almost surely finite.

Proof. We start with $\sigma$. Consider the function $g(y) = \mathbb{P}_y[\sigma < \infty]$ on $[-1, 1]$. It is a weak solution of the Dirichlet problem $\Delta^{\mathbb{R}}_{\alpha, \beta} g = 0$ on $(-1, 1)$ with boundary values 1 at $\pm 1$. By Theorem 5.9 in [4] (in a simplified version, because here we are dealing with the infinite line as a metric graph), $g$ is a strong solution. Thus, $g$ satisfies the following “broken” differential equation, where we have to use $\Delta^{\mathbb{R}}_{\alpha, \beta} p$:

$$
\frac{1}{(\log q)^2} g'' + \frac{\alpha - 1}{\log q} g' = 0, \quad g'(0-) = \beta p g'(0+),
$$

with $g(\pm 1) = 1$. The unique solution is $g \equiv 1$, whence $\mathbb{P}_y[\sigma < \infty] = g(y) = 1$.

Now let us consider $\tau(1)$. By (3.8), the transition density of $(Y_t)$ is invariant under translation by integers. Therefore we may suppose that the starting point is in $[0, 1)$. If it is 0 then $\tau(1) = \sigma$, so we restrict to starting points $y \in (0, 1)$.

Set $h(y) = \mathbb{P}_y[\tau(1) < \infty]$. Then $h$ satisfies the differential equation

$$
\frac{1}{(\log q)^2} h'' + \frac{\alpha - 1}{\log q} h' = 0
$$

on the interval $(0, 1)$, with boundary values $h(0) = h(1) = 1$. Again, the unique solution is $h \equiv 1$, whence $\mathbb{P}_y[\tau(1) < \infty] = 1$.

We shall need detailed computations regarding the two integer random variables $\tau = \tau(2) - \tau(1)$ and $Y = Y_{\tau(2)} - Y_{\tau(1)}$, in particular their expected values and variances. Note that $Y$ takes the values $\pm 1$.

Proposition 4.6. (a) The increments $\tau(n) - \tau(n-1)$, $n \geq 1$, are independent and almost surely finite.

(b) They are identically distributed for $n \geq 2$, and when $Z_0 \in \Lambda T$, then also $\tau(1)$ has the same distribution.

(c) Let $b = (\alpha - 1) \log q/2$ and consider the real functions $s(\lambda) = b^2 + (\log q)^2 \lambda$ and

$$
\begin{align*}
\lambda &= (\beta p + 1) \sum_{n=0}^{\infty} \frac{s(\lambda)^n}{(2n)!} + (\beta p - 1) b \sum_{n=0}^{\infty} \frac{s(\lambda)^n}{(2n + 1)!} \quad \lambda \in \mathbb{R}.
\end{align*}
$$

Then the random variables $Y$ and $\tau$ defined above are independent,

$$
\begin{align*}
\mathbb{E}(e^{-\lambda \tau 1_{[Y=1]}}) &= \beta p e^b / r(\lambda) \quad \text{and} \quad \mathbb{E}(e^{-\lambda \tau 1_{[Y=-1]}}) = e^{-b} / r(\lambda).
\end{align*}
$$

(d) In particular, setting $a = \beta p q^{\alpha - 1}$,

$$
\begin{align*}
\mathbb{P}[Y = 1] &= \frac{a}{a + 1}, \quad \mathbb{P}[Y = -1] = \frac{1}{a + 1},
\end{align*}
$$

$$
\begin{align*}
\mathbb{E}(Y) &= \frac{a - 1}{a + 1}, \quad \text{and} \quad \mathbb{V}(Y) = \frac{4a}{(a + 1)^2}.
\end{align*}
$$
(e) The Laplace transform $\lambda \mapsto \mathbb{E}(e^{-\lambda \tau}) = (e + 1)e^{-\theta}/r(\lambda)$ is analytic in a
neighbourhood of 0, so that $\tau$ has finite exponential moment $\mathbb{E}(e^{\lambda \tau})$ for
some $\lambda_0 > 0$.

(f) The expectation and variance of $\tau$ are

$$
\mathbb{E}(\tau) = r'(0)e^b/(a + 1) \quad \text{and} \quad \text{Var}(\tau) = \mathbb{E}(\tau)^2 - r''(0)e^b/(a + 1).
$$

Proof. (a) and (b) are clear.

For (c), we fix $\lambda \geq 0$ and consider again the exit time $\sigma$ of the process $(Y_t)$
from the interval $[-1, 1]$. We let $f_{\pm 1}(y) = \mathbb{E}_y(e^{-\lambda \sigma 1_{[y, \pm 1]}})$, respectively, defined
for the starting point $Y_0 = y \in [-1, 1]$. We note that $\mathbb{E}(e^{-\lambda \tau 1_{[y, \pm 1]}}) = f_{\pm 1}(0)$.
Each of the two functions $f_{\pm 1}$ is a weak, whence strong ([4], Theorem 5.9) solution of the Dirichlet problem $\Delta f_{\pm 1} = \lambda \cdot f_{\pm 1}$ on the interval $[-1, 1]$ with boundary
values 0 and 1, resp. 1 and 0 at the endpoints $-1$ and 1. Thus, $f_{-1}$ and $f_1$ satisfy the “broken” differential equation

$$
\frac{1}{(\log q)^2} f_{\pm 1}'' + \frac{\alpha - 1}{\log q} f_{\pm 1}' = \lambda \cdot f_{\pm 1}, \quad f_{\pm 1}'(0^-) = \beta p f_{\pm 1}'(0^+),
$$

$$
f_{-1}(-1) = 0, \quad f_1(1) = 1, \quad \text{resp.} \quad f_{-1}(-1) = 1, \quad f_{-1}(1) = 0.
$$

The computation of the solutions is a lengthy, but basic exercise that leads to (e); it may be useful here to note that

$$
r(\lambda) = \begin{cases}
(\beta p + 1) \cosh \sqrt{s(\lambda)} + (\beta p - 1) b \sinh \sqrt{s(\lambda)} \sqrt{-s(\lambda)}, & s(\lambda) \geq 0, \\
(\beta p + 1) \cos \sqrt{-s(\lambda)} + (\beta p - 1) b \sin \sqrt{-s(\lambda)} \sqrt{-s(\lambda)}, & s(\lambda) \leq 0.
\end{cases}
$$

Statement (d) is obtained by setting $\lambda = 0$ in (e).

A short computation now shows that $\mathbb{E}(e^{-\lambda \tau 1_{[y, \pm 1]}}) = \mathbb{E}(e^{-\lambda \tau}) \text{Pr}([Y = \pm 1])$
for all $\lambda \geq 0$, which yields independence of $Y$ and $\tau$.

Statement (e) is obvious from the form of the Laplace transform.

Statement (f) is obtained by direct computations of the first and second derivatives of the transform. \(\square\)

We can compute

$$
(4.7) \quad \mathbb{E}(\tau) = \frac{(\log q)^2}{\frac{2b^2}{2}} \frac{(\beta p - 1)b \cosh b + [(\beta p + 1)b - (\beta p - 1)] \sinh b}{(\beta p + 1) \cosh b + (\beta p - 1) \sinh b}, \quad \text{if } \alpha \neq 1, \quad \text{if } \alpha = 1,
$$

However, we omit the lengthy formula for $\text{Var}(\tau)$, which can be obtained by tedious
computation but provides no specific insight.

The following is obtained by completely similar, but simpler computations. (Namely, we have to solve the same differential equation as above for computing $\mathbb{E}_p(e^{-\lambda \tau(1)})$, but it is not “broken”.)
Lemma 4.8. For any \( y \in \mathbb{R} \), there is \( \lambda = \lambda(y) > 0 \) (depending only on the fractional part of \( y \)) such that for the process \( (Y_t) \) starting at \( y \)

\[
E_y(e^{\lambda \tau(1)}) < \infty.
\]

We now clarify the nature of the induced processes on \( \mathbb{Z} \) and on \( \mathbb{T} \), respectively.

Corollary 4.9. With \( a = \beta pq^{a-1} \) as in Proposition 4.6(d),

(a) the process \( (Y_{\tau(n)})_{n \geq 1} \) is a nearest neighbour random walk on \( \mathbb{Z} \) with transition probabilities

\[
p_Z(k,l) = \Pr[Y_{\tau(n+1)} = l \mid Y_{\tau(n)} = k] = \begin{cases} 
\frac{a}{1+a}, & \text{if } l = k + 1, \\
\frac{1}{1+a}, & \text{if } l = k - 1, \\
0, & \text{otherwise}.
\end{cases}
\]

(b) The process \( (W_{\tau(n)})_{n \geq 1} \) is a transient nearest neighbour random walk on (the vertex set of) \( \mathbb{T} \) with transition probabilities

\[
p_T(v,w) = \Pr[W_{\tau(n+1)} = w \mid W_{\tau(n)} = v] = \begin{cases} 
\frac{a}{(1+a)p}, & \text{if } w^- = v, \\
\frac{1}{1+a}, & \text{if } w = v^-, \\
0, & \text{otherwise,}
\end{cases}
\]

where \( v, w \in V(\mathbb{T}) \).

Proof. (a) is immediate from Proposition 4.6(d).

Part (b) is an immediate consequence of (a), because for any \( v \in V(\mathbb{T}) \), we must have \( p_T(v,v^-) = p_Z(k,k) \), while \( p_T(v,w) \) must be the same for all successors \( w \) of \( v \), with sum \( p_Z(k,k+1) \). It is well-known and easy to prove that this random walk on \( V(\mathbb{T}) \) is transient (visits any finite set only finitely often a.s.), compare with [15] or [33]. \( \square \)

The transition kernel of the induced processes on \( HT \), resp. \( H \), cannot be computed as explicitly. We need to consider the non-compact set

\[
\Omega_v = \{(z, w) \in HT : w \in N(v) \} \subset HT,
\]

where \( N(v) \) is the “neighbourhood star” in \( T \) at \( v \in V(\mathbb{T}) \). That is, \( N(v) \) is the union of all edges (i.e., intervals!) of \( T \) which have \( v \) as one endpoint. It is a compact metric subtree of \( T \), whose boundary \( \partial N(v) \) consists of all neighbours of \( v \) in \( V(\mathbb{T}) \). We write \( \partial^+ N(v) = \partial N(v) \setminus \{v^-\} \) (the forward neighbours of \( v \)).
For any starting point \( w \in \Omega_v \), the exit time \( \tau_{w} \) is almost surely finite by (3.8) and Lemma 4.4. Thus, we have the probability measure \( \mu_{w}^{\Omega} \) on the boundary of \( \Omega_v \) in \( HT \),
\[
\partial \Omega_v = \bigcup_{w \in \partial N(v)} L_w.
\]
For \( w \in L_v \), this is the transition probability of the Markov chain \( (X_{\tau(n)}) \) on \( LT \) : for any \( w \in L_v \) \((v \in V(\mathbb{T}))\) and Borel set \( B \subset \partial \Omega_v \),
\[
(4.11) \quad \Pr[w \in X_{\tau(n+1)} | X_{\tau(n)} = w] = \mu_{w}^{\Omega}(B).
\]
**Lemma 4.12.** For any \( w \in \Omega_v \), the measure \( \mu_{w}^{\Omega} \) is supported by whole of the boundary of \( \Omega_v \) in \( HT \),
\[
\partial \Omega_v = L_v^- \cup \bigcup_{w \in V(\mathbb{T}); w^- = v} L_w.
\]
In particular, the process \( (X_{\tau(n)}) \) is irreducible on \( LT \) : for any starting point \( w \in LT \) and any non-empty open interval \( I \) that lies on one of the bifurcation lines,
\[
\Pr_{w}[\exists n : X_{\tau(n)} \in I] > 0.
\]
**Proof.** The second statement follows from the first one. The first one follows from ellipticity of \( \Delta_{\alpha, \beta} \). More specifically, we can also see this as follows. A boundary point \( z \) of any open domain \( \Omega \subset HT \) is regular for the Dirichlet problem with respect to \( \partial \Omega \) if and only if \( \Pr_{z}[\tau_{\Omega} = 0] = 1 \) (a general fact from potential theory).

Every boundary point of \( \Omega_v \) is regular. This follows from the fact that \( \tau_{\Omega_v} \) is the same as the exit time of the process \( (W_t) \) on \( \mathbb{T} \) from the neighbourhood star \( N(v)^{\circ} \). But the Dirichlet problem for the latter (with boundary values at the finitely many neighbours of \( v \) in \( V(\mathbb{T}) \)) is obviously solvable, as one can verify by direct, elementary computations similar to those used in the proof of Lemma 4.4.

To conclude, recall that every regular point has to be in the support of the first exit measure. \( \square \)

We choose the point \( o = (1, o) \in HT \) as the origin of treebolic space. Let
\[
\mu = \mu_{o}^{\Omega}, \quad \text{where} \quad \Omega = \Omega_o.
\]
By group invariance (3.13), we have
\[
(4.13) \quad \mu_{w}^{\Omega} = \delta_g * \mu, \quad \text{when} \quad g \in A, \quad go = w \in L_v.
\]
The convolution of the Dirac measure at \( g \) with \( \mu \) is defined via the action of the group \( A \), which is transitive on \( LT \). That is, \( LT \) is a homogeneous space of \( A \) (the stabilizer of \( o \) in \( A \) is a non-trivial compact subgroup), and \( (X_{\tau(n)}) \) is a random walk on that homogeneous space.

The transition kernel of \( (Z_{\tau(n)}) \) can be obtained analogously. That process evolves on
\[
L_{H} = \bigcup_{k \in \mathbb{Z}} L_k \subset H.
\]
We have
\[ \tilde{\Omega}_k = (S_{k-1} \cup S_k)^\circ = \pi^\mathbb{H}(\Omega_v) \quad \text{for any } v \in H_k \subset V(T). \]
Its boundary within \( \mathbb{H} \) is \( \partial \tilde{\Omega}_k = L_{k-1} \cup L_{k+1} \). For any starting point \( z \in \tilde{\Omega}_k \), we let \( \tilde{\mu}_z^\Omega \) be the exit distribution from \( \tilde{\Omega}_k \). In analogy with Lemma 4.12, it is supported by the whole of \( \partial \tilde{\Omega}_k \), and for any \( z \in L_k \) and Borel set \( B \subset \partial \tilde{\Omega}_k \),
\[
\Pr[Z_{\tau(n)}+1] \in B \mid Z_{\tau(n)} = z] = \tilde{\mu}_z^\Omega(B).
\]
We set
\[
\tilde{\mu} = \tilde{\mu}_1^\Omega, \quad \text{where} \quad \tilde{\Omega} = \tilde{\Omega}_0.
\]
This is the image of \( \mu \) under the projection \( \pi^\mathbb{H} \). Once more by group invariance (3.13), we have
\[
\tilde{\mu}_z^\Omega = \delta_g * \tilde{\mu}, \quad \text{when} \quad g \in \text{Aff}(\mathbb{H}, q), \quad g\iota = z \in L_k.
\]
Now we note that the group \( \text{Aff}(\mathbb{H}, q) \) acts simply transitively on \( L\mathbb{H} \). Indeed, \( L\mathbb{H} \) can be identified with \( \text{Aff}(\mathbb{H}, q) \) via the homeomorphic one-to-one correspondence
\[
(4.16) \quad z = x + i q^k \leftrightarrow g = \begin{pmatrix} q^k & x \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad g\iota = z.
\]
Thus, group invariance tells us that we can consider the process \( (Z_{\tau(n)}) \) as the right random walk on \( \text{Aff}(\mathbb{H}, q) \) with law \( \tilde{\mu} \). In other words, the increments \( Z_{\tau(n-1)}^{-1} Z_{\tau(n)} \), \( n \geq 2 \) (resp. \( n \geq 1 \), when \( Z_0 \in \text{HT} \)) are i.i.d. random variables with distribution \( \tilde{\mu} \), when we consider inverses in \( \text{Aff}(\mathbb{H}, q) \) via the identification (4.16).

**Corollary 4.17.** The random walk \( (Z_{\tau(n)}) \) on \( \text{Aff}(\mathbb{H}, q) \) is transient.

**Proof.** The support of the probability measure \( \tilde{\mu} \) on \( \text{Aff}(\mathbb{H}, q) \) generates that group as a semigroup, that is, the random walk is irreducible (every open set is reached with positive probability). We know from (2.12) that the group \( \text{Aff}(\mathbb{H}, q) \) is non-unimodular. Now, any irreducible random walk on a non-unimodular group must be transient, see [27], or, for a shorter proof, [32].

The remainder of this section is dedicated to a study of properties of the probability measures \( \mu \) on \( \partial \tilde{\Omega} = L_0 - \bigcup_{v \in s_0} L_v \subset LT \) and \( \tilde{\mu} \) on \( \partial \tilde{\Omega} = L_{-1} \cup L_1 \subset L\mathbb{H} \), respectively.

An important step is to show that in between two successive times \( \tau(n) \) and \( \tau(n+1) \), the processes \( (Z_t) \) on \( \mathbb{H} \) and thus also \( (X_t) \) on \( \text{HT} \) cannot escape too far "sideways" within the current strip (i.e., the strip to which the process is confined between those two times).

**Proposition 4.18.** Suppose \( (Z_t) \) starts in \( \tilde{\Omega} \). There is \( \rho < 1 \) such that for every \( n \in \mathbb{N} \), we have for the exit time \( \sigma \) from \( \tilde{\Omega} \)
\[
\Pr_0 \left[ \max \{ |Re Z_t - Re z_0| : 0 \leq t \leq \sigma \} \geq n \right] \leq 2 \rho^n \quad \text{for every } \quad z_0 \in \tilde{\Omega}.
\]
(Recall again that \( \sigma = \tau(1) \) when \( z_0 \in L_0 \).)
Proof. By invariance under horizontal translations, we may assume that \( Re z_0 = 0 \).

Consider the vertical segments, resp. open sets,

\[ J_n = \{ n + iy : q^{-1} < y < q \} \subset \bar{\Omega} \quad \text{and} \quad \bar{\Omega}^{(n)} = \{ z \in \bar{\Omega} : Re z < n \}, \quad n \in \mathbb{Z}, \]

so that \( J_n \) is the right-hand side boundary of \( \bar{\Omega}^{(n)} \). For any starting point in \((S_0 \cup S_1)^c\), the exit time of \((Z_t)\) from \( \bar{\Omega} \) is the \( \sigma \) from (4.3), and when \( Z_0 \in L_0 \) then \( \sigma = \tau(1) \). Analogously, we let \( \sigma(n) \) be the exit time of \((Z_t)\) from \( \bar{\Omega}^{(n)} \).

Now our argument will be as follows: if \((Z_t)\) starts at \( z_0 \) and there is some \( t \leq \tau(1) \) such that \( Re Z_t \geq n \) then \((Z_t)_{t < \sigma} \) must pass through each \( J_k \), \( k = 1, \ldots, n \).

\[
\begin{array}{ccccccc}
L_1 & & J_{-3} & J_{-2} & J_{-1} & & J_1 & J_2 & J_3 & S_1 \\
L_0 & & & & & & & & & \\
L_{-1} & & & & & & i & & & S_0
\end{array}
\]

Figure 6. The set \( \bar{\Omega} \subset \mathbb{H} \), subdivided by the bifurcation line \( L_0 \) and the vertical segments \( J_k \).

The function \( z \mapsto \Pr_z[Z_{\sigma(1)} \in L_{-1} \cup L_1] \) is weakly harmonic (harmonic in the sense of distributions) on \( \bar{\Omega}^{(1)} \), whence strongly harmonic by Theorem 5.9 in [4], and thus continuous. We consider this function on \( J_0 \). At the endpoint of that segment, it is \( \rho = 1 \), while inside \( J_0 \) it is \( < 1 \). Thus, there is \( z_0 \in J_0 \) where our function attains its minimum, and

\[
\rho = 1 - \Pr_{z_0}[Z_{\sigma(1)} \in L_{-1} \cup L_1] < 1.
\]

But then

\[
\Pr_z[Z_{\sigma(1)} \in J_1] = 1 - \Pr_z[Z_{\sigma(1)} \in L_{-1} \cup L_1] \leq \rho \quad \text{for every} \; z \in J_0.
\]

By invariance under the group \( \text{Aff}(\mathbb{H}, q) \), and in particular under translations by reals, we also have for all \( k \geq 1 \)

\[
\Pr_z[Z_{\sigma(k)} \in J_k] \leq \rho \quad \text{for every} \; z \in J_{k-1}.
\]

We now use “balayage” in probabilistic terms. Just for the next lines, consider the measure \( \tilde{\nu}(B) = \Pr_{\tilde{i}}[Z_{\sigma(k-1)} \in B] \) for Borel sets \( B \subset J_{k-1} \). If \( Z_0 = i \) and \( Z_{\sigma(k)} \in J_k \) then we must have \( Z_{\sigma(k-1)} \in J_{k-1} \). Therefore (by the strong Markov property),

\[
\Pr_{z_0}[Z_{\sigma(k)} \in J_k] = \Pr_{z_0}[Z_{\sigma(k)} \in J_k, Z_{\sigma(k-1)} \in J_{k-1}] = \int_{J_{k-1}} \Pr_z[Z_{\sigma(k)} \in J_k] d\tilde{\nu}(z) \\
\leq \rho \cdot \tilde{\nu}(J_{k-1}) = \rho \cdot \Pr_{z_0}[Z_{\sigma(k-1)} \in J_{k-1}].
\]
Inductively, 
\[ P_{Z_0}[Z_{\sigma(k)} \in J_k] \leq \rho^k \quad \text{for every } k \geq 1. \]
If \( Z_0 = z_0 \) and \( \text{Re } Z_t \geq n \) for some \( t \leq \sigma \) then a visit to \( J_n \) must have occurred before time \( t \). That is, \( \sigma(n) \leq t \), whence
\[ (4.19) \quad P_{z_0}[\max\{\text{Re } Z_t : 0 \leq t \leq \sigma\} \geq n] \leq \rho^n. \]
Now observe that our process is also invariant under the reflection \( x + iy \mapsto -x + iy \). Therefore
\[ \text{Pr}_t[\min\{\text{Re } Z_t : 0 \leq t \leq \sigma\} \leq -n] \leq \rho^n. \]
The proposed inequality follows. \( \Box \)

Relying again on group invariance (3.8), we deduce the following.

**Corollary 4.20.** The random variables
\[ D_n = \max\{d_{HT}(X_t, X_{\tau(n)}) : \tau(n) \leq t \leq \tau(n + 1)\} \]
\[ = \max\{d_{H}(Z_t, Z_{\tau(n)}) : \tau(n) \leq t \leq \tau(n + 1)\}, \quad n \in \mathbb{N}, \]
are i.i.d. and
\[ \limsup_{n \to \infty} \frac{D_n}{\log \log n} \leq 2 \quad \text{almost surely.} \]
In particular,
\[ \mathbb{E}\left(\exp(\exp(D_n/3))\right) < \infty. \]

**Proof.** It is clear that the \( D_n \) are i.i.d. For the purpose of the proofs of this and the next corollary, set
\[ M_n = \max\{|\text{Re } Z_t - \text{Re } Z_{\tau(n-1)}| : \tau(n-1) \leq t \leq \tau(n)\}. \]
These random variables are also i.i.d. With \( \rho \) as in Proposition 4.18, and for arbitrary \( \varepsilon > 0 \),
\[ \sum_{n=2}^{\infty} \mathbb{P}[M_n \geq \frac{1+\varepsilon}{\log(1/\rho)} \log n] \leq 2 \sum_{n=2}^{\infty} \exp\left(\left(\log \rho\right)\left[\frac{1+\varepsilon}{\log(1/\rho)} \log n\right]\right) \leq 2 \rho \sum_{n=2}^{\infty} \frac{1}{n^{1+\varepsilon}}, \]
which is finite. By the Borel–Cantelli lemma,
\[ \limsup_{n \to \infty} \frac{M_n}{\log \log n} \leq \frac{1}{\log(1/\rho)} \quad \text{almost surely.} \]
We also see that
\[ (4.21) \quad \mathbb{E}(e^{\lambda_1 M_1}) < \infty \quad \text{for } 0 < \lambda_1 < \log(1/\rho). \]
By simple computations with the hyperbolic metric, for any \( \zeta = (z, w) \in \partial \Omega \), and thus \( z \in \partial \Omega = L_1 \cup L_{-1} \), one has
\[ (4.22) \quad \log(1 + |\text{Re } z|^2) - \log q \leq d_{HT}(\zeta, \sigma) = d_{H}(z, i) \leq \log q + 2 \log(1 + |\text{Re } z|). \]
Therefore $D_n \leq \log q + 2 \log(1 + M_n)$, whence as above,

$$\sum_{n=3}^{\infty} \Pr[D_n \geq (2 + \varepsilon) \log \log n] < \infty$$

for every $\varepsilon > 0$. We get $\limsup D_n / \log \log n \leq 2$ a.s. Also, for some $c > 0$, $e^{D_{1/3}} \leq q^{1/3}(1 + M_1)^{2/3} \leq c + \lambda_1 M_1$. Now (4.21) yields the doubly exponential moment condition for $D_1$.

From the last corollary and (4.22), we also get the following.

**Corollary 4.23.** With $\lambda_1 > 0$ as in (4.21),

$$\int_{\partial \tilde{\Omega}} \exp(\lambda_1 \Re z) d\tilde{\mu}(z) < \infty.$$

In particular, $\mu$ and $\tilde{\mu}$ satisfy doubly exponential moment conditions,

$$\int_{\partial \tilde{\Omega}} \exp(\exp(d_{\tilde{H}}(i, z)/3)) d\tilde{\mu}(z) = \int_{\partial \Omega} \exp(\exp(d_{HT}(i, z)/3)) d\mu(z) < \infty.$$

Finally, we anticipate a result from [5] which appears very natural, but whose proof is quite subtle.

**Proposition 4.24.** Let $\Omega = \Omega_v$ or $\Omega = S^0_\circ \subset HT$ ($v \in V(\mathbb{T})$). Then for any starting point $z \in \Omega$, the exit measure $\mu^0_\Omega$ has a continuous, strictly positive density with respect to Lebesgue measure on the finitely many bifurcation lines that make up $\partial \Omega$.

The analogous statement holds on “sliced” hyperbolic plane.

5. Rate of escape and convergence to the boundary at infinity

**Theorem 5.1.** In the natural metric of $HT$, the Brownian motion $(X_t)$ on $HT$ generated by $\Delta_{\alpha, \beta}$ has the following rate of escape.

$$\lim_{t \to \infty} \frac{1}{t} d_{HT}(X_t, X_0) = |\ell(\alpha, \beta)|$$

almost surely, where $\ell(\alpha, \beta) = \frac{\log q}{E(\tau)} \frac{a-1}{a+1}$.

with $a$ and $E(\tau)$ given by Proposition 4.6(d) and (4.7), respectively.

The proof of this theorem will go hand in hand with the one of Theorem 5.5 below, conceming convergence of $(X_t)$ to the boundary.

The tree $\mathbb{T}$ has its natural geometric compactification $\hat{\mathbb{T}}$ with boundary at infinity $\partial \mathbb{T} = \partial^* \mathbb{T} \cup \{ \infty \}$, see Figure 2. Analogously, the hyperbolic plane $\mathbb{H}$ has its standard hyperbolic compactification $\hat{\mathbb{H}}$ with boundary $\partial \mathbb{H} = \partial^* \mathbb{H} \cup \{ \infty \}$, where $\partial^* \mathbb{H} = \mathbb{R}$, see Figure 3. Since $HT$ is a topological subspace of $\mathbb{H} \times \mathbb{T}$, we can compactify it as follows.
Definition 5.2. The geometric compactification $\overline{HT}$ of HT is the closure of HT in the compact space $\hat{H} \times \hat{T}$. The geometric boundary at infinity of HT is

$$\partial HT = \overline{HT} \setminus HT.$$ 

The boundary consists of the following five pieces:

$$(5.3) \quad \partial HT = \left(\{\infty\} \times \partial^* T\right) \cup \left(\partial^* \hat{H} \times \{\infty\}\right) \cup \left((\infty) \times \hat{T}\right) \cup (\hat{H} \times \{\infty\}) \cup \left((\infty, \infty)\right).$$

For a better understanding (and future use), we describe convergence to the boundary.

5.4. Consider a sequence $z_n = (z_n, w_n)$ in HT, with $z_n = x_n + iy_n$.

(a) $z_n \to (\infty, \xi) \in \{\infty\} \times \partial^* T$ if $w_n \to \xi$ in $\hat{T}$, in which case necessarily $z_n \to \infty$.

(b) $z_n \to (\zeta, \omega) \in \partial^* \hat{H} \times \{\infty\}$ if $z_n \to \zeta$ in $\hat{H}$, that is, $x_n \to \zeta$ and $y_n \to 0$ as sequences in $\mathbb{R}$. In this case necessarily $w_n \to \omega$.

(c) $z_n \to (\infty, w) \in \{\infty\} \times T$ if $w_n \to w$ in $T$ and $z_n \to \infty$ in $\hat{H}$, that is, $|x_n| \to +\infty$ and $y_n \to q^{w}$ as sequences in $\mathbb{R}$.

(d) $z_n \to (z, \omega) \in \mathbb{H} \times \{\infty\}$ if $z_n \to z$ in $\mathbb{H}$ and $w_n \to \omega$ in $\hat{T}$, that is, $d(o, w_n, o) \to +\infty$ and $h(w_n) \to \log q(\text{Im} z)$.

(e) $z_n \to (\infty, \omega)$ if $z_n \to \infty$ and $w_n \to \omega$. In this case, up to passing to a sub-sequence, we may assume in addition that there is $\tau \in [-\infty, +\infty]$ such that $h(w_n) \to \tau$ and $y_n \to q^{\tau} \in [0, +\infty]$. (Each value $\tau$ can be attained in the limit by some sequence $z_n$.)

Theorem 5.5. In the topology of $\overline{HT}$, the Brownian motion $X_t = (Z_t, W_t)$ on HT generated by $\Delta_{\alpha, \beta}$ converges almost surely to a boundary-valued limit random variable $X_\infty = (Z_\infty, W_\infty)$. Writing $\nu_3$ for its distribution when $X_0 = 3$, we have the following.

(i) If $\ell(\alpha, \beta) > 0$, then $X_\infty \in \{\infty\} \times \partial^* T$, and all of the latter set is charged by $\nu_3$.

(ii) If $\ell(\alpha, \beta) < 0$, then $X_\infty \in \partial^* \mathbb{H} \times \{\infty\}$, and all of the latter set is charged by $\nu_3$.

(iii) If $\ell(\alpha, \beta) = 0$, then $X_\infty = (\infty, \omega)$, a deterministic limit.

The most useful tool for proving the last two theorems is the notion of regular sequences of Kaimanovich [28], which we formulate here just for hyperbolic plane and tree.

Definition 5.6. Let $X = \mathbb{H}$ or $X = T$. A sequence $(z_n)$ in $X$ is called regular with rate $r \geq 0$ if there is a geodesic ray $(\pi_t)_{t \geq 0}$ in $X$ (that is, $d_X(\pi_t, \pi_s) = |t - s|$ for all $s, t \geq 0$) such that

$$d_X(x_n, \pi_{rn})/n \to 0 \quad \text{as} \quad n \to \infty.$$
The following was shown in [28].

**Lemma 5.7.** A sequence \((z_n)\) in \(\mathbb{H}\) is regular if and only if there is \(b \in \mathbb{R}\) such that
\[
\log \text{Im}(z_n)/n \to b \quad \text{and} \quad d_B(z_{n+1}, z_n)/n \to 0.
\]
In this case, \(r = |b|\) and \(d_B(z_n, z_0)/n \to r\).

Furthermore, if \(b > 0\) then \(z_n \to \infty\) in the topology of \(\hat{\mathbb{H}}\), while if \(b < 0\) then there is some \(\zeta \in \partial^* \mathbb{H}\) such that \(z_n \to \zeta\) in the topology of \(\hat{\mathbb{H}}\). (There is no general statement of this form when \(b = 0\).)

The analogue for trees was proved in [15].

**Lemma 5.8.** A sequence \((w_n)\) in \(T\) is regular if and only if there is \(b \in \mathbb{R}\) such that
\[
b(w_n)/n \to b \quad \text{and} \quad d_T(w_{n+1}, w_n)/n \to 0.
\]
In this case, \(r = |b|\) and \(d_T(w_n, w_0)/n \to r\).

Furthermore, if \(b > 0\) then \(w_n \to \infty\) in the topology of \(\hat{T}\), while if \(b < 0\) then there is some \(\xi \in \partial^* T\) such that \(w_n \to \xi\) in the topology of \(\hat{T}\). (Again, there is no general statement of this form when \(b = 0\).)

Before embarking on the proofs of the above two theorems, we also need the following.

**Lemma 5.9.** \(\lim_{t \to \infty} Y_t/t = \ell(\alpha, \beta)/\log q\) almost surely, where (recall) \(Y_t = \pi^B(X_t)\).

**Proof.** Corollary 4.9 and the law of large numbers imply that \(\frac{1}{n} Y_{\tau(n)} \sim \frac{n-1}{n} + 1\) almost surely. Again by the law of large numbers, Proposition 4.6 tells us that \(\tau(n)/n \to E(\tau)\) almost surely. Combining these two facts, we get that \(\frac{Y_{\tau(n)}}{\tau(n)} \to \frac{2}{\log q} / E(\tau)\) almost surely. Given \(t > 0\), let the random \(n_t \in \mathbb{N}\) be as in (4.2). Then \(n_t \to \infty\) and \(\tau(n_t)/t \to 1\) almost surely, as \(t \to \infty\). By construction, \(Y_t\) lies between \(Y_{\tau(n_t)}\) and \(Y_{\tau(n_t+1)}\), which differ by 1. Therefore the almost sure limit
\[
\lim_{t \to \infty} \frac{Y_t}{t} = \lim_{t \to \infty} \frac{Y_{\tau(n_t)}}{t} = \lim_{t \to \infty} \frac{Y_{\tau(n_t)}}{\tau(n_t)} \frac{\tau(n_t)}{t}
\]
exists and has the proposed value. \(\square\)

Let us now consider the process \((W_t)\) on \(T\).

**Proposition 5.10.** Let \(a\) be as in Proposition 4.6 (d) and \(\ell(\alpha, \beta)\) as in Theorem 5.1. Then
\[
\lim_{t \to \infty} \frac{1}{t} d_T(W_t, W_0) = \frac{1}{\log q} |\ell(\alpha, \beta)| \quad \text{almost surely.}
\]

If \(\ell(\alpha, \beta) \leq 0 \iff a \leq 1\) then for any starting point \(w \in T\),
\[
\lim_{t \to \infty} W_t = \infty \quad \text{almost surely in the topology of } \hat{T}.
\]
If $\ell(\alpha, \beta) > 0 \iff \alpha > 1$ then there is a $\partial^* T$-valued random variable $W_\infty$ such that for any starting point $w \in T$, we have almost surely that

$$\lim_{t \to \infty} W_t = W_\infty \quad \text{in the topology of } \hat{T}.$$ 

In this case, let $\nu_w^{\partial T}$ be the distribution of $W_\infty$, given that $W_0 = w$. This is a probability measure that has a strictly positive, continuous, bounded density with respect to the “Lebesgue” measure $\lambda$ on $\partial^* T$ explained in Remark 2.23.

Proof. Consider first the random walk $(W_{\tau(n)})$ on $V(T)$. As $d_\tau(W_{\tau(n+1)}, W_{\tau(n)}) = 1$, lemmas 5.9 and 5.8 yield that the sequence $(W_{\tau(n)})$ is almost surely regular. We obtain that first of all,

$$\frac{1}{n} d_\tau(W_{\tau(n)}, W_0) \to \left| \frac{a - 1}{a + 1} \right| \quad \text{almost surely.}$$

The proof now proceeds as the one of Lemma 5.9: with $n_t$ as in (4.2), we have that $W_t$ lies on the edge between $W_{\tau(n_t)}$ and $W_{\tau(n_t+1)}$, whence $d_\tau(W_t, W_{\tau(n_t)}) \leq 1$. Therefore

$$\lim_{t \to \infty} \frac{d_\tau(W_t, W_0)}{t} = \lim_{t \to \infty} \frac{d_\tau(W_{\tau(n_t)}, W_0)}{n_t} \frac{n_t}{\tau(n_t)} \frac{\tau(n_t)}{t} = \left| \frac{a - 1}{a + 1} \right| \frac{1}{E(\tau)},$$

as proposed.

Second, again by Lemma 5.8, $(W_{\tau(n)})$ converges a.s. to $\omega$, when $a < 1$.

When $a > 1$, it converges a.s. to a $\partial^* T$-valued random variable $W_\infty$. Using the formulas that are displayed in Proposition 9.23 of [34], one can compute the limit distribution $\nu_v^{\partial T}$ of that random walk, when $W_0 = v \in V(T)$. Explicit computations can be found in [33]. One sees that $\nu_v^{\partial T}$ has the stated properties. In particular, it carries no point mass and is supported by the whole of $\partial^* T$ (or equivalently, $\partial T$).

If we replace the starting point $v \in V(T)$ by a starting point $w$ that lies in the interior of some edge $[v, v']$ then the process starting at $w$ also must converge to $\partial^* T$, and we have

$$\nu_w^{\partial T} = \Pr[W_{\tau(1)} = v | W_0 = w] \nu_v^{\partial T} + \Pr[W_{\tau(1)} = v^- | W_0 = w] \nu_v^{\partial T}.$$ 

We still have to show that $W_t \to \omega$ almost surely, when $a = 1$. This is obtained by the following simple argument. Being a transient nearest neighbour random walk, $(W_{\tau(n)})$ must converge almost surely to some random end of $T$, see Theorem 9.18 in [34]. But the projection $h(W_{\tau(n)}) = Y_{\tau(n)}$ is a recurrent random walk on $Z$, when $a = 1$. Thus, there is a random subsequence $(n')$ along which $h(W_{\tau(n')}) = 0$. This subsequence must converge to $\omega$, whence $\omega$ is the limit of the entire sequence. $\square$

In fact, the last proposition provides the simplest class of cases to which the results of [15] apply (but explaining how to apply those general results would consume more space and energy than the above direct arguments.) We next want to present the analogous proposition concerning the process $(Z_t)$ on $H$. Recall that
we can interpret the random walk \((Z_{\tau(n)})\) on \(LH\) as a right random walk on the group \(\text{Aff}(H, q)\) which is identified with \(LH\) via (4.16). With this identification, the law of that random walk is the probability \(\tilde{\mu}\) of (4.15). We know that in the notation of the group operation, the increments \(Z_{\tau(n)}^{-1}Z_{\tau(n+1)}\), \(n \geq 2\), are i.i.d. \(\text{Aff}(H, q)\)-valued random variables with common distribution \(\tilde{\mu}\), so that they can be written as random affine transformations \((A_n B_n)\), where \(A_n = q^{Y_{\tau(n)} - Y_{\tau(n-1)}}\), the associated transformation of \(H\) is \(z \mapsto A_n z + B_n\). While \(A_n\) only takes the two values \(q\) and \(1/q\), the common distribution of the real random variables \(B_n\) has a continuous density with respect to Lebesgue measure by Proposition 4.24.

\[ \text{Proposition 5.11.} \quad \lim_{t \to \infty} \frac{1}{t} d_H(Z_t, Z_0) = |\ell(\alpha, \beta)| \quad \text{almost surely.} \]

If \(\ell(\alpha, \beta) \geq 0 (\iff a \geq 1)\) then for any starting point \(z \in H\), we have almost surely that
\[ \lim_{t \to \infty} Z_t = \infty \quad \text{almost surely in the topology of } \hat{H}. \]

If \(\ell(\alpha, \beta) < 0 (\iff a < 1)\) then there is a random variable \(Z_{\infty}\) taking values in \(\partial^*H = \mathbb{R}\) such that for any starting point \(z \in H\), we have almost surely that
\[ \lim_{t \to \infty} Z_t = Z_{\infty} \quad \text{in the topology of } \hat{H}. \]

In this case, let \(\nu^H_{\partial^*H}\) be the distribution of \(Z_{\infty}\), given that \(Z_0 = z\). This is a probability measure on \(\partial^*H \equiv \mathbb{R}\) that has a continuous, strictly positive density with respect to Lebesgue measure.

\[ \text{Proof.} \quad \text{By Corollary 4.20,} \]
\[ \frac{1}{n} d_H(Z_{\tau(n+1)}, Z_{\tau(n)})/n \to 0 \quad \text{almost surely.} \]

By Lemma 5.9,
\[ (5.12) \quad \frac{1}{n} \log \text{Im}(Z_{\tau(n)}) = \frac{\log q}{n} Y_{\tau(n)} \to \log q \frac{a-1}{a+1} \quad \text{almost surely.} \]

Thus, by Lemma 5.7, the sequence \((Z_{\tau(n)})\) is almost surely regular in \(H\), with rate \(\log q \left| \frac{a-1}{a+1} \right|\). When \(a > 1\) it converges to \(\infty\) in the topology of \(\hat{H}\), while when \(a < 1\), it converges in that topology to a random element of \(\partial^*H\).

The more difficult situation is the one where the rate of the sequence is 0. In that case, it was proved by Brofferio [10] that \(Z_{\tau(n)} \to \infty\) almost surely in the topology of \(\hat{H}\). This is not yet enough to guarantee that also \(Z_{t} \to \infty\) almost surely. We take inspiration from [10]. For any \(g \in \text{Aff}(H, q)\), a neighbourhood base of \(\infty\) in \(\hat{H}\) is given by the collection of all sets \(\hat{H} \setminus g^{-1}V_r\), where
\[ V_r = \{z = x + iy : |x| \leq r \text{ and } 0 \leq y \leq q^r\}, \quad r \in \mathbb{N}. \]
Our argument will not depend on the starting point, but only on what happens from time \( \tau(1) \) onwards. Thus, we may assume that \( Z_0 \in LH \), which can be identified with Aff(\( \mathbb{H} \), \( q \)). We know from [10] that for any \( r \) and for any starting point in LH, we have almost surely that \( Z_{\tau(n)} \in \mathbb{H} \setminus V_r \) for all but finitely many \( n \). Equivalently, for starting point \( 1 \) and for some \( g \in \text{Aff}(\mathbb{H}, q) \), for any \( r \) we have \( Z_{\tau(n)} \in \mathbb{H} \setminus g^{-1}V_r \) for all but finitely many \( n \).

Thus, we need an element \( g \in \text{Aff}(\mathbb{H}, q) \) such that with probability 1, in between the times \( \tau(n) \) and \( \tau(n+1) \), the process \((X_t)\) does not enter into \( g^{-1}V_r \), if \( n \) is sufficiently large. This will follow from the Borel–Cantelli lemma after showing that

\[
(5.13) \sum_{n=1}^{\infty} \Pr \left[ Z_{\tau(n)} \in \mathbb{H} \setminus g^{-1}V_r, \ Z_t \in g^{-1}V_r \text{ for some } t \text{ with } \tau(n) < t < \tau(n+1) \right] < \infty.
\]

Again, we use the identification (4.16) of LH with Aff(\( \mathbb{H} \), \( q \)) and consider the potential measure \( \mathcal{U} = \sum_{n=0}^{\infty} \tilde{\mu}^{(n)} \), where \( \tilde{\mu}^{(n)} \) is the nth convolution power of the measure \( \tilde{\mu} \) on Aff(\( \mathbb{H} \), \( q \)). By transience of \((Z_{\tau(n)})\), this \( \mathcal{U} \) is a Radon measure on Aff(\( H \), \( q \)) \( \equiv \) LH. For \( z \in \mathbb{H} \), let

\[
f_r(z) = 1_{\mathbb{H} \setminus V_r}(z) \Pr_z[Z_t \in V_r \text{ for some } t \text{ with } 0 < t < \tau(1)].
\]

Then for any \( g \in \text{Aff}(H, q) \),

\[
\sum_{n=1}^{\infty} \Pr \left[ Z_{\tau(n)} \in H \setminus g^{-1}V_r, \ Z_t \in g^{-1}V_r \text{ for some } t \text{ with } \tau(n) < t < \tau(n+1) \right] = \int_{\mathbb{H}} f_r(gz) \, d \mathcal{U}(z).
\]

Let \( z = b + i q^m \in \mathbb{H} \setminus V_r \), with \( m \in \mathbb{Z} \) and \( b \in \mathbb{R} \). Write \( z = g_z i \), where \( g_z = \left( \begin{array}{cc} p^m & b \\ 0 & 1 \end{array} \right) \in \text{Aff}(\mathbb{H}, q) \). Then

\[
f_r(z) = 1_{\mathbb{H} \setminus g_z^{-1}V_r}(i) \Pr_z[Z_t \in g_z^{-1}V_r \text{ for some } t \text{ with } 0 < t < \tau(1)].
\]

We have

\[
g_z^{-1}V_r = \{ x + iy : |x + q^{-m}b| \leq q^{-m}r \text{ and } 0 \leq y \leq q^{r-m} \}.
\]

We must have \( i \in \mathbb{H} \setminus g_z^{-1}V_r \). Starting at \( i \), the process \((Z_t)\) does not leave \( S_0 \cup S_1 \) before time \( \tau(1) \). Thus, in order to be able to enter into \( g_z^{-1}V_r \) before that time, we must have \( r - m \geq 0 \); otherwise \( f_r(z) = 0 \).

Suppose that we do have \( r - m \geq 0 \), and that \( i \) stays to the left of \( g_z^{-1}V_r \), so that \( -r - b > 0 \). Then in order to enter into \( g_z^{-1}V_r \) before \( \tau(1) \), the process must cross the vertical line where \( x = -q^{-m}(r + b) \). Setting \( k = [-q^{-m}(r + b)] \) (next lower integer), this means that \( Z_t \) must pass through the segment \( J_k \) of Figure 6. By Proposition 4.18, resp. (4.19) in its proof, \( f_r(z) \leq \rho^k \). Analogously, if \( i \) stays to the right of \( g_z^{-1}V_r \), which means that \( r - b < 0 \), then \( f_r(z) \leq \rho^k \), where \( k = [q^{-m}(b - r)] \). Setting \( \lambda = -\log \rho \), we find that

\[
f_r(b + i q^m) = \begin{cases} 0, & \text{if } m > r \text{ or } |b| < r \\ \leq \exp(-\lambda(q^{-m}(|b| - r) + 1)), & \text{if } m \leq r \text{ and } |b| \geq r. \end{cases}
\]
The right Haar measure on $\text{Aff}(\mathbb{H}, q) \equiv L^1 \mathbb{H}$ is one-dimensional Lebesgue measure on each of the lines $L_k$, compare with (2.12). Thus, the integral of $f_r$ with respect to right Haar measure is

$$\sum_{m \leq r} \int_{|b| \geq r} f_r(b + i q^m) \, db < \infty.$$ 

Lemma 1 in [10] yields that in this case, $\int_{\mathbb{H}} f_r(gz) \, dU(z) < \infty$ for $dg$-almost all $g \in \text{Aff}(\mathbb{H}, q)$. This is true for all $r \in \mathbb{N}$. Thus, there is some fixed $g \in \text{Aff}(\mathbb{H}, q)$ such that the last integral is finite for every $r \in \mathbb{N}$. For this $g$, (5.13) holds for every $r \in \mathbb{N}$, so that $Z_t \to \infty$ almost surely.

We finally have to explain that in the case $a < 1$, the limit random variable on $\partial^* \mathbb{H}$ has a distribution with continuous, positive density with respect to Lebesgue measure. Let us write $Z_{\tau(1)} = \left( \begin{array}{cc} A_1 & B_1 \\ 0 & 1 \end{array} \right)$, which is independent of the other $\left( \begin{array}{cc} A_2 & B_2 \\ 0 & 1 \end{array} \right)$ but does in general not have the same distribution. We know from Proposition 4.24 that for arbitrary starting point $z \in \mathbb{H}$, the distribution of $B_1$ has a continuous density with respect to Lebesgue measure. We then have

$$Z_{\tau(n)} = \left( \begin{array}{cc} A_1 \cdots A_n & \sum_{k=1}^n A_1 \cdots A_{k-1} B_k \\ 0 & 1 \end{array} \right).$$

When $a < 1$, it is very well known and quite easy to verify that in $\mathbb{R}$, the upper right matrix element of $Z_{\tau(n)}$ converges almost surely to

$$Z_{\infty} = \sum_{k=1}^{\infty} A_1 \cdots A_{k-1} B_k,$$

as $n \to \infty$. Recalling the identification (4.16), we see that this is the limit of $Z_{\tau(n)}$ in $\tilde{\mathbb{H}}$, since $A_1 \cdots A_n \to 0$ almost surely. It is now easy to verify that along with all the $B_n$ (including $B_1$), for arbitrary starting point also the distribution of $Z_{\infty}$ has a continuous density with respect to Lebesgue measure on $\mathbb{R}$.

**Remark 5.14.** Our result on almost sure convergence of $(Z_t)$ to $\infty$ in the critical case $a = 1$ also applies to Brownian motion with vertical drift on $\mathbb{H}$ without any bifurcation lines. Indeed, this corresponds just to the case when $\beta p = 1$. This closes a small gap left open in the proof of Proposition 4.2(iii) in [11], concerning the passage from discrete to continuous time.

**Proof of Theorems 5.1 and 5.5.** Theorem 5.1 regarding the rate of escape of $(X_t)$ now follows by combining the inequalities of Proposition 2.8 with the rates of escape of $(Y_t)$, $(W_t)$ and $(Z_t)$, as provided by Lemma 5.9 and Propositions 5.10 and 5.11, respectively.

Theorem 5.5 follows by combining those two propositions with the description (5.4) of convergence to the boundary in treebolic space. □
Theorem 5.5 provides the following, which (as mentioned) was only indicated in [4].

Corollary 5.15. The processes \((X_t)\) on \(HT\), \((Z_t)\) on \(\mathbb{H}\) and \((W_t)\) on \(\mathbb{T}_p\) \((p \geq 2)\), as defined in Proposition 3.11, are transient.

6. Central limit theorem

The proof of a CLT for \(d(X_t, X_0) \to \infty\) depends significantly on the sign of the drift \(\ell(\alpha, \beta)\). It will follow from the CLT for the random walk \((X_{\tau(n)})\). Here we shall work with \(d(X_t, 0)\) instead of \(d(X_t, X_0)\), which makes no difference, as we divide by \(\sqrt{t}\). In any case, before that we need the CLT for the vertical component \(Y_t\) of \(X_t\).

Lemma 6.1. With \(\text{Var}(Y)\) and \(\text{Var}(\tau)\) as in Proposition 4.6, set

\[
\sigma^2 = \sigma^2(\alpha, \beta) = \frac{1}{E(\tau)} \text{Var}(Y) + \frac{\ell(\alpha, \beta)^2}{E(\tau) \log^2 q} \text{Var}(\tau).
\]

Then

\[
\frac{1}{\sqrt{t}} \left( Y_t - t \frac{\ell(\alpha, \beta)}{\log q} \right) \to N(0, \sigma^2) \text{ in law, as } t \to \infty.
\]

Proof. The \(\mathbb{R}^2\)-valued random variables \((Y_{\tau(n)} - Y_{\tau(n-1)}, \tau(n) - \tau(n-1))_{n \geq 2}\) are i.i.d., see Proposition 4.6. By the two-dimensional CLT,

\[
(6.2) \quad \frac{1}{\sqrt{n}} \left( Y_{\tau(n)} - n \frac{a - 1}{a + 1}, \tau(n) - nE(\tau) \right) \to N(0, \Sigma^2) \text{ in law,}
\]

where \(E(\tau)\) is as in Proposition 4.6(e) and \(N(0, \Sigma^2)\) is the two-dimensional normal distribution with mean vector 0 and \(\Sigma^2\) is the covariance matrix of \((Y_{\tau(2)} - Y_{\tau(1)}, \tau(2) - \tau(1))\), which is just the diagonal matrix with diagonal entries \(\text{Var}(Y)\) and \(\text{Var}(\tau)\).

As in the proof of Lemma 5.9, with the \(n_t\) of (4.2), we know that

\[
(6.3) \quad \frac{n_t}{t} = \frac{n_t}{\tau(n_t) - t} \to \frac{1}{E(\tau)} \text{ almost surely, as } t \to \infty,
\]

and that \(|Y_t - Y_{\tau(n_t)}| < 1\). Now we decompose

\[
\frac{Y_t - t \frac{\ell(\alpha, \beta)}{\log q}}{\sqrt{t}} = \frac{Y_t - Y_{\tau(n_t)}}{\sqrt{t}} + \sqrt{\frac{n_t}{t}} \frac{Y_{\tau(n_t)} - n_t \frac{2}{\log q}}{\sqrt{n_t}} - \frac{\ell(\alpha, \beta)}{\log q} t - \tau(n_t) \sqrt{\frac{n_t}{t}} \frac{\ell(\alpha, \beta)}{\log q} \frac{\tau(n_t) - n_t E(\tau)}{\sqrt{n_t}}.
\]
The first term of the sum on the right-hand side tends to 0 because \(0 \leq Y_t - Y_{\tau(n_t)} < 1\) almost surely. The third term tends to 0 almost surely, because

\[
\frac{t - \tau(n_t)}{\sqrt{t}} \rightarrow \frac{\tau(n_t + 1) - \tau(n_t)}{\sqrt{n_t}} \quad \text{almost surely},
\]

and \((\tau(n + 1) - \tau(n))/\sqrt{n} \rightarrow 0\) by Proposition 4.6(d). Also, we know that \(n_t/t \rightarrow 1/E(\tau)\) almost surely. Hence,

\[
\frac{Y_t - t \ell(\alpha, \beta)/\log q}{\sqrt{t}} \xrightarrow{\text{in law}} \frac{1}{\sqrt{E(\tau)}} \frac{Y_{\tau(n_t)} - n_t \frac{z - 1}{z + 1}}{\sqrt{n_t}} \frac{\ell(\alpha, \beta)}{\sqrt{E(\tau) \log q}} \frac{\tau(n_t) - n_t E(\tau)}{\sqrt{n_t}}
\]

as \(t \rightarrow \infty\). It follows from (6.2) that this converges in law to the centred normal distribution with variance \(\sigma^2(\alpha, \beta)\), as proposed. \(\square\)

**Lemma 6.4.** (a) If \(\ell(\alpha, \beta) > 0\), then

\[
\limsup_{t \to \infty} \left( d_{HT}(X_t, o) - d_{Z,t}(Z_t, i) \right) < \infty \quad \text{almost surely.}
\]

(b) If \(\ell(\alpha, \beta) < 0\), then

\[
\limsup_{t \to \infty} \left( d_{HT}(X_t, o) - (\log q) d_{T}(W_t, o) \right) < \infty \quad \text{almost surely.}
\]

(The two appearing differences are always non-negative.)

**Proof.** (a) By Proposition 5.10, \(W_t \to W_\infty \in \partial^* \mathcal{T}\) almost surely. Therefore \(h(o \wedge W_t) \to h(o \wedge W_\infty)\) a.s., that is, the two (finite!) random variables coincide from some random \(t_0\) onwards, and in particular \(h(W_t) = Y_t \geq 0\) for \(t \geq t_0\). By (2.2),

\[
d_{T}(W_t, o) = h(W_t) - 2 h(o \wedge W_t) = Y_t - 2 h(o \wedge W_\infty)
\]

for all \(t \geq t_0\),

and for those \(t\), the first inequality of Proposition 2.8 yields

\[
d_{HT}(X_t, o) \leq d_{Z,t}(Z_t, i) - 2 (\log q) h(o \wedge W_\infty).
\]

(We note here that \(h(o \wedge W_\infty) \leq 0\).) This yields (a).

(b) This time, we use Proposition 5.11 and get that \(i \wedge Z_t \to i \wedge Z_\infty \in \mathbb{H}\) (a.s. convergence in \(\mathbb{H}\)). Therefore, by (2.3),

\[
\limsup_{t \to \infty} \left| d_{Z,t}(Z_t, i) - (2 \log \Im(i \wedge Z_\infty)) - \log \Im Z_t \right| < \infty \quad \text{almost surely.}
\]

Note that \(\log \Im Z_t < 0\) if \(t\) is sufficiently large. Thus, in the same way as in (a), Proposition 2.8 yields statement (b). \(\square\)

We now consider \((X_{\tau(n_t)})\). The group \(\mathcal{A}\) acts transitively on the set \(\mathcal{L}T\) defined in (3.2) of all bifurcation lines in \(\mathcal{H}T\). In part (2) of the proof of Theorem 2.15, we have introduced the coordinates \([b, \gamma]\) for the elements of \(\mathcal{A}\). In the same way,
Theorem 6.6. If the Haar measure (2.17) on \( \mathbb{R} \) is analogous to the notation used in the statement of Theorem 2.15, while in the statement of Theorem 2.15, the transition probabilities of \( (g, \gamma) \) are given by the probability measure \( \pi \). From Lemma 3.1 in [31] (see also [27], Remarque 6, p. 5). Statements (ii)–(iv) are immediate consequences.

The product is of course taken in the group \( \mathcal{A} \). The (simple) proof of part (i) of the following lemma is omitted; it follows from Lemma 3.1 in [31] (see also [27], Remarque 6, p. 5). Statements (ii)–(iv) are immediate consequences.

**Lemma 6.5.** (i) For any \( g \in \mathcal{A} \), the sequence \( (gR_n, o) \) is a realisation of the induced random walk \( (X^g_{\tau(n)})_{n \geq 0} \) on \( \mathcal{A} \) starting at \( g o \). That is, it is an \( \mathcal{A} \)-valued Markov chain with transition probabilities (4.11).

(ii) Via the identification (4.16) of \( \mathbb{LH} \) with \( \text{Aff}(\mathbb{H}, q) \), the random walk \( \pi^\mathbb{H}(R_n) \) is a realisation of the process \( (Z^g_{\tau(n)}) \) on \( \mathbb{LH} \) starting at \( i \).

(iii) \( R_n = \pi^\mathbb{H}(R_n) \) is a right random walk on the group \( \text{Aff}(\mathbb{T}) \), and the process \( (R_n, o)_{n \geq 0} \) is a realisation of the random walk \( (W^g_{\tau(n)})_{n \geq 0} \) on the vertex set of \( \mathbb{T} \) as described in Corollary 4.9 (b), with \( R_0 o = o \).

(iv) With \( \Phi \) as in (2.14), the sequence \( \left( \Phi(R_n) \right)_{n \geq 0} \) is a realisation of the random walk \( (Y^g_{\tau(n)})_{n \geq 0} \) on \( \mathbb{Z} \) as described in Corollary 4.9 (a), with starting point 0.

**Theorem 6.6.** If \( \ell(\alpha, \beta) \neq 0 \) and \( \sigma^2 \) is as in Lemma 6.1,

\[
\frac{1}{\sqrt{t}} \left( d_{\mathbb{HT}}(X_t, o) - t |\ell(\alpha, \beta)| \right) \rightarrow N(0, \sigma^2) \quad \text{in law, as } t \rightarrow \infty.
\]
Proof. Case 1. \( \ell(\alpha, \beta) > 0 \).

Lemma 6.4(a) tells us that we just have to consider \( d_H(Z_t, i) \). Using the same notation as before Proposition 5.11, we write \( Z_{\tau(n)}^{-1} \) as independent \( \tilde{\mu} \)-distributed group elements of \( \text{Aff}(\mathbb{H}, q) \) for \( n \geq 2 \), as well as \( Z_{\tau(1)} = (A_0, B_1) \), which is independent of the other ones (but may have a different distribution, according to the starting point). The group inverses are \( (A_n, B_n)^{-1} = (1/A_n, -B_n/A_n) \). Since \( A_n \) only takes values \( q \) and \( 1/q \), also \( -B_n/A_n \) has exponential moments. Taking products in that group, \( Z_{\tau(n)} = (A_0, B_1) \cdots (A_n, B_2) \), so that

\[
Z_{\tau(n)}^{-1} = \left( \begin{array}{cc}
A_0 & B_0 \\
0 & 1
\end{array} \right)^{-1} \cdots \left( \begin{array}{cc}
A_2 & B_2 \\
0 & 1
\end{array} \right)^{-1} \left( \begin{array}{cc}
A_1 & B_1 \\
0 & 1
\end{array} \right)^{-1} \equiv \left( \begin{array}{cc}
A_2 & B_2 \\
0 & 1
\end{array} \right)^{-1} \cdots \left( \begin{array}{cc}
A_n & B_n \\
0 & 1
\end{array} \right)^{-1} \left( \begin{array}{cc}
A_1 & B_1 \\
0 & 1
\end{array} \right)^{-1}
\]

Note that in \( \mathbb{R} \), we have \( A_1 \cdots A_n = q^{Y_{\tau(n)}} \). Now \( (Z_{\tau(n)}^{*}) \) is again a right random walk on \( \text{Aff}(\mathbb{H}, q) \), and returning to the identification with \( LH \), we have that

\[
d_H(Z_{\tau(n)}^{*}, Z_{\tau(n-1)}^{*})/n \to 0 \quad \text{and} \quad \frac{1}{n} \log \text{Im} Z_{\tau(n)}^{*} = - \frac{(\log q)}{n} (Y_{\tau(n)} - Y_{\tau(1)}) \to - \log q \frac{a - 1}{a + 1} \quad \text{almost surely.}
\]

Since the last limit is \( < 0 \), by Lemma 5.7 our sequence is a.s. regular and converges to a random variable \( Z_{\tau(n)}^{*} \in \partial^* \mathbb{H} = \mathbb{R} \) almost surely in the topology of \( \mathbb{H} \). But then, using (2.3) as in Lemma 6.4,

\[
d_H(Z_{\tau(n)}, i) = d_H(Z_{\tau(n)}^{-1}, i) \xrightarrow{\text{in law}} d_H(Z_{\tau(n)}^{*}, i)
\]

\[
\asymp 2 \log \text{Im}(Z_{\tau(n)}^{*} \wedge i) - \log \text{Im} Z_{\tau(n)}^{*} \asymp Y_{\tau(n)}
\]

where \( \asymp \) means that the difference between the left and right-hand sides is bounded in absolute value.

Therefore, combining Lemma 6.4(a) with Corollary 4.20,

\[
\frac{1}{\sqrt{t}} d_H(X_t, o) \stackrel{a.s.}{\sim} \frac{1}{\sqrt{t}} d_H(Z_{\tau(n)}, i) \xrightarrow{\text{in law}} \frac{1}{\sqrt{t}} Y_{\tau(n)} \stackrel{a.s.}{\sim} \frac{1}{\sqrt{t}} Y_t,
\]

as \( t \to \infty \). Now Lemma 6.1 yields the result, when \( \ell(\alpha, \beta) > 0 \).

Case 2. \( \ell(\alpha, \beta) < 0 \).

Here, Lemma 6.4(b) tells us that \( d_H(X_t, o)/\sqrt{t} \) behaves in law like \( d_T(W_t, o)/\sqrt{t} \) on \( T \), which in turn in view of Lemma 6.5 behaves like \( d_T(R_t, o)/\sqrt{t} \). One can proceed as in Case 1. This time we can use the proof of the CLT for \( (R_t, o) \) that is given in [15]; we get that

\[
d_T(R_t, o)/\sqrt{t} \xrightarrow{\text{in law}} -\Phi(R_t)/\sqrt{t} \sim -Y_t/\sqrt{t},
\]

and the result follows. \( \square \)
The central limit theorem in the drift-free case requires some subtle input from [24] and [25]; it will be modelled after [6], which in turn relies on [15]. (In [6], the weak limit is incorrect, due to a small error in the last step.)

We need standard Brownian motion \((B_t)_{t \geq 0}\) on \(\mathbb{R}\) starting at 0, and the associated random variables

\[
\underline{M} = \min\{B_t : 0 \leq t \leq 1\}, \quad \overline{M} = \max\{B_t : 0 \leq t \leq 1\}, \quad \text{and} \quad \mathcal{N} = B_1,
\]
so that \(\mathcal{N}\) has standard normal distribution.

**Theorem 6.7.** If \(\ell(\alpha, \beta) = 0\), then

\[
\frac{1}{\sqrt{t}} d_{HT}(X_t, \sigma) \to \frac{\log q}{\sqrt{E(\tau)}} \left(2\overline{M} - 2\underline{M} - |\mathcal{N}|\right) \quad \text{in law, as } t \to \infty.
\]

**Proof.** In the proof, we suppose that \((X_t)\) starts at \(\sigma\), so that \(Z_0 = i\), \(W_0 = \sigma\) and \(Y_0 = 0\). The passage to arbitrary starting point is a simple exercise that we leave to the reader. By Corollary 4.20 and (6.3),

\[
\frac{1}{\sqrt{t}} d_{HT}(X_t, \sigma) \sim \frac{1}{\sqrt{t}} d_{HT}(X_{\tau(n)}, \sigma) \sim \frac{1}{E(\tau)} \frac{1}{\sqrt{n}} d_{HT}(X_{\tau(n)}, \sigma) \quad \text{almost surely,}
\]
as \(t \to \infty\). Thus, we want to show that \(d_{HT}(X_{\tau(n)}, \sigma)/\sqrt{n} \to (\log q) U_0\) in law, as \(n \to \infty\). By Proposition 2.8,

\[
(6.8) \quad \frac{1}{\sqrt{n}} d_{HT}(X_{\tau(n)}, \sigma) \sim \frac{1}{\sqrt{n}} d_{HT}(Z_{\tau(n)}, i) + \frac{\log q}{\sqrt{n}} d_{HT}(W_{\tau(n)}, \sigma) - \frac{\log q}{\sqrt{n}} |Y_{\tau(n)}|.
\]

By Lemma 6.5, we can identify

\[
X_{\tau(n)} = R_n, \sigma, \quad W_{\tau(n)} = R_n, \sigma, \quad \text{and} \quad Y_{\tau(n)} = \Phi(R_n).
\]

In the drift-free case, \((\Phi(R_n))\) is nothing but classical simple random walk on \(\mathbb{Z}\) starting at 0. Define

\[
\overline{M}_n = \max\{\Phi(R_k) : k = 0, \ldots, n\} \quad \text{and} \quad \underline{M}_n = \min\{\Phi(R_k) : k = 0, \ldots, n\}, \quad \overline{T}(n) = \max\{k \leq n : \Phi(R_k) = \overline{M}_n\} \quad \text{and} \quad \underline{T}(n) = \max\{k \leq n : \Phi(R_k) = \underline{M}_n\}
\]

It is well known that

\[
(6.9) \quad \frac{1}{\sqrt{n}} \left(\overline{M}_n, \underline{M}_n, \Phi(R_n)\right) \to \left(\overline{M}, \underline{M}, \mathcal{N}\right) \quad \text{in law.}
\]

See e.g. Billingsley [7], (9.2)+(9.8), for this result and the joint distribution of the limiting triple, and Borodin and Salminen [8], 1.15.8(2) on page 174, for the joint distribution of \((\overline{M} - \underline{M}, \mathcal{N})\).

Each \(Z_{\tau(n)}\) is an element of \(\text{Aff}(\mathbb{H}, q)\) and can be inverted in that group. Since we are assuming that \(Z_0 = i\), all increments \(Z_{\tau(n-1)}^{-1} Z_{\tau(n)}, n \geq 1\), are i.i.d. and
have the distribution $\tilde{\mu}$ of (4.15). The support of $\tilde{\mu}$ generates the whole of $\text{Aff}(\mathbb{H}, q)$, and it has finite moments of exponential order by Corollary 4.23. We can now invoke the method and result of [24]. Since its reformulation in our setting is not completely transparent, we provide a brief “translation”. The reference [24] comprises the following, where we set $\tau(n) = \tau(T(n))$.

- The sequence of pairs of random variables

$$(Z_{\tau(n)}, Z_{\tau(n)}^1)$$

with values in $\text{Aff}(\mathbb{H}, q) \times \text{Aff}(\mathbb{H}, q) \equiv \mathbb{LH} \times \mathbb{LH}$ converges in law (i.e., weakly) in $\mathbb{H} \times \mathbb{H}$ to a pair of independent random variables $(Z^\dagger, Z^\dagger)$ with values in $\partial^* \mathbb{H} \times \partial^* \mathbb{H} \equiv \mathbb{R}^2$, both of which have continuous distributions (in fact, continuous densities with respect to Lebesgue measure). Thus, $Z^\dagger \neq Z^\dagger$ with probability 1, so that

$$Z_{\tau(n)}^{-1} \wedge Z_{\tau(n)}^{-1} \overset{\text{in law}}{\to} Z^\dagger \wedge Z^\dagger \in \mathbb{H}.$$  

Note that

$$\log \text{Im}(Z_{\tau(n)}^{-1}) = - (\log q) M_n \quad \text{and} \quad \log \text{Im}(Z_{\tau(n)}^{-1} Z_{\tau(n)}) = (\log q)(\Phi(R_n) - M_n).$$

Using (2.3), we get that

$$d_{\mathbb{H}}(Z_{\tau(n)}, i) = d_{\mathbb{H}}(Z_{\tau(n)}^{-1}, Z_{\tau(n)}^{-1} Z_{\tau(n)})$$

$$\sim 2 \log \text{Im}(Z_{\tau(n)}^{-1} \wedge Z_{\tau(n)}^{-1} Z_{\tau(n)}) - \log \text{Im}(Z_{\tau(n)}^{-1}) - \log \text{Im}(Z_{\tau(n)}^{-1} Z_{\tau(n)})$$

$$\overset{\text{in law}}{\to} \text{Im}(Z^\dagger \wedge Z^\dagger) + (\log q)(2M_n - \Phi(R_n)).$$

On the tree, similarly to the above, the following is proved in [15].

- The sequence of pairs of random variables

$$(R_{T(n)}^{-1} o, R_{T(n)}^{-1} R_n o)$$

with values in $T \times T$ converges in law (i.e., weakly) in $\hat{T} \times \hat{T}$ to a pair of independent random variables $(R^\dagger, R^\dagger)$ with values in $\partial^* T \times \partial^* T$, both of which have continuous distributions ($\equiv$ without point masses). Thus, $R^\dagger \neq R^\dagger$ with probability 1, so that

$$R_{T(n)}^{-1} o \wedge R_{T(n)}^{-1} R_n o \overset{\text{in law}}{\to} R^\dagger \wedge R^\dagger \in T.$$  

This time we use (2.2). Noting that

$$\eta(R_{T(n)}^{-1} o) = -M_n \quad \text{and} \quad \eta(R_{T(n)}^{-1} R_n o) = \Phi(R_n) - M_n,$$
we get
\[
d_T(R_n o, o) = d_T(R_{n o}^{-1}, R_{n o}^{-1} R_{n} o)
\]
\[
= h(R_{n o}^{-1}) + h(R_{n}^{-1} R_{n o}) - 2 h(R_{n o}^{-1} R_{n} o)
\]
\[
in \text{law} \Phi(R_n) - 2 M_n - 2 h(R^1 \land R^1)
\]
Putting things together (which is legitimate because our discrete time processes
are all modelled via $R_n$ on the same probability space), we get, from (6.8),
\[
\frac{1}{\sqrt{n}} d_{HT}(X_{\tau(n)}, o) \xrightarrow{\text{in law}} \log q \left( \frac{2 M_n}{\sqrt{n}} - 2 M_n - |\Phi(R_n)| \right).
\]
Now (6.9) yields the theorem. \qed

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