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Hardy spaces and regularity for the inhomogeneous Dirichlet and Neumann problems

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Abstract. This article has three aims. First, we study Hardy spaces, $h^p_L(\Omega)$, associated with an operator L which is either the Dirichlet Laplacian Δ_D or the Neumann Laplacian Δ_N on a bounded Lipschitz domain Ω in $\mathbb{R}^n$, for $0 < p \leq 1$. We obtain equivalent characterizations of these function spaces in terms of maximal functions and atomic decompositions. Second, we establish regularity results for the Green operators, regarded as the inverses of the Dirichlet and Neumann Laplacians, in the context of Hardy spaces associated with these operators on a bounded semiconvex domain Ω in $\mathbb{R}^n$. Third, we study relations between the Hardy spaces associated with operators and the standard Hardy spaces $h^p_r(\Omega)$ and $h^p_z(\Omega)$, then establish regularity of the Green operators for the Dirichlet problem on a bounded semiconvex domain Ω in $\mathbb{R}^n$, and for the Neumann problem on a bounded convex domain Ω in $\mathbb{R}^n$, in the context of the standard Hardy spaces $h^p_r(\Omega)$ and $h^p_z(\Omega)$. This gives a new solution to the conjecture made by D.-C. Chang, S. Krantz and E. M. Stein regarding the regularity of Green operators for the Dirichlet and Neumann problems on $h^p_r(\Omega)$ and $h^p_z(\Omega)$, respectively, for all $\frac{n}{n+1} < p \leq 1$.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with sufficiently smooth boundary. Consider the inhomogeneous Dirichlet problem for the Laplacian, i.e.,

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Denote by $\mathbb{G}_D$ the Green operator for the Dirichlet problem (1.1), i.e. the solution operator $f \mapsto \mathbb{G}_D(f) := u$. We also consider the inhomogeneous Neumann problem for the Laplacian and denote the corresponding Green operator by $\mathbb{G}_N$.

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The regularity of the operators $\mathbb{G}_D$ and $\mathbb{G}_N$ on the L^p-scale, for $1 < p < \infty$, in the context of a bounded C^∞ domain Ω in $\mathbb{R}^n$ is well understood. A classical reference is the paper [4] by S. Agmon, A. Douglis and L. Nirenberg; see also [49]. More recent developments include extensions to Hardy spaces H^p for $0 < p \leq 1$, due to D-C. Chang, G. Dafni, G. Krantz and E.M. Stein; see [14], [15], [13] and the references therein.

A natural question is to study the regularity of these Green operators on the L^p-scale for $1 < p < \infty$ on a bounded domain $\Omega \subseteq \mathbb{R}^n$ under weaker smoothness hypotheses on $\partial\Omega$. A similar question can be asked when the L^p-scale is replaced by the scale of Hardy spaces, H^p, for $0 < p \leq 1$. Depending on the nature of the assumptions regarding the smoothness of the boundary of Ω and the range of the integrability index p, there are various nuances of the answers presently available in the literature. The following brief summary gives an overview of the progress so far in this direction of research.

(i) One early result in this line of work, due to J. Kadlec (cf. [46]) in the 1960's, is that the mappings

$$f \mapsto \frac{\partial^2 \mathbb{G}_D(f)}{\partial x_i \partial x_j}, \quad 1 \leq i, j \leq n, \tag{1.2}$$

are well defined and bounded on $L^2(\Omega)$ whenever Ω is a bounded convex domain in $\mathbb{R}^n$. See also [37].

(ii) The mappings in (1.2) were shown to be of weak type $(1,1)$ by B. Dahlberg, G. Verchota and T. Wolff [23] in the 1990's, and by Fromm [33], and to be bounded on a suitable Hardy space by Adolfsson [2], still under the assumption that the domain Ω is bounded and convex. By interpolation, these mappings are bounded on L^p for $1 < p < 2$.

(iii) L^2-boundedness of the Green operators in the case of the Neumann boundary condition has been known since the mid 1970's ([36]), but optimal L^p estimates, valid in the range $1 < p \leq 2$, have only been proved in the 1990's by Adolfsson and D. Jerison [3]. Their strategy was to obtain an endpoint estimate for atoms in a suitable Hardy space $H^1(\Omega)$, and then to use interpolation with the L^2 results.

(iv) For $p \leq 1$, the regularity of the Green operators on scales of local Hardy spaces, h^p, have been recently studied in [50] when Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, and the results were formulated in terms of a pair of Hardy spaces, $h^p_r(\Omega)$ and $h^p_z(\Omega)$ (see (5.1) and (5.3) for definitions), for the range $\frac{n}{n+\epsilon} < p < 1$ for some $0 < \epsilon \leq 1$.

(v) In relation to (ii) and (iii) above, it should be mentioned that the aforementioned L^p continuity of two derivatives on Green potentials may fail in the class of Lipschitz domains for any $p \in (1,\infty)$ and in the class of convex domains for any $p \in (2,\infty)$ (see [2], [3], [22], [42] and [51] for counterexamples; recall that every convex domain is Lipschitz).

(vi) Quite recently, in [55], the authors have studied the mapping properties of operators of the form $\partial^2 \mathbb{G}_D / \partial x_i \partial x_j$, $1 \leq i, j \leq n$, on Besov and Triebel–Lizorkin scales in a bounded Lipschitz domain satisfying a uniform exterior ball condition (henceforth abbreviated as UEBC). When specialized to the class of local Hardy spaces this gives that $\partial^2\mathbb{G}_D/\partial x_i\partial x_j$ is a bounded mapping from $h^p_r(\Omega)$ into itself whenever $\frac{n}{n+1} < p \le 1$.

The present paper can be viewed as a continuation of the above body of work. Our main results answer the question whether the regularity of the Green operator on the L^p-scale of a semiconvex bounded domain Ω in the case of Dirichlet condition, and a convex bounded domain Ω in the case of Neumann condition, has a satisfactory analogue at the level of Hardy spaces h^p. Our results also clarify the nature of these h^p spaces as well as the range of p's for which such regularity results hold. Here we also report further progress on a question posed by D.-C. Chang, S. Krantz and E. Stein [14] pertaining to the regularity of Green operators, $\mathbb{G}_D$ and $\mathbb{G}_N$, associated with the Dirichlet and Neumann Laplacians on Hardy spaces. As far as this question is concerned, in the class of arbitrary bounded Lipschitz domains, a solution (which is optimal relative to this class) has been given in [50] for the range $1-\varepsilon < p \le 1$, for $\varepsilon > 0$ depending on the domain, while [55] has addressed the case of the Dirichlet Green potential in a bounded semiconvex domain Ω for the scale $h^p_r(\Omega)$ indexed by $p \in (\frac{n}{n+1}, 1]$. In the present paper we explore the range $0 < p \le 1$ and consider both the Dirichlet and the Neumann Green potentials.

In order to discuss the strategy adopted in this work, we need to review some background facts and set some notation. We start by recalling the definition of the local Hardy spaces introduced by Goldberg (see [35]). Let $\phi \in \mathscr{S}(\mathbb{R}^n)$ be a function with the property that $\int_{\mathbb{R}^n}\phi(x)\,dx = 1$ and, for each $t > 0$, define $\phi_t(x) := t^{-n}\phi(x/t)$. For $0 < p < \infty$, the local Hardy space $h^p(\mathbb{R}^n)$ is defined as the space of tempered distribution $f \in \mathscr{S}'(\mathbb{R}^n)$ for which the maximal function

$$\mathcal{M}_{\rm loc}f(x) := \sup_{0<t\le 1}\big|\phi_t * f(x)\big| \tag{1.3}$$

belongs to $L^p(\mathbb{R}^n)$. If this is the case, define

$$\|f\|_{h^p(\mathbb{R}^n)} := \|\mathcal{M}_{\rm loc}f\|_{L^p(\mathbb{R}^n)}. \tag{1.4}$$

An equivalent definition of $h^p(\mathbb{R}^n)$ involves the non-tangential maximal function associated with the heat semigroup (or Poisson semigroup) generated by Δ, the Laplace operator on $\mathbb{R}^n$. If $f \in \mathscr{S}'(\mathbb{R}^n)$, then (see Theorem 1 in [35])

$$f \in h^p(\mathbb{R}^n) \Longleftrightarrow \sup_{|y-x|<t\le 1}\big|e^{-t^2\Delta}f(y)\big| \in L^p(\mathbb{R}^n). \tag{1.5}$$

In this article we are in fact considering two Hardy spaces, namely $h^p_{\Delta_D}(\Omega)$ and $h^p_{\Delta_N}(\Omega)$, in the spirit of the work in [9], [30], [7] and [39]. Roughly speaking, for a reasonable functional f,

$$f \in h^p_L(\Omega) \quad \Longleftrightarrow \sup_{y\in\Omega:\,|y-x|<t\le 1}\big|e^{-t^2L}f(y)\big| \in L^p(\Omega), \tag{1.6}$$

where $\{e^{-tL}\}$ is the heat semigroup generated by L, and L is either the Dirichlet Laplacian Δ_D or the Neumann Laplacian Δ_N on Ω, respectively (for more precise definitions see §2).

Both spaces $h^p_{\Delta_D}(\Omega)$ and $h^p_{\Delta_N}(\Omega)$ are useful because they are particularly well adapted to the Dirichlet and Neumann Laplacians in Ω. For example, their elements have atomic decompositions with the atoms exhibiting cancellation properties customized to the specific nature of the partial differential operator in question (something occasionally referred to as L-cancellation, if a generic operator L is employed). Such atomic decompositions play an important role in the proofs of the following regularity results:

Main Result 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, simply connected, semiconvex domain.*

(i) *The operators $\frac{\partial^2 \mathbb{G}_D}{\partial x_i \partial x_j}$, initially defined on $L^2(\Omega) \cap h^p_{\Delta_D}(\Omega)$, can be extended as bounded linear mappings from $h^p_{\Delta_D}(\Omega)$ into $L^p(\Omega)$ whenever $0 < p \leq 1$. Also, the operators $\frac{\partial^2 \mathbb{G}_N}{\partial x_i \partial x_j}$, initially defined on $L^2(\Omega) \cap h^p_{\Delta_N}(\Omega)$, can be extended as bounded linear mappings from $h^p_{\Delta_N}(\Omega)$ into $L^p(\Omega)$ whenever $0 < p \leq 1$.*

(ii) *Both operators $\frac{\partial^2 \mathbb{G}_D}{\partial x_i \partial x_j}$ and $\frac{\partial^2 \mathbb{G}_N}{\partial x_i \partial x_j}$ are of type weak $(1,1)$. Hence, by Marcinkiewicz interpolation, they can be extended from $L^2(\Omega) \cap L^p(\Omega)$ to bounded linear mappings on $L^p(\Omega)$ whenever $1 < p \leq 2$.*

From suitable estimates for heat kernels associated to the semigroups $\{e^{-t\Delta_D}\}$ and $\{e^{-t\Delta_N}\}$, we prove that for all $\frac{n}{n+1} < p \leq 1$, the spaces $h^p_r(\Omega)$ and $h^p_{\Delta_D}(\Omega)$ coincide as sets[1] for any bounded semiconvex domain $\Omega \subset \mathbb{R}^n$ while the spaces $h^p_z(\Omega)$ and $h^p_{\Delta_N}(\Omega)$ coincide for any bounded convex domain $\Omega \subset \mathbb{R}^n$, and their norms are equivalent in both cases (See Proposition 5.3). Then, by relying on an equivalent characterization of the Hardy space $h^p_r(\Omega)$ from [56] as well as atomic decompositions, we prove the following regularity result for the Dirichlet and Neumann Green potentials for the Laplacian.

Main Result 1.2. (i) *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, simply connected, semiconvex domain. Then the operators $\frac{\partial^2 \mathbb{G}_D}{\partial x_i \partial x_j}$, $i, j \in \{1, \dots, n\}$, are bounded on $h^p_r(\Omega)$ for all $\frac{n}{n+1} < p \leq 1$.*

(ii) *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded, simply connected, convex domain. Then the operators $\frac{\partial^2 \mathbb{G}_N}{\partial x_i \partial x_j}$, $i, j \in \{1, \dots, n\}$, are bounded linear mappings from $h^p_z(\Omega)$ into $h^p_r(\Omega)$ whenever $\frac{n}{n+1} < p \leq 1$.*

Note that for the Dirichlet problem on a smooth domain Ω in $\mathbb{R}^n$, the mapping $f \mapsto \frac{\partial^2 \mathbb{G}_D(f)}{\partial x_i \partial x_j}$ does not extend from $h^p_r(\Omega)$ to $h^p_r(\Omega)$ if $0 < p \leq \frac{n}{n+1}$ (see [15] for a discussion in this regard). This indicates that $h^p_r(\Omega)$ is the appropriate version of the Hardy space for the Dirichlet problem only when $\frac{n}{n+1} < p \leq 1$ and that this range of p's is sharp. This also substantiates the need to find a suitable replacement for $h^p_r(\Omega)$ for small values of p.

Part (i) of Main Result 1.2 is essentially the same as that of [55] but obtained by a different method. As a whole, our main results thus extend and complete the results in [2], [3], [33], and [15], as well as some of the results in [50] and [55]. Our

[1] We thank the referee for suggesting the proof of the right-to-left inclusion.

approach is new and conceptually different from the previously known ones. First, we develop appropriate machinery to treat operators which fall beyond the scope of the classical Calderón–Zygmund theory, following the line of study initiated in [26], [19] and [5]. Indeed, the operators that we are naturally led to consider need not have Hölder continuous kernels, a regularity condition which often proves too restrictive for certain applications. For example, the kernel of the operator $\nabla^2 L^{-1}$ is formally given by

$$\int_0^\infty \nabla_x^2 p_t(x,y)\,dt, \tag{1.7}$$

and the second order derivative $\nabla_x^2 p_t(x,y)$ of the kernel $p_t(x,y)$, associated with the heat semigroup e^{-tL}, does not satisfy a Hölder condition in the spatial variable. The loss of Hölder continuity is compensated by a more subtle built-in regularity property inherited from the semigroup e^{-tL}, and by the availability of certain regularity estimates for the solutions to the Dirichlet and Neumann problems which are specific to semiconvex domains (see Section 4 for a discussion). Second, for the full range $0 < p \le 1$ of the integrability index p, we prove our results by making use of the properties of heat semigroups, and the fact that elements of the Hardy spaces $h^p_{\Delta_D}(\Omega)$ and $h^p_{\Delta_N}(\Omega)$ have an atomic decomposition with atoms having sufficient "L-cancellation" property. This provides more flexibility since, in principle, this method does not differentiate between Dirichlet and Neumann boundary conditions.

The layout of the paper is as follows. In Section 2, we establish suitable upper bounds as well as Hölder continuity estimates for the heat kernels of the Dirichlet and Neumann Laplacians in subdomains of $\mathbb{R}^n$. In Section 3, we introduce the Hardy spaces $h^p_{\Delta_D}(\Omega)$ and $h^p_{\Delta_N}(\Omega)$, $0 < p \le 1$, associated to the Dirichlet and Neumann Laplacians, respectively, and show that the adapted Hardy spaces defined in terms of atoms, and in terms of maximal functions using the heat semigroups, are all equivalent, assuming sufficient "L-cancellation" of our atoms. Our Main Results 1.1 and 1.2 are proved in Sections 4 and 5 by using suitable estimates for singular integrals with non-smooth kernels and an optimal on-diagonal heat kernel estimate.

In closing, we wish to note that while the results in Sections 4 and 5 are based on the Gaussian heat kernel bounds for the Dirichlet Laplacian Δ_D and the Neumann Laplacian Δ_N on a domain $\Omega \subset \mathbb{R}^n$, which are derived in Section 2 for $n \ge 3$, only minor modifications of our proofs are required in order to treat the two-dimensional case.

2. Heat kernels and Green functions

In this section, we assume that Ω is a bounded convex domain in $\mathbb{R}^n$ where $n \ge 3$, equipped with Euclidean distance and Lebesgue measure. Note that every bounded convex domain is a Lipschitz domain, but the Lipschitz property is not sufficient.

We now describe the Dirichlet and Neumann Laplacians on a bounded domain Ω in $\mathbb{R}^n$. Denote by $W^{1,2}(\Omega)$ the usual Sobolev space on Ω equipped with the norm $(\|f\|_2^2 + \|\nabla f\|_2^2)^{1/2}$, and let $W_0^{1,2}(\Omega)$ stand for the closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$. To proceed, let V be a closed subspace of $W^{1,2}(\Omega)$ and denote by Q the quadratic form on the space V given by

$$Q(u) := \int_\Omega \sum_i \frac{\partial u}{\partial x_i} \frac{\partial \overline{u}}{\partial x_i}\, dx, \quad u \in V. \tag{2.1}$$

Then Q is closable and the self-adjoint operator Δ associated with the obtained closed form is called the (minus) Laplacian on Ω. Different choices of the space V induce different boundary conditions for the operator Δ. Most notably, when V is $W_0^{1,2}(\Omega)$ the corresponding boundary condition is Dirichlet, while the choice $V = W^{1,2}(\Omega)$ corresponds to a Neumann boundary condition. Under Dirichlet and Neumann boundary conditions, Δ will be denoted in the sequel by Δ_D and Δ_N, respectively. More specifically,

$$\begin{aligned} &\Delta_D : \mathcal{D}(\Delta_D) \subseteq L^2(\Omega) \longrightarrow L^2(\Omega), \quad \Delta_D(u) := -\textstyle\sum_i \partial_i^2 u \\ &\text{for every } u \in \mathcal{D}(\Delta_D) := \{u \in W_0^{1,2}(\Omega) : \textstyle\sum_i \partial_i^2 u \in L^2(\Omega)\}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\Delta_N : \mathcal{D}(\Delta_N) \subseteq L^2(\Omega) \longrightarrow L^2(\Omega), \qquad \Delta_N(u) := -\textstyle\sum_i \partial_i^2 u \quad \text{for every} \\ &u \in \mathcal{D}(\Delta_N) := \Big\{ u \in W^{1,2}(\Omega) : \textstyle\sum_i \partial_i^2 u \in L^2(\Omega) \quad \text{and} \\ &\qquad \int_\Omega \sum_i \frac{\partial u}{\partial x_i} \frac{\partial \overline{w}}{\partial x_i} dx = -\int_\Omega \Big(\sum_i \partial_i^2 u \Big) w\, dx,\ \forall\, w \in W^{1,2}(\Omega) \Big\}. \end{aligned} \tag{2.3}$$

The Dirichlet and Neumann Laplacians on Ω are nonnegative self-adjoint operators on $L^2(\Omega)$. Using spectral theory one can define the semigroups $\{e^{-t\Delta_D}\}_{t>0}$ and $\{e^{-t\Delta_N}\}_{t>0}$ generated by these operators on $L^2(\Omega)$. It is well known (see for example, [24]) that for an arbitrary open set $\Omega \subseteq \mathbb{R}^n$, the semigroup kernel $p_{t,\Delta_D}(x, y)$ associated to $e^{-t\Delta_D}$ satisfies the Gaussian upper bound

$$0 < p_{t,\Delta_D}(x, y) \le (4\pi t)^{-n/2} \exp\Big(-\frac{|x-y|^2}{4t} \Big) \quad \forall\, x, y \in \Omega,\ \forall\, t > 0. \tag{2.4}$$

We will say that an open set $\Omega \subseteq \mathbb{R}^n$ has the extension property if there exists a bounded linear map $E : W^{1,2}(\Omega) \to W^{1,2}(\mathbb{R}^n)$ such that Eu is an extension of u from Ω to $\mathbb{R}^n$ for all $u \in W^{1,2}(\Omega)$. Then (see Theorem 3.2.9 in [24]), if Ω has the extension property, there exists a constant $C > 0$ such that the kernels $p_{t,\Delta_N}(x, y)$ of the semigroup $e^{-t\Delta_N}$ satisfy

$$0 < p_{t,\Delta_N}(x, y) \le \frac{C}{|B^\Omega(x, \sqrt{t})|} \exp\Big(-\frac{|x-y|^2}{4\alpha t} \Big) \tag{2.5}$$

for every $t > 0$, every $x, y \in \Omega$ and each $\alpha > 1$, where $|E|$ denotes the Lebesgue measure of a measurable set $E \subseteq \mathbb{R}^n$, and for $x \in \mathbb{R}^n$ and $r > 0$ we have set

$$B^\Omega(x, r) := \{y \in \Omega : |x - y| < r\}. \tag{2.6}$$

It is well known that any open set $\Omega \subseteq \mathbb{R}^n$ having the extension property also satisfies the doubling property

$$|B^{\Omega}(x,2r)| \leq C|B^{\Omega}(x,r)|, \quad \forall\, x \in \Omega,\ \forall\, r \in (0, \operatorname{diam}(\Omega)). \tag{2.7}$$

Let us also note here that the operator Δ_N conserves probability, that is

$$e^{-t\Delta_N} 1 = 1. \tag{2.8}$$

This conservative property does not hold for Δ_D (see Chapter 4 of [57]).

It is useful to observe that Gaussian upper bounds carry over from heat kernels to the time derivatives of their kernels. More specifically, we have the following lemma (see Lemma 2.5 on page 352 in [20]).

Lemma 2.1. *Let Ω be an open set in $\mathbb{R}^n$ which satisfies the doubling property (2.7). Let T_t be a uniformly bounded analytic semigroup on $L^2(\Omega)$ and assume that T_t, $t > 0$, has a kernel $p_t(x,y)$ satisfying*

$$|p_t(x,y)| \leq \frac{C}{|B^{\Omega}(x,\sqrt{t})|} \exp\Big(-\frac{|x-y|^2}{ct}\Big). \tag{2.9}$$

Then for every $k \in \mathbb{N}$, there exist two positive constants c_k and C_k such that the time derivatives of the kernel $p_t(x,y)$ satisfy

$$\Big|\frac{d^k}{dt^k} p_t(x,y)\Big| \leq \frac{C_k}{t^k |B^{\Omega}(x,\sqrt{t})|} \exp\Big(-\frac{|x-y|^2}{c_k t}\Big), \ \ \forall\, x,y \in \Omega, \forall\, t > 0. \tag{2.10}$$

2.1. The Dirichlet Laplacian

In this section we establish suitable Hölder continuity estimates for the kernel of the heat semigroup $\{e^{-t\Delta_D}\}$. We begin with the following lemma:

Lemma 2.2. *Suppose $\Omega \subseteq \mathbb{R}^n$ is an open set. Let $H(z,x,y)$ be the complex time heat kernel of Δ_D for $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, i.e., $H(z,x,y)$ is the integral kernel of the semigroup $e^{-z\Delta_D}$:*

$$e^{-z\Delta_D} u(x) = \int_{\Omega} H(z,x,y) u(y)\, dy, \quad \textit{for } u \in L^2(\Omega),\ x \in \Omega. \tag{2.11}$$

Then for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that the upper bound

$$\big|H(z,x,y)\big| \leq C_\epsilon (\operatorname{Re} z)^{-n/2} \exp\Big\{-\operatorname{Re}\Big(\frac{|x-y|^2}{4(1+\epsilon)z}\Big)\Big\} \tag{2.12}$$

holds for all $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ and all $x,y \in \Omega$.

Proof. Since Δ_D generates a holomorphic semigroup $e^{-z\Delta_D}$, and the kernel $p_t(x,y)$ of the heat semigroup $\{e^{-t\Delta_D}\}$ satisfies the Gaussian upper bound (2.4), the bound (2.12) for $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ can be obtained by analytic continuation as in the proof of Theorem 3.4.8 on page 103 in [24]. $\square$

It is important to have bounds on the heat kernel for the complex number z in the right half plane because, as we shall see momentarily, this estimate implies bounds on the Green function $G_\lambda(x,y)$ for complex values of λ with $\arg\lambda$ arbitrarily small. We now proceed to establish such upper bounds on the Green function.

Lemma 2.3. *Assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded open set. For each $\lambda \in \mathbb{C}\backslash(-\infty,0)$, let $G_\lambda(\cdot,\cdot)$ be the Green function associated with the operator $\Delta_D - \lambda I$, i.e., $G_\lambda(x,y)$ is the integral kernel of the resolvent $(\Delta_D - \lambda I)^{-1}$,*

$$(\Delta_D - \lambda I)^{-1}u(x) = \int_\Omega G_\lambda(x,y)u(y)\,dy, \quad u \in L^2(\Omega), \;\; x \in \Omega. \tag{2.13}$$

Fix $0 < \mu < \pi$. Then there exist positive constants C and γ, depending only on Ω, n and μ, such that

$$\big|G_\lambda(x,y)\big| \le Ce^{-\gamma\sqrt{|\lambda|}|x-y|}\,\frac{1}{|x-y|^{n-2}} \qquad \textit{for all } x,y \in \Omega,\;\; x \ne y, \tag{2.14}$$

provided either $\lambda = 0$, or $0 < \mu < \arg\lambda < 2\pi - \mu$.

Proof. The existence and uniqueness of the Green function are known from work in [38]. To deduce the stated bound on the Green function $G_\lambda(x,y)$ we make use of the estimates for the heat kernel $H(z,x,y)$ from Lemma 2.2. In the case when $\mu < \arg\lambda \le \pi$, we choose Γ to be the ray $\{z \in \mathbb{C} : z = |z|e^{i\theta_0}\}$, where θ_0 is chosen such that $0 < \theta_0 < \frac{\pi}{2}$ and $\mu + \theta_0 > \frac{\pi}{2}$. We then have

$$(\Delta_D - \lambda I)^{-1} = \int_\Gamma e^{\lambda z} e^{-z\Delta_D}\,dz. \tag{2.15}$$

Hence the kernel $G_\lambda(x,y)$ of $(\Delta_D - \lambda I)^{-1}$ can be written as

$$G_\lambda(x,y) = \int_\Gamma e^{\lambda z} H(z,x,y)\,dz. \tag{2.16}$$

Since $\mathrm{Re}\,(z\lambda) < 0$ and $|\mathrm{Re}\,(z\lambda)| \le C_\mu |z\lambda|$, it follows that the bound (2.12) holds. Thus, for $x,y \in \Omega$, $x \ne y$, we obtain

$$\begin{aligned}
\big|G_\lambda(x,y)\big| &\le C\int_\Gamma e^{-c_1|z\lambda|}\frac{1}{|z|^{n/2}}e^{-c_2\frac{|x-y|^2}{|z|}}\,d|z| \\
&\le C\int_\Gamma e^{-2\sqrt{\frac{c_1c_2|\lambda||x-y|^2}{2}}}\,\frac{1}{|z|^{n/2}}e^{-b_1\frac{|x-y|^2}{2|z|}}\,d|z| \\
&\le Ce^{-\gamma\sqrt{|\lambda|}|x-y|}\int_\Gamma \frac{1}{|z|^{n/2}}e^{-c_2\frac{|x-y|^2}{2|z|}}\,d|z| \\
&\le Ce^{-\gamma\sqrt{|\lambda|}|x-y|}\int_0^\infty r^{\frac{(n-4)}{2}}e^{-c_2|x-y|^2 r}\,dr \\
&\le Ce^{-\gamma\sqrt{|\lambda|}|x-y|}\frac{1}{|x-y|^{n-2}}.
\end{aligned} \tag{2.17}$$

This proves Lemma 2.3 in the case when $\mu < \arg\lambda \le \pi$. The case when $\pi < \arg\lambda < 2\pi - \mu$ is similar. $\square$

Definition 2.4. A set $E \subseteq \mathbb{R}^n$ is said to satisfy an exterior ball condition at $x \in \partial E$ if there exist $v \in S^{n-1}$ and $r > 0$ such that

$$B(x + rv, r) \subseteq \mathbb{R}^n \setminus E. \tag{2.18}$$

Set $r(x) := \sup\{r > 0 : \text{(2.18) holds for some } v \in S^{n-1}\}$, whenever E satisfies an exterior ball condition at $x \in \partial E$.

We say that E satisfies a uniform exterior ball condition with radius $r > 0$ provided

$$\inf_{x\in\partial E} r(x) \geq r, \tag{2.19}$$

and the value of r in (2.19) will be referred to as the UEBC constant. We say that E satisfies a UEBC provided there exists $r > 0$ with the property that E satisfies a uniform exterior ball condition with radius r.

The following lemma establishes the Hölder continuity of the Green functions.

Lemma 2.5. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set satisfying a* UEBC*. Fix $\mu \in (0, \pi)$ and $\lambda \in \mathbb{C}$. Then the Green function $G_\lambda(x, y)$ satisfies the following estimate:*

$$\big|\nabla_y G_\lambda(x, y)\big| \leq C \left\{ \frac{1}{|x-y|^{n-1}} + \frac{\sqrt{|\lambda|}}{|x-y|^{n-2}} \right\}, \quad \forall\, x, y \in \Omega,\ x \neq y. \tag{2.20}$$

Also, for every $\alpha \in (0, 1)$, there exist positive constants C and β such that, for every $x, y_1, y_2 \in \Omega$ with $x \neq y_j$, $j = 1, 2$, there holds

$$\begin{aligned} &\big|G_\lambda(x, y_1) - G_\lambda(x, y_2)\big| \\ &\quad\leq C \max_{i=1,2} e^{-\beta\sqrt{|\lambda|}|x-y_i|} |y_1 - y_2|^\alpha \left\{ \frac{1}{|x-y_i|^{n-2+\alpha}} + \frac{|\lambda|^{\alpha/2}}{|x-y_i|^{n-2}} \right\} \end{aligned} \tag{2.21}$$

for all λ with $|\arg\lambda| \geq \mu > 0$. The constant C depends on the value μ, the dimension n, $\operatorname{diam}\Omega$*, and the* UEBC *constant of Ω, and β depends on μ and α.*

Proof. Given that Ω satisfies a UEBC, Grüter and Widman have proved (see [38]) that the Green function $G(x, y)$ $(:= G_0(x, y))$ satisfies

$$\big|\nabla_y G(x, y)\big| \leq \frac{C}{|x-y|^{n-1}}, \quad \forall\, x, y \in \Omega,\ x \neq y. \tag{2.22}$$

To obtain the bounds for $G_\lambda(x, y)$, we use the resolvent identity which implies

$$G_\lambda(x, y) = G(x, y) + \lambda \int_\Omega G(z, y) G_\lambda(x, z)\, dz, \quad \forall\, x, y \in \Omega,\ x \neq y. \tag{2.23}$$

Hence by (2.22) and (2.14), we have that for every $x, y \in \Omega$, $x \neq y$,

$$\begin{aligned} \big|\nabla_y G_\lambda(x, y)\big| &\leq \big|\nabla_y G(x, y)\big| + |\lambda| \int_\Omega \big|\nabla_y G(z, y)\big| \big|G_\lambda(x, z)\big|\, dz \\ &\leq \frac{C}{|x-y|^{n-1}} + C|\lambda| \int_\Omega \frac{e^{-\gamma\sqrt{|\lambda|}|x-z|}}{|x-z|^{n-2}} \frac{1}{|z-y|^{n-1}}\, dz. \end{aligned} \tag{2.24}$$

To estimate the second term, break the integral over the domain Ω into two parts, an integral over Ω_1 and an integral over Ω_2, where

$$\Omega_1 := \big\{z \in \Omega : |x-z| \le |y-z|\big\} \text{ and } \Omega_2 := \big\{z \in \Omega : |x-z| > |y-z|\big\}. \tag{2.25}$$

Corresponding to this, we have

$$\begin{aligned} \int_{\Omega_1} \frac{e^{-\gamma\sqrt{|\lambda|}|x-z|}}{|x-z|^{n-2}} \frac{1}{|z-y|^{n-1}}\, dz &\le 2 \int_{\Omega_1} \frac{e^{-\gamma\sqrt{|\lambda|}|x-z|}}{|x-z|^{n-2}} \frac{1}{|x-y|^{n-1}}\, dz \\ &= \frac{2}{|x-y|^{n-1}} \int_{\Omega_1} \frac{e^{-\gamma\sqrt{|\lambda|}|x-z|}}{|x-z|^{n-2}}\, dz \le \frac{C}{|x-y|^{n-1}} \frac{1}{|\lambda|}, \end{aligned} \tag{2.26}$$

and

$$\begin{aligned} \int_{\Omega_2} \frac{e^{-\gamma\sqrt{|\lambda|}|x-z|}}{|x-z|^{n-2}} \frac{1}{|z-y|^{n-1}}\, dz &\le 2 \int_{\Omega_2} \frac{e^{-\gamma\sqrt{|\lambda|}|x-z|}}{|x-z|^{n-2}} \frac{1}{|z-y|^{n-1}}\, dz \\ &\le \frac{2}{|x-y|^{n-2}} \int_{\Omega_2} \frac{e^{-\gamma\sqrt{|\lambda|}|z-y|}}{|z-y|^{n-1}}\, dz \le \frac{C}{|x-y|^{n-2}} \frac{1}{\sqrt{|\lambda|}}. \end{aligned} \tag{2.27}$$

These estimates, in combination with (2.24), show that (2.20) holds. To obtain (2.21), fix x, y_1, $y_2 \in \Omega$, with $x \ne y_j$, $j = 1, 2$, and, without loss of generality, suppose that $|x - y_1| \le |x - y_2|$. It follows from Lemma 2.3 that

$$\begin{aligned} \big|G_\lambda(x, y_1) - G_\lambda(x, y_2)\big| &\le C \sum_{i=1}^{2} e^{-\gamma\sqrt{|\lambda|}|x-y_i|} \frac{1}{|x-y_i|^{n-2}} \\ &\le C_1 e^{-\gamma\sqrt{|\lambda|}|x-y_1|} \frac{1}{|x-y_1|^{n-2}}. \end{aligned} \tag{2.28}$$

Also, from (2.20) and the Mean Value Theorem we have that

$$\begin{aligned} \big|G_\lambda(x, y_1) - G_\lambda(x, y_2)\big| &\le C|y_1 - y_2| \bigg\{ \sum_{i=1}^{2} \Big(\frac{1}{|x-y_i|^{n-1}} + \frac{\sqrt{|\lambda|}}{|x-y_i|^{n-2}} \Big) \bigg\} \\ &\le C_2 |y_1 - y_2| \Big\{ \frac{1}{|x-y_1|^{n-1}} + \sqrt{|\lambda|} \frac{1}{|x-y_1|^{n-2}} \Big\}. \end{aligned} \tag{2.29}$$

Therefore, for every $\alpha \in (0, 1)$, it follows from (2.28) and (2.29) that

$$\begin{aligned} \big|G_\lambda(x, y_1) - G_\lambda(x, y_2)\big| &\le \bigg\{ C_2 |y_1 - y_2| \Big(\frac{1}{|x-y_1|^{n-1}} + \sqrt{|\lambda|} \frac{1}{|x-y_1|^{n-2}} \Big) \bigg\}^{\alpha} \\ &\qquad \times \Big\{ C_1 e^{-\gamma\sqrt{|\lambda|}|x-y_1|} \frac{1}{|x-y_1|^{n-2}} \Big\}^{1-\alpha} \\ &\le C_3 e^{-\beta\sqrt{|\lambda|}|x-y_1|} |y_1 - y_2|^{\alpha} \Big\{ \frac{1}{|x-y_1|^{n-2+\alpha}} + \frac{|\lambda|^{\alpha/2}}{|x-y_1|^{n-2}} \Big\}. \end{aligned} \tag{2.30}$$

This proves (2.21) and completes the proof of Lemma 2.5. $\square$

Next, we augment the previous result with estimates on the kernels of powers of the resolvents.

Lemma 2.6. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set satisfying a* UEBC*. Fix $\mu \in (0, \pi)$ and $\lambda \in \mathbb{C}$ with $|\arg \lambda| \geq \mu > 0$. Then for every $\alpha \in (0,1)$ and large enough integer $m \in \mathbb{N}$, $(\Delta_D - \lambda I)^{-m}$ has a kernel $R_{\lambda,\,m}(x, y)$ which satisfies*

$$(2.31)\quad \big|R_{\lambda,\,m}(x, y_1) - R_{\lambda,\,m}(x, y_2)\big| \leq C \max_{i=1,2} |\lambda|^{-m+\frac{n}{2}+\frac{\alpha}{2}} |y_1 - y_2|^{\alpha} e^{-\gamma\sqrt{|\lambda|}|x-y_i|}$$

for every $x, y_1, y_2 \in \Omega$, $x \neq y_1$, $x \neq y_2$. The constant C depends on μ, α, the dimension n, diam (Ω)*, and the* UEBC *constant of Ω, while γ depends on μ and α.*

Proof. For all λ with $|\arg \lambda| \geq \mu > 0$, it is known (see for instance Theorem 1 on page 37 in [28]) that for all large enough integers $m \in \mathbb{N}$ (say, for example, $m > n/2$), $(\Delta_D - \lambda I)^{-m}$ has a kernel $R_{\lambda,\,m}(x, y)$ which, for some $c > 0$, satisfies

$$(2.32)\qquad \big|R_{\lambda,\,m}(x, y)\big| \leq C|\lambda|^{-m+\frac{n}{2}} e^{-c\sqrt{|\lambda|}|x-y|}, \quad \forall\, x, y \in \Omega,\ x \neq y.$$

To obtain the bounds for $R_{\lambda,\,m}(x, y)$ stated in (2.31), we use the resolvent identity which implies

$$(2.33)\qquad R_{\lambda,\,m+1}(x, y) = \int_{\Omega} R_{\lambda,\,m}(x, z) G_{\lambda}(z, y)\, dz, \quad x, y \in \Omega,\ x \neq y.$$

It follows from the above equality and the estimates (2.32) and (2.21) that for every $\alpha \in (0, 1)$,

$$\begin{aligned}
&\big|R_{\lambda,\,m+1}(x, y_1) - R_{\lambda,\,m+1}(x, y_2)\big| \\
&\quad = \Big|\int_{\Omega} R_{\lambda,\,m}(x, z)\big(G_{\lambda}(z, y_1) - G_{\lambda}(z, y_2)\big)\, dz\Big| \\
&\quad \leq C|\lambda|^{-m+\frac{n}{2}} |y_1 - y_2|^{\alpha} \int_{\Omega} e^{-c\sqrt{|\lambda|}|x-z|} e^{-\beta\sqrt{|\lambda|}|z-y_1|} \frac{1}{|z - y_1|^{n-2+\alpha}}\, dz \\
&\qquad + C|\lambda|^{-m+\frac{n}{2}} |y_1 - y_2|^{\alpha} \int_{\Omega} e^{-c\sqrt{|\lambda|}|x-z|} e^{-\beta\sqrt{|\lambda|}|z-y_1|} \frac{|\lambda|^{\alpha/2}}{|x - y_1|^{n-2}} \\
&\qquad + C|\lambda|^{-m+\frac{n}{2}} |y_1 - y_2|^{\alpha} \int_{\Omega} e^{-c\sqrt{|\lambda|}|x-z|} e^{-\beta\sqrt{|\lambda|}|z-y_2|} \frac{1}{|z - y_2|^{n-2+\alpha}} dz \\
&\qquad + C|\lambda|^{-m+\frac{n}{2}} |y_1 - y_2|^{\alpha} \int_{\Omega} e^{-c\sqrt{|\lambda|}|x-z|} e^{-\beta\sqrt{|\lambda|}|z-y_2|} \frac{|\lambda|^{\alpha/2}}{|x - y_2|^{n-2}} \\
(2.34)&\quad =: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},
\end{aligned}$$

for every $x, y_1, y_2 \in \Omega$ with $x \neq y_1$ and $x \neq y_2$.

To estimate term I, we set $\gamma := \min(c, \beta/2)$ (with $c > 0$ as in (2.32)), and use the fact that

$$|x - z| + |z - y_1| \geq |x - y_1| \quad \text{and} \quad e^{-|x|} \leq c_k |x|^{-k} \quad \text{for all } k > 0,$$

to obtain

$$
\begin{aligned}
\int_\Omega & e^{-c\sqrt{|\lambda|}|x-z|}\, e^{-\beta\sqrt{|\lambda|}|z-y_1|}\frac{1}{|z-y_1|^{n-2+\alpha}}\,dz \\
&\le \int_\Omega e^{-\gamma\sqrt{|\lambda|}\big(|x-z|+|z-y_1|\big)}\, e^{-\frac{1}{2}\beta\sqrt{|\lambda|}|z-y_1|}\frac{1}{|z-y_1|^{n-2+\alpha}}\,dz \\
&\le e^{-\gamma\sqrt{|\lambda|}|x-y_1|}\int_\Omega e^{-\frac{1}{2}\beta\sqrt{|\lambda|}|z-y_1|}\frac{1}{|z-y_1|^{n-2+\alpha}}\,dz \\
&\le C_1 e^{-\gamma\sqrt{|\lambda|}|x-y_1|}|\lambda|^{-1+\frac{\alpha}{2}}\int_\Omega e^{-\frac{1}{2}\beta|z-y_1|}\frac{1}{|z-y_1|^{n-2+\alpha}}\,dz \\
&\le C_2 e^{-\gamma\sqrt{|\lambda|}|x-y_1|}|\lambda|^{-1+\frac{\alpha}{2}}. \qquad (2.35)
\end{aligned}
$$

This, in turn, gives

$$
\mathrm{I} \le C|\lambda|^{-(m+1)+\frac{n}{2}+\frac{\alpha}{2}}|y_1-y_2|^\alpha e^{-\gamma\sqrt{|\lambda|}|x-y_1|}. \tag{2.36}
$$

The estimates for II, III and IV are similar. Combining all the estimates obtained, (2.31) follows. Hence, the proof of Lemma 2.6 is complete. □

Finally, we prove the following Hölder continuity estimate for the heat kernel of the semigroup $\{e^{-t\Delta_D}\}$ (see [8]).

Lemma 2.7. *Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set satisfying a* UEBC. *Then for each $\alpha \in (0,1)$, there exist positive constants C and c depending only on α and n, such that the kernel $p_{t,\Delta_D}(x,y)$ of the heat semigroup $e^{-t\Delta_D}$ satisfies the following Hölder continuity estimate:*

$$
\big|p_{t,\Delta_D}(x,y_1)-p_{t,\Delta_D}(x,y_2)\big| \le Ct^{-n/2}\Big(\max_{i=1,2} e^{-c\frac{|x-y_i|^2}{t}}\Big)\Big(\frac{|y_1-y_2|}{\sqrt{t}}\Big)^\alpha \tag{2.37}
$$

for all $t>0$ and $x, y_1, y_2 \in \Omega$.

Proof. Fix some small $\mu \in (0,\pi)$ and for each $\theta \in [-\pi,\pi)\setminus(-\mu,\mu)$ and $R>0$ define

$$
\begin{aligned}
&\Gamma_1 := \{re^{-i\theta} : r \ge R\}, \quad \Gamma_2 := \{Re^{i\phi} : \phi \in [-\pi,\pi)\setminus(-\theta,\theta)\}, \\
&\Gamma_3 := \{re^{i\theta} : r \ge R\}.
\end{aligned} \tag{2.38}
$$

Furthermore, assume that $m \in \mathbb{N}$ is a large integer (e.g., $m \ge \frac{n+3}{2}$ will do). Then using the inverse Laplace transform we have that for each fixed $x, y \in \Omega$, $x \ne y$, and $t>0$,

$$
p_{t,\Delta_D}(x,y) = (-1)^m\frac{(m-1)!}{2\pi i t^{m-1}}\int_{\Gamma_R} e^{\lambda t}R_{\lambda,m}(x,y)\,d\lambda, \tag{2.39}
$$

where

$$
R \ge R(x,y,t) := \max\Big\{\frac{1}{t}, \frac{|x-y|^2}{t^2}\Big\} \quad \text{and} \quad \Gamma_R := \Gamma_1\cup\Gamma_2\cup\Gamma_3. \tag{2.40}
$$

Fix $x, y_1, y_2 \in \Omega$ such that $x \neq y_1$ and $x \neq y_2$, as well as $t > 0$, and choose $R \geq \max\{R(x, y_1, t), R(x, y_2, t)\}$. Using estimate (2.31) in Lemma 2.6, we obtain that for some constants C and γ independent of R one has

$$
\begin{aligned}
\big|p_{t,\Delta_D}(x,y_1) - p_{t,\Delta_D}(x,y_2)\big| &= \left|\frac{(m-1)!}{2\pi i t^{m-1}} \int_{\Gamma_R} e^{\lambda t}\big(R_{\lambda,m}(x,y_1) - R_{\lambda,m}(x,y_2)\big) d\lambda\right| \\
&\leq \frac{C}{t^{m-1}} |y_1 - y_2|^\alpha \int_{\Gamma_R} e^{\mathrm{Re}(\lambda t)} |\lambda|^{-m+\frac{n}{2}+\frac{\alpha}{2}} e^{-\gamma\sqrt{|\lambda|}|x-y_1|}\, d|\lambda| \\
&\quad + \frac{C}{t^{m-1}} |y_1 - y_2|^\alpha \int_{\Gamma_R} e^{\mathrm{Re}(\lambda t)} |\lambda|^{-m+\frac{n}{2}+\frac{\alpha}{2}} e^{-\gamma\sqrt{|\lambda|}|x-y_2|}\, d|\lambda| \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned} \tag{2.41}
$$

Observe that

$$
\begin{aligned}
&\frac{1}{t^{m-1}} \int_{\Gamma_1 \cup \Gamma_3} e^{\mathrm{Re}(\lambda t)} |\lambda|^{-m+\frac{n}{2}+\frac{\alpha}{2}} e^{-\gamma\sqrt{|\lambda|}|x-y_1|}\, d|\lambda| \\
&\qquad \leq \frac{C}{t^{\frac{n+\alpha}{2}}} \int_R^\infty e^{-c'|\lambda| t} \big(|\lambda| t\big)^{-m+\frac{n}{2}+\frac{\alpha}{2}+1} e^{-\gamma\sqrt{|\lambda|}|x-y_1|} \frac{d|\lambda|}{|\lambda|} \\
&\qquad \leq \frac{C}{t^{\frac{n+\alpha}{2}}} e^{-\gamma\sqrt{R}|x-y_1|} e^{-\frac{1}{2}c' Rt} \int_1^\infty e^{-\frac{1}{2}c' s} s^{-m+\frac{n}{2}+\frac{\alpha}{2}}\, ds \\
&\qquad \leq \frac{C}{t^{\frac{n+\alpha}{2}}} e^{-\gamma\sqrt{R}|x-y_1|} e^{-\frac{1}{2}c' Rt}.
\end{aligned} \tag{2.42}
$$

Using the fact that $R \geq \max\big(\frac{1}{t}, \frac{|x-y_1|^2}{t^2}\big)$, the last term is dominated by

$$
\frac{C}{t^{\frac{n+\alpha}{2}}} e^{-c\frac{|x-y_1|^2}{t}}. \tag{2.43}
$$

Again we can bound the third term (i.e., $\int_{\Gamma_2}$) in the term I of (2.41) by

$$
\frac{C}{t^{\frac{n+\alpha}{2}}} \int_{|\lambda|=R} e^{-c' Rt} \big(Rt\big)^{-m+\frac{n}{2}+\frac{\alpha}{2}+1} e^{-\gamma\sqrt{R}|x-y_1|} \frac{d|\lambda|}{|\lambda|}. \tag{2.44}
$$

This term is clearly dominated by $Ct^{-(n+\alpha)/2}\big(Rt\big)^{-m+\frac{n}{2}+\frac{\alpha}{2}+1} e^{-\gamma\frac{|x-y_1|^2}{t}}$ which, in light of the fact that $-m + \frac{n}{2} + \frac{\alpha}{2} + 1 < 0$, yields the following bound on term I:

$$
\mathrm{I} \leq Ct^{-n/2} e^{-c\frac{|x-y_1|^2}{t}} \Big(\frac{|y_1 - y_2|}{\sqrt{t}}\Big)^\alpha, \quad \text{for some } c > 0. \tag{2.45}
$$

Given the goal we have in mind, this is of the right order. Term II is estimated in a similar fashion. When these estimates are combined, (2.37) follows, completing the proof of Lemma 2.7. □

2.2. The Neumann Laplacian

In this subsection we quote without proof from [64] the following estimate for the spatial derivatives of the heat kernel of the semigroup $\{e^{-t\Delta_N}\}$.

Lemma 2.8 ([64]). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex domain. Then the heat semigroup $e^{-t\Delta_N}$ has a kernel $p_t(x,y)$ which, for some positive constants C and c, satisfies*

$$\big|\nabla_x p_{t,\Delta_N}(x,y)\big| \leq \frac{C}{\sqrt{t}\,\big|B^\Omega(x,\sqrt{t})\big|}\exp\Big\{-\frac{|x-y|^2}{ct}\Big\} \tag{2.46}$$

for all $t > 0$ and all $x, y \in \Omega$.

Remark. The proof in [64] is based on reflecting Brownian motion.

3. Hardy spaces associated with the Dirichlet and Neumann Laplacians

Hardy spaces associated with operators were studied by many authors recently, see for examples [6], [7], [9], [30], [39], [40], [41], [45] and [65]. More specifically, we refer the reader to [9] for an extensive study of the Hardy space $h^1_L(\Omega)$ adapted to an operator L which is either the Dirichlet or Neumann Laplacian on a Lipschitz domain Ω of $\mathbb{R}^n$. In this section, we introduce Hardy spaces $h^p_L(\Omega)$ for the range $0 < p \leq 1$, where L is either the Dirichlet or Neumann Laplacian on a doubling open subset Ω of $\mathbb{R}^n$ which can be regarded as a space of homogeneous type when equipped with the Euclidean distance and the n-dimensional Lebesgue measure. While many of the results in this section are already known from [9] in the case $p = 1$, our study here extends the range of p to $0 < p < 1$ which is essential for our work on boundedness of the Green operators. While the estimates for Hardy spaces $h^p_L(\Omega)$ with p being close enough to 1 might not be much different from those of the case $p = 1$, estimates for the case p being close to 0 need to be carried out carefully because we have to work with distributions rather than functions. To overcome this difficulty, we use certain techniques developed recently in [39].

3.1. Hardy spaces via atoms

Let L be either the operator Δ_D or Δ_N on some open subset Ω of $\mathbb{R}^n$. The aim of this section is to define for each index $p \in (0,1]$ a Hardy space, $h^p_L(\Omega)$, associated with the operator L on Ω. In what follows, we assume that $0 < p \leq 1$ and that

$$M \in \mathbb{N} \quad \text{and} \quad M > \Big[\frac{n}{2}\Big(\frac{1}{p} - 1\Big)\Big], \tag{3.1}$$

where $[a]$ is the integer part of $a \in \mathbb{R}$. Let us denote by $\mathcal{D}(T)$ the domain of an unbounded operator T, and by T^k the k-fold composition of T with itself, in the sense of unbounded operators. To simplify notation we shall often just use B^Ω for

$B(x_B, r_B) \cap \Omega$. Also given $\lambda > 0$, we shall write λB^Ω for the λ-dilated ball, which is the ball with the same center as B^Ω and with radius $r_{\lambda B} := \lambda r_B$. Set

$$U_0^\Omega(B) := B^\Omega, \quad \text{and} \quad U_j^\Omega(B) := 2^j B^\Omega \backslash 2^{j-1} B^\Omega \quad \text{for } j = 1, 2, \dots \tag{3.2}$$

Fix a bounded open set $\Omega \subseteq \mathbb{R}^n$ and fix $\kappa_o \in (0, \operatorname{diam}(\Omega)/10)$. We first introduce the notion of a local $(p, 2, M)$-*atom*, $0 < p \leq 1$, associated with a general nonnegative, (possibly unbounded) self-adjoint operator L on $L^2(\Omega)$ with domain $\mathcal{D}(L)$.

Definition 3.1. Let $0 < p \leq 1$. A bounded, measurable function a supported in Ω is called a local $(p, 2, M)$-atom if there exists a ball B of $\mathbb{R}^n$ centered in Ω (but not necessarily included in Ω) with radius $r_B \leq 2 \operatorname{diam}(\Omega)$ such that $\|a\|_{L^2(\mathbb{R}^n)} \leq |B \cap \Omega|^{1/2-1/p}$ and either

(i) $r_B > \kappa_o$; or

(ii) $r_B \leq \kappa_o$ and a is a $(p, 2, M)$-atom, that is, there exists a function $b \in \mathcal{D}(L^M)$ such that $a = L^M b$, $\operatorname{supp}(L^k b) \subset B \cap \overline{\Omega}$, $k = 0, 1, \dots, M$, and

$$\left\|(r_B^2 L)^k b\right\|_{L^2(\mathbb{R}^n)} \leq r_B^{2M} |B \cap \Omega|^{1/2-1/p} \quad \text{for all } k = 0, 1, \dots, M. \tag{3.3}$$

In the sequel we always assume that $\kappa_o = 1$ which can be arranged via an appropriate dilation.

Generally speaking, the elements in the space $h^p(\mathbb{R}^n)$ with $p < 1$ (see, for instance, [35] for a definition), are not functions but rather tempered distributions in $\mathbb{R}^n$. A similar phenomenon occurs for the type of Hardy spaces we have in mind, so it is necessary to introduce an appropriate space of linear functionals prior to defining these Hardy spaces.

Definition 3.2. If $\alpha \geq 0$ and we consider an integer $s \geq \left[\frac{n\alpha}{2}\right]$, then an L^2-integrable function on Ω is said to belong to $\Lambda_L^{\alpha,s}(\Omega)$ if

$$\begin{aligned} \|\ell\|_{\Lambda_L^{\alpha,s}(\Omega)} := & \sup_{\substack{B(x_B, r_B) \\ 0 < r_B < 1, x_B \in \Omega}} \left[\frac{1}{|B \cap \Omega|^{1+2\alpha}} \int_{B\cap\Omega} \left|(I - (I + r_B^2 L)^{-1})^s \ell(x)\right|^2 dx\right]^{1/2} \\ & + \sup_{\substack{B(x_B, r_B) \\ r_B \geq 1, x_B \in \Omega}} \left[\frac{1}{|B \cap \Omega|^{1+2\alpha}} \int_{B\cap\Omega} \left|\ell(x)\right|^2 dx\right]^{1/2} < \infty. \end{aligned} \tag{3.4}$$

In the sequel, we will often write $\mathrm{bmo}_L(\Omega)$ in place of $\Lambda_L^{0,1}(\Omega)$, the adapted space of functions with bounded mean oscillations on Ω.

Remark. We caution the reader that our Λ^α notation differs from the classical one: the "order of smoothness" of our $\Lambda_L^{\alpha,s}$ space is $n\alpha$, not α.

In this case the mapping $\ell \mapsto \|\ell\|_{\Lambda_L^{\alpha,s}(\Omega)}$ is a norm and, as such, $\Lambda_L^{\alpha,s}(\Omega)$ is a normed space. Compared to the classical definition (see, for example, [12] and [43])) the resolvent $(I + r_B^2 L)^{-1}$ plays the role of averaging over the ball, and the power $M > \left[\frac{n}{2}\left(\frac{1}{p} - 1\right)\right]$ provides the necessary "L-cancellation".

Assume that $0 < p < 1$ and that a is a local $(p, 2, M)$-atom supported in a ball B in $\mathbb{R}^n$ which is centered at a point in Ω and has radius $r_B \leq 2\,\mathrm{diam}(\Omega)$. When $r_B < 1$, observe that

$$\frac{(r_B^2 L)^M}{\big(I - \big(1 + r_B^2 L\big)^{-1}\big)^M} = (I + r_B^2 L)^M = \sum_{k=0}^{M} \frac{M!}{(M-k)!\,k!} \big(r_B^2 L\big)^{M-k}, \tag{3.5}$$

which, together with the condition $a = L^M b$ and the fact that L is self-adjoint, gives

$$\begin{aligned}
\big|\langle \ell, a\rangle\big| &= r_B^{-2M}\big|\big\langle \ell,\, (r_B^2 L)^M b\big\rangle\big| \\
&\leq C r_B^{-2M} \sum_{k=0}^{M} \Big| \int_{B\cap\Omega} (I - (I + r_B^2 L)^{-1})^M \ell(x) \overline{(r_B^2 L)^{M-k} b(x)}\, dx \Big| \\
&\leq C r_B^{-2M} \sum_{k=0}^{M} \Big(\int_{B\cap\Omega} |(I - (I + r_B^2 L)^{-1})^M \ell(x)|^2\, dx \Big)^{1/2} \\
&\qquad\qquad \cdot \Big(\int_{B\cap\Omega} |(r_B^2 L)^{M-k} b(x)|^2\, dx \Big)^{1/2} \\
&\leq C \|\ell\|_{\Lambda_L^{1/p-1, M}(\Omega)}.
\end{aligned} \tag{3.6}$$

Above, $\langle \cdot, \cdot \rangle$ denotes the duality pairing in Ω which is compatible with the inner product in $L^2(\Omega)$, and we have used Cauchy–Schwarz's inequality and the L^2-normalization of a. In the case when $r_B \geq 1$, it is clear that we have

$$\big|\langle \ell, a\rangle\big| \leq C \|\ell\|_{\Lambda_L^{1/p-1, M}(\Omega)}.$$

That is, the mapping $\ell \mapsto \int_\Omega a\ell\, dx$ is a bounded linear functional on $\Lambda_L^{1/p-1, M}(\Omega)$ with norm not exceeding a fixed constant C.

We are now ready to introduce the atomic Hardy space $h^p_{L,at,M}(\Omega)$, for $0 < p \leq 1$, associated to the operator L, considered to be as before.

Definition 3.3. Given $p \in (0, 1]$ and $M > \big[\frac{n}{2}\big(\frac{1}{p} - 1\big)\big]$, let $f \in \big(\Lambda_L^{1/p-1, M}(\Omega)\big)^*$. An atomic $(p, 2, M)$-representation of f is a series $f = \sum_j \lambda_j a_j$ where the sequence $\{\lambda_j\}_{j=0}^\infty$ belongs to ℓ^p, a_j is a local $(p, 2, M)$-atom for each nonnegative integer j, and the sum converges in $L^2(\Omega)$. Set

$$\mathscr{S}^p_{L,\mathrm{at},M}(\Omega) := \big\{ f \in L^2(\Omega) :\ f \text{ has an atomic } (p, 2, M)\text{-representation} \big\},$$

with the "norm" (it is a true norm only when $p = 1$), given by

$$||f||_{\mathscr{S}^p_{L,\mathrm{at},M}(\Omega)} = \inf\Big\{ \Big(\sum_{j=0}^\infty |\lambda_j|^p \Big)^{1/p} : f = \sum_{j=0}^\infty \lambda_j a_j \text{ is an atomic } (p, 2, M)\text{-representation} \Big\}.$$

The atomic Hardy space $h^p_{L,at,M}(\Omega)$ is then defined as the completion of the space $\mathscr{S}^p_{L,\mathrm{at},M}(\Omega)$ in $(\Lambda_L^{1/p-1, M}(\Omega))^*$ with respect to the metric induced by $||f||_{\mathscr{S}^p_{L,\mathrm{at},M}(\Omega)}$.

In the above setting the mapping $h \mapsto \|h\|_{h^p_{L,at,M}(\Omega)}$, $0 < p < 1$, clearly fails to be a norm, but $d(h, g) := \|h - g\|_{h^p_{L,at,M}(\Omega)}$ is, nonetheless, a metric. For $p = 1$, the mapping $h \mapsto \|h\|_{h^p_{L,at,M}(\Omega)}$ is indeed a norm. A straightforward argument shows that $h^p_{L,at,M}(\Omega)$ is complete. In particular, $h^1_{L,at,M}(\Omega)$ is a Banach space. It also follows easily from the above definitions that

$$h^p_{L,at,M_2}(\Omega) \subseteq h^p_{L,at,M_1}(\Omega) \tag{3.7}$$

if $0 < p \leq 1$ and $M_1, M_2 \in \mathbb{N}$ satisfy the condition $[\frac{n}{2}(\frac{1}{p} - 1)] < M_1 \leq M_2 < \infty$.

3.2. Hardy spaces via the maximal function

Here the goal is to characterize our previously introduced Hardy spaces by means of the maximal function. Fix an open, bounded subset Ω of $\mathbb{R}^n$ and consider an operator L as before. Given a function $f \in L^2(\Omega)$, consider the following local version of the non-tangential maximal operator associated with the heat semigroup generated by the operator L:

$$N_{\mathrm{loc,\,h}} f(x) := \sup_{y\in\Omega,\, |y-x|<t\leq 1} |e^{-t^2 L} f(y)|, \quad x \in \Omega. \tag{3.8}$$

For $0 < p \leq 1$, the space $h^p_{L,N_{\mathrm{loc,\,h}}}(\Omega)$ is then defined as the completion of $L^2(\Omega)$ in the quasi-norm

$$\|f\|_{h^p_{L,N_{\mathrm{loc,\,h}}}(\Omega)} := \|N_{\mathrm{loc,\,h}} f\|_{L^p(\Omega)}. \tag{3.9}$$

Before stating our first result in this section, we remind the reader of the definition of the class of Lipschitz domains.

Definition 3.4. Let Ω be a nonempty, proper open subset of $\mathbb{R}^n$. Also, fix $x_0 \in \partial\Omega$. Call Ω a Lipschitz domain near x_0 if there exist $b, c > 0$ with the following significance. There exist an $(n-1)$-dimensional affine variety $H \subset \mathbb{R}^n$ passing through x_0, a choice N of the unit normal to H, and an open set

$$\mathcal{C} = \mathcal{C}(x_0, H, N, b, c) := \{x' + tN : x' \in H,\, |x' - x_0| < b,\ |t| < c\}, \tag{3.10}$$

called a local coordinate cylinder near x_0 (with axis along N), such that

$$\mathcal{C} \cap \Omega = \mathcal{C} \cap \{x' + tN : x' \in H,\ t > \varphi(x')\}, \tag{3.11}$$

for some Lipschitz function $\varphi : H \to \mathbb{R}$ satisfying

$$\varphi(x_0) = 0 \quad \text{and} \quad |\varphi(x')| < c/2 \ \text{ if } \ |x' - x_0| \leq b. \tag{3.12}$$

We shall call a bounded open set $\Omega \subseteq \mathbb{R}^n$ a bounded Lipschitz domain if it is a Lipschitz domain near every point $x \in \partial\Omega$.

Then the following result holds:

Theorem 3.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let L be one of the operators Δ_D and Δ_N. Then for every $0 < p \leq 1$ and every $M \in \mathbb{N}$ with $M > \big[\frac{n}{2}\big(\frac{1}{p} - 1\big)\big]$, the spaces $h^p_{L,at,M}(\Omega)$ and $h^p_{L,N_{\mathrm{loc,\,h}}}(\Omega)$ coincide (algebraically and topologically). In particular,*

$$\|f\|_{h^p_{L,at,M}(\Omega)} \approx \|f\|_{h^p_{L,N_{\mathrm{loc,\,h}}}(\Omega)}. \tag{3.13}$$

To prove Theorem 3.5, we need the following lemma. Its proof is similar to that of Lemma 4.3 in [39] and we omit it here.

Lemma 3.6. *Suppose that Ω is an open set in $\mathbb{R}^n$ which satisfies the doubling property* (2.7). *Let $0 < p \le 1$ and fix an integer $M > \left[\frac{n}{2}\left(\frac{1}{p}-1\right)\right]$. Assume that T is a bounded nonnegative sublinear* (*resp. linear*) *operator on $L^2(\Omega)$ which, for every $(p,2,M)$-atom a, satisfies*

$$\|Ta\|_{L^p(\Omega)} \le C, \tag{3.14}$$

with constant C independent of a. Then T extends to a bounded sublinear (*resp. linear*) *operator from $h^p_{L,at,M}(\Omega)$ to $L^p(\Omega)$, and*

$$\|Tf\|_{L^p(\Omega)} \le C\|f\|_{h^p_{L,at,M}(\Omega)}. \tag{3.15}$$

Proof of Theorem 3.5. We proceed in two steps, starting with:

Step I. *Proof of the inclusion of $h^p_{L,at,M}(\Omega) \subseteq h^p_{L,N_{\mathrm{loc,h}}}(\Omega)$ in the case when $M > [\frac{n(1-p)}{2p}]$.* By Lemma 3.6, it suffices to show that there exists $C > 0$ such that for every $(p,2,M)$ atom a associated to a ball $B = B(x_B, r_B)$ in $\mathbb{R}^n$ centered in Ω, we have

$$\|N_{\mathrm{loc,h}}a\|_{L^p(\Omega)} \le C. \tag{3.16}$$

From condition (2.4) (or (2.5)), we have that $N_{\mathrm{loc,h}}a(x) \le C\mathcal{M}_\Omega a(x)$ for almost every $x \in \Omega$, where $\mathcal{M}_\Omega$ denotes the Hardy–Littlewood maximal operator on Ω. By Hölder's inequality,

$$\|N_{\mathrm{loc,h}}a\|^p_{L^p(4B)} \le |4B|^{1-p/2}\|\mathcal{M}_\Omega a\|^p_{L^2(\Omega)} \le C|B|^{1-p/2}\|a\|^p_{L^2(B)} \le C. \tag{3.17}$$

We now proceed to estimate $N_{\mathrm{loc,h}}a(x)$ with $x \notin 4B$ by examining several cases. Introduce $\epsilon := 2M + n(1-\frac{1}{p}) > 0$. To facilitate the subsequent presentation, define

$$N^{(1)}_{\mathrm{loc,h}}f(x) := \sup_{\substack{|x-y|<t\\ 0<t\le r_B}} |e^{-t^2L}f(y)|, \tag{3.18}$$

$$N^{(2)}_{\mathrm{loc,h}}f(x) := \sup_{\substack{|x-y|<t\\ r_B<t<|x-x_B|/4}} |e^{-t^2L}f(y)|, \tag{3.19}$$

$$N^{(3)}_{\mathrm{loc,h}}f(x) := \sup_{\substack{|x-y|<t\\ t\ge|x-x_B|/4}} |e^{-t^2L}f(y)|. \tag{3.20}$$

Case 1. $0 < t \le r_B$. In this scenario we note that for $x \notin 4B$, if we have $|x-y| \le t < |x-x_B|/4$ and $z \in B$, then necessarily $|y-z| \ge |x-x_B|/2$. This permits us to estimate

$$\begin{aligned} N^{(1)}_{\mathrm{loc,h}}a(x) &\le C \sup_{\substack{|x-y|<t\le 1\\ 0<t\le r_B}} \int_B t^{-n}\exp\Big(-\frac{|y-z|^2}{ct^2}\Big)|a(z)|\,dz \\ &\le C \sup_{0<t\le r_B} t^{-n}\exp\Big(-\frac{|x-x_B|^2}{ct^2}\Big)\|a\|_{L^1(B)} \le C\frac{r_B^\epsilon}{|x-x_B|^{\frac{n}{p}+\epsilon}}. \end{aligned} \tag{3.21}$$

Case 2. $r_B < t < |x - x_B|/4$. Since a is a $(p, 2, M)$-atom, we can write $a = L^M b$ for some $b \in \mathcal{D}(L^M)$ satisfying (ii) and (iii) of Definition 3.1. It follows that $\|b\|_{L^1(B)} \le C r_B^{2M+n(1-1/p)}$. On account of (2.10) we then have

$$
\begin{aligned}
N^{(2)}_{\mathrm{loc,h}} a(x) &= \sup_{\substack{|x-y|<t\le 1\\ r_B<t<|x-x_B|/4}} t^{-2M} |(t^2 L)^M e^{-t^2 L} b(y)| \\
&\le C \sup_{\substack{|x-y|<t\le 1\\ r_B<t<|x-x_B|/4}} t^{-2M} \int_B t^{-n} \exp\Big(- \frac{|y-z|^2}{ct^2}\Big) |b(z)|\, dz \\
&\le C \|b\|_{L^1(B)} \sup_{r_B<t<|x-x_B|/4} t^{-2M-n} \exp\Big(- \frac{|x-x_B|^2}{ct^2}\Big) \\
&\le C \frac{r_B^{2M+n(1-\frac{1}{p})}}{|x-x_B|^{\frac{n}{p}+2M+n(1-\frac{1}{p})}} = C \frac{r_B^{\epsilon}}{|x-x_B|^{\frac{n}{p}+\epsilon}}. \qquad (3.22)
\end{aligned}
$$

Case 3. $t \ge |x - x_B|/4$. In this case, $t^{-n} \le C|x - x_B|^{-n}$ for every $z \in B$, and then by (2.10) again,

$$
\begin{aligned}
N^{(3)}_{\mathrm{loc,h}} a(x) &= \sup_{\substack{|x-y|<t\le 1\\ t\ge|x-x_B|/4}} |L^M e^{-t^2 L} b(y)| \\
&\le C \sup_{t\ge|x-x_B|/4} t^{-2M} |x-x_B|^{-n} \|b\|_{L^1(B)} \le C \frac{r_B^{\epsilon}}{|x-x_B|^{\frac{n}{p}+\epsilon}}. \qquad (3.23)
\end{aligned}
$$

Combining the estimates obtained in *Cases* 1, 2 and 3, we therefore conclude that

$$
N_{\mathrm{loc,h}} a(x) \le C \frac{r_B^{\epsilon}}{|x-x_B|^{\frac{n}{p}+\epsilon}}. \qquad (3.24)
$$

Integrating both sides of (3.24) over Ω yields (3.16). This concludes the proof of the fact that the inclusion $h^p_{L,at,M}(\Omega) \subseteq h^p_{L,N_{\mathrm{loc,\,h}}}(\Omega)$ is well defined and continuous.

Step II. *Proof of the inclusion* $h^p_{L,N_{\mathrm{loc,\,h}}}(\Omega) \subseteq h^p_{L,at,M}(\Omega)$. To get started, we first review some preliminary results. Given a function $f \in L^2(\Omega)$, consider the following local version of the quadratic operator associated with the heat semigroup generated by the operator L:

$$
S_{\mathrm{loc,\,h}} f(x) := \Big(\int_0^1 \int_{y\in\Omega,\, |y-x|<t} |t^2 L e^{-t^2 L} f(y)|^2 \frac{dy\, dt}{t^{n+1}} \Big)^{1/2}, \quad x \in \Omega. \qquad (3.25)
$$

Then for every $f \in L^2(\Omega)$,

$$
\|S_{\mathrm{loc,\,h}} f\|_{L^p(\Omega)} \le C \|N_{\mathrm{loc,\,h}} f\|_{L^p(\Omega)}. \qquad (3.26)
$$

The proof of (3.26) follows from an argument analogous to the one given in the proof of Proposition 19 of [9] for the case $p = 1$ (see also [32], [39] and [40])). We omit the details but we wish to stress that it is here that we use the fact that Ω is a bounded Lipschitz domain.

Let us recall that, if L is a nonnegative and self-adjoint operator on $L^2(\Omega)$, and $E_L(\lambda)$ denotes its spectral decomposition, then for every bounded Borel function $F:[0,\infty)\to\mathbb{C}$, one defines the bounded operator $F(L)$, mapping $L^2(\Omega)$ into $L^2(\Omega)$, by the formula

$$F(L) := \int_0^\infty F(\lambda)\,dE_L(\lambda). \tag{3.27}$$

In particular, the operator $\cos(t\sqrt{L})$ is then well defined and bounded on $L^2(\Omega)$. Moreover, it follows from Theorem 3 of [21] and conditions (2.4) and (2.5) that there exists a finite, positive constant c_0 with the property that the Schwartz kernel $K_{\cos(t\sqrt{L})}$ of $\cos(t\sqrt{L})$ satisfies

$$\operatorname{supp} K_{\cos(t\sqrt{L})} \subseteq \big\{(x,y)\in\Omega\times\Omega : |x-y|\le c_0 t\big\}. \tag{3.28}$$

See also [16] and [61]. By the Fourier inversion formula, whenever F is an even, bounded, Borel function with $\hat F\in L^1(\mathbb{R})$, we can write $F(\sqrt{L})$ in terms of $\cos(t\sqrt{L})$. Concretely, by recalling (3.27) we have

$$F(\sqrt{L}) = (2\pi)^{-1}\int_{-\infty}^{\infty}\hat F(t)\cos(t\sqrt{L})\,dt, \tag{3.29}$$

which, when combined with (3.28), gives

$$K_{F(\sqrt{L})}(x,y) = (2\pi)^{-1}\int_{|t|\ge c_0^{-1}|x-y|}\hat F(t)K_{\cos(t\sqrt{L})}(x,y)\,dt, \qquad \forall\, x,y\in\Omega. \tag{3.30}$$

Above, $\hat F$ denotes the Fourier transform of F.

Going further, we state the following useful result (see Lemma 3.5 in [39]).

Lemma 3.7. *Suppose that Ω is an open set in $\mathbb{R}^n$ which satisfies the doubling property (2.7). Let $\varphi\in C_0^\infty(\mathbb{R})$ be an even function satisfying $\int\varphi = 2\pi$ and $\operatorname{supp}\varphi\subset(-c_0^{-1},c_0^{-1})$, where c_0 is the constant in (3.28). For every $m=0,1,2,\dots$, set*

$$\Phi^{(m)}(\xi) := \frac{d^m}{d\xi^m}\Phi(\xi) \quad \text{where} \quad \Phi(\xi) := \hat\varphi(\xi).$$

Then for every integers $\kappa,m=0,1,2,\dots$, and for every $t>0$, the kernel $K_{(t\sqrt{L})^{2\kappa+m}\Phi^{(m)}(t\sqrt{L})}$ of $(t\sqrt{L})^{2\kappa+m}\Phi^{(m)}(t\sqrt{L})$ satisfies

$$\operatorname{supp} K_{(t\sqrt{L})^{2\kappa+m}\Phi^{(m)}(t\sqrt{L})} \subseteq \big\{(x,y)\in\Omega\times\Omega :\ |x-y|\le t\big\}. \tag{3.31}$$

Proof. For all $\kappa,m=0,1,2,\dots$, we set $\Psi^{(m)}_{\kappa,t}(\zeta) := (t\zeta)^{2\kappa+m}\Phi^{(m)}(t\zeta)$. Using the definition of the Fourier transform, it can be verified that

$$\widehat{\Psi^{(m)}_{\kappa,t}}(s) = (-1)^{m+\kappa}\frac{1}{t}\psi^{(m)}_\kappa\Big(\frac{s}{t}\Big),$$

where we have set $\psi^{(m)}_\kappa(s) = \frac{d^{2\kappa+m}}{ds^{2\kappa+m}}\big(s^m\varphi(s)\big)$. Observe that for all numbers $\kappa,m = 0,1,2,\dots$, the function $\Psi^{(m)}_{\kappa,t}\in\mathscr{S}(\mathbb{R})$ is an even function.

It follows from formula (3.30) that

$$K_{(t\sqrt{L})^{2\kappa+m}\Phi^{(m)}(t\sqrt{L})}(x,y)$$
$$= (-1)^{m+\kappa}\frac{1}{2\pi}\int_{|st|\geq c_0^{-1}d(x,y)}\frac{d^{2\kappa+m}}{ds^{2\kappa+m}}\Big(s^m\varphi(s)\Big)K_{\cos(st\sqrt{L})}(x,y)\,ds.$$

Since $\varphi\in C_0^\infty(\mathbb{R})$ and $\operatorname{supp}\varphi\subset(-c_0^{-1},c_0^{-1})$, the claim (3.31) follows readily from this. □

Next we include a brief review of tent spaces on Ω following [17] (see also [58]). If O is an open subset of Ω, then the "tent" over O, denoted by $\widehat{O}$, is defined as

$$\widehat{O}:=\{(x,t)\in\Omega\times(0,\infty):\,d(x,O^c)\geq t\}. \tag{3.32}$$

For a measurable function F defined on $\Omega\times(0,\infty)$, consider

$$\mathcal{A}F(x):=\Big(\int_0^\infty\int_{y\in\Omega,\,|y-x|<t}|F(y,t)|^2\frac{dy\,dt}{t|B^\Omega(y,t)|}\Big)^{1/2},\quad x\in\Omega. \tag{3.33}$$

Given $0<p<\infty$, the "tent space" $T_2^p(\Omega)$ is defined as the space of measurable functions F on $\Omega\times(0,\infty)$, for which $\mathcal{A}F\in L^p(\Omega)$. This space is equipped with the quasi-norm $\|F\|_{T_2^p(\Omega)}:=\|\mathcal{A}F\|_{L^p(\Omega)}$. Observe that $T_2^p(\Omega)$ is a Banach space when $p\in[1,\infty)$, and if $0<p<\infty$ then $T_2^p(\Omega)\cap T_2^2(\Omega)$ is dense in $T_2^p(\Omega)$. A measurable function A on $\Omega\times(0,\infty)$ is said to be a T_2^p-atom if there exists a ball $B^\Omega\subseteq\Omega$ such that A is supported in $\widehat{B^\Omega}$ (defined in (3.32)) and

$$\iint_{\Omega\times(0,\infty)}|A(x,t)|^2\frac{dxdt}{t}\leq|B^\Omega|^{1-\frac{2}{p}}. \tag{3.34}$$

By [58] (which extends the results of [17] to the setting of spaces of homogeneous type), every $F\in T_2^p(\Omega)$ has an atomic decomposition. For further reference, we formally state this result below.

Lemma 3.8. *Let Ω be an open set in $\mathbb{R}^n$ which satisfies the doubling property (2.7) and let $p\in(0,1]$. Then for every element $F\in T_2^p(\Omega)$, there exist a numerical sequence $\{\lambda_j\}_{j=0}^\infty$ and a sequence of T_2^p-atoms $\{A_j\}_{j=0}^\infty$ such that*

$$F=\sum_{j=0}^\infty\lambda_jA_j\quad\text{in } T_2^p(\Omega) \tag{3.35}$$

and

$$\sum_{j=0}^\infty|\lambda_j|^p\leq C\|F\|^p_{T_2^p(\Omega)}. \tag{3.36}$$

For a proof, we refer the reader to Theorem 1.2 in [58].

After this preamble, we begin in earnest the proof of the inclusion $h^p_{L,N_{\mathrm{loc,\,h}}}(\Omega) \subseteq h^p_{L,at,M}(\Omega)$. Let $f \in h^p_{L,N_{\mathrm{loc,\,h}}}(\Omega) \cap L^2(\Omega)$. For every $M \geq 1$, we shall show that there exist a family of local $(p,2,M)$-atoms $\{a_j\}_{j=0}^\infty$ and a sequence of numbers $\{\lambda_j\}_{j=0}^\infty$ such that f can be represented in the form $f = \sum_{j=0}^\infty \lambda_j a_j$, with

$$\|f\|^p_{h^p_{L,at,M}(\Omega)} \leq C \sum_{j=0}^\infty |\lambda_j|^p \leq C\|S_{\mathrm{loc,\,h}} f\|^p_{L^p(\Omega)} \leq C\|f\|^p_{h^p_{L,N_{\mathrm{loc,\,h}}}(\Omega)}, \tag{3.37}$$

where C is independent of f.

Next, let φ, c_0, and Φ be as in Lemma 3.7; recall that $\int \varphi = 2\pi$, and thus $\Phi(0) = \hat{\varphi}(0) = 1$. Using the Faà di Bruno formula (see [63]), we have, for every $m \in \mathbb{Z}_+$,

$$\frac{d^m}{ds^m} e^{-s^2} = e^{-s^2} \sum_{j=0}^{[m/2]} (-1)^{m-j}\, 2^{m-2j} \frac{m!}{(m-2j)!\, j!}\, s^{m-2j}, \tag{3.38}$$

where $[m/2]$ denotes the integer part of $m/2$. The equality (3.38), together with the fact that

$$s^m \frac{d^m}{ds^m}\big(\Phi(s)e^{-s^2}\big) = s^m \sum_{\ell=0}^m \frac{m!}{(m-\ell)!\,\ell!} \frac{d^\ell}{ds^\ell}\Phi(s)\, \frac{d^{m-\ell}}{ds^{m-\ell}} e^{-s^2},$$

shows that

$$\begin{aligned} s^m \frac{d^m}{ds^m}\Big(\Phi(s)e^{-s^2}\Big) &= \sum_{\ell=0}^m \sum_{j=0}^{\left[\frac{m-\ell}{2}\right]} (-1)^{m-\ell-j} 2^{m-\ell-2j} \frac{m!}{(m-\ell)!\,\ell!}\, \frac{(m-\ell)!}{(m-\ell-2j)!\, j!} \\ &\qquad\qquad \times \Big(s^{2(m-j)-\ell} \frac{d^\ell}{ds^\ell}\Phi(s)\Big) e^{-s^2} \\ &= \sum_{\ell=0}^m \sum_{j=0}^{\left[\frac{m-\ell}{2}\right]} c_1(m,\ell,j) \Psi^{(\ell)}_{m,j}(s) e^{-s^2}, \end{aligned} \tag{3.39}$$

where

$$c_1(m,\ell,j) := (-1)^{m-\ell-j} 2^{m-\ell-2j} \frac{m!}{(m-\ell)!\,\ell!}\, \frac{(m-\ell)!}{(m-\ell-2j)!\, j!} \tag{3.40}$$

and

$$\Psi^{(\ell)}_{m,j}(s) := s^{2(m-j)-\ell} \frac{d^\ell}{ds^\ell}\Phi(s). \tag{3.41}$$

A direct computation based on integration by parts, further shows that for every $M \in \mathbb{N}$, we have

$$\begin{aligned} &\int_0^1 \Big(s^{2(M+1)} \frac{d^{2(M+1)}}{ds^{2(M+1)}}\big(\Phi(s)e^{-s^2}\big)\Big)\Big|_{s=tz^{1/2}} \frac{dt}{t} \\ &= \sum_{m=0}^{2M+1} \sum_{\ell=0}^{2M+1-m} \sum_{j=0}^{\left[\frac{2M+1-m-\ell}{2}\right]} c_2(M+1,m,\ell,j) \Psi^{(\ell)}_{2M+1-m,j}(z^{1/2}) e^{-z} + (2M+1)!\,, \end{aligned} \tag{3.42}$$

valid for all $z \neq 0$ in a sector $|\arg z| < \mu$ with $\mu \in (0,\pi)$, and

$$c_2(M+1,m,\ell,j) = (-1)^m \frac{(2M+1)!}{(2M+1-m)!}\, c_1(2M+1-m,\ell,j). \tag{3.43}$$

Then we have the following result:

Lemma 3.9 (Inhomogeneous Calderón type reproducing formula). *For every $M \in \mathbb{N}$ and every $f \in L^2(\Omega)$, we have*

$$f = \frac{1}{(2M+1)!}\,(f_1 - f_2),$$

where

$$f_1 := \int_0^1 \left(s^{2(M+1)} \frac{d^{2(M+1)}}{ds^{2(M+1)}} \big(\Phi(s) e^{-s^2}\big)\right)\Big|_{s=tL^{1/2}} \frac{dt}{t} \tag{3.44}$$

$$= \sum_{\ell=0}^{2(M+1)} \sum_{j=0}^{\left[\frac{2(M+1)-\ell}{2}\right]} c_1(2M+2,\ell,j) \times \int_0^1 (t^2L)^M \Psi^{(\ell)}_{2M+2,\, j+M+1}(tL^{1/2}) t^2 L e^{-t^2L} f \frac{dt}{t}$$

and

$$f_2 := \sum_{m=0}^{2M+1} \sum_{\ell=0}^{2M+1-m} \sum_{j=0}^{\left[\frac{2M+1-m-\ell}{2}\right]} c_2(M+1,m,\ell,j) \Psi^{(\ell)}_{2M+1-m,j}(L^{1/2}) e^{-L} f. \tag{3.45}$$

Here, $c_1(2M+2,\ell,j)$ and $c_2(M+1,m,\ell,j)$ are the constants in (3.40) and (3.43), respectively.

Proof. The proof of the lemma is a consequence of (3.39) and (3.42), by making use of the L^2-functional calculus for L([52]), and is omitted here. □

Also, define

$$F(y,t) := \begin{cases} t^2Le^{-t^2L} f(y) & \text{for } 0 < t \le 1 \text{ and } y \in \Omega, \\ 0 & \text{for } t > 1 \text{ and } y \in \Omega. \end{cases} \tag{3.46}$$

Then $\operatorname{supp} F \subseteq \Omega \times (0,1]$. Since $f \in h^p_{L,N_{\mathrm{loc,\,h}}}(\Omega) \cap L^2(\Omega)$, we have that $F \in T^p_2(\Omega)$. By Lemma 3.8, F has a T^p_2-atomic decomposition, say

$$F = \sum_{i=0}^{\infty} \lambda_i A_i, \tag{3.47}$$

where $\lambda_i \in \mathbb{C}$,

$$\sum_{i=0}^{\infty} |\lambda_i|^p \le C\|F\|^p_{T^p_2(\Omega)} \le C\|S_{\mathrm{loc,\,h}} f\|^p_{L^p(\Omega)} \le C\|f\|^p_{h^p_{L,N_{\mathrm{loc,\,h}}}(\Omega)},$$

and A_i are T_2^p-atoms, i.e., functions supported in $\widehat{B_i^\Omega}$ and

$$\int_{\Omega\times(0,\infty)} |A_i(x,t)|^2 dxdt/t \leq |B_i|^{1-\frac{2}{p}}. \tag{3.48}$$

Therefore, by (3.44)

$$\begin{aligned} f_1 &= \sum_{\ell=0}^{2(M+1)} \sum_{j=0}^{\left[\frac{2(M+1)-\ell}{2}\right]} \sum_{i=0}^{\infty} c_1(2M+2,\ell,j)\lambda_i \\ &\qquad\qquad \times \int_0^1 (t^2L)^M \Psi^{(\ell)}_{2M+2,\,j+M+1}\big(t\sqrt{L}\big)\big(A_i(\cdot,t)\big)\frac{dt}{t} \\ &= \sum_{\ell=0}^{2(M+1)} \sum_{j=0}^{\left[\frac{2(M+1)-\ell}{2}\right]} \sum_{i=0}^{\infty} c_1(2M+2,\ell,j)\lambda_i\, a_{\ell,j,i}, \end{aligned} \tag{3.49}$$

where we have set

$$a_{\ell,j,i} := L^M b_{\ell,j,i} \tag{3.50}$$

and

$$b_{\ell,j,i} := \int_0^1 t^{2M} \Psi^{(\ell)}_{2M+2,\,j+M+1}\big(t\sqrt{L}\big)\big(A_i(\cdot,t)\big)\frac{dt}{t}. \tag{3.51}$$

We claim that, up to normalization by a fixed multiplicative constant, the $a_{\ell,j,i}$'s are $(p,2,M)$-atoms for L. To get started with the proof of the claim, we note that for every $k=0,1,\dots,M$, there holds

$$L^k b_{\ell,j,i} = \int_0^1 t^{2M} L^k \Psi^{(\ell)}_{2M+2,\,j+M+1}\big(t\sqrt{L}\big)\big(A_i(\cdot,t)\big)\frac{dt}{t}. \tag{3.52}$$

By Lemma 3.7, the integral kernel $K_{(t^2L)^k\Psi^{(\ell)}_{2M+2,j+M+1}(t\sqrt{L})}(x,y)$ of the operator $(t^2L)^k\Psi^{(\ell)}_{2M+2,j+M+1}(t\sqrt{L})$ satisfies

$$\operatorname{supp} K_{(t^2L)^k\Psi^{(\ell)}_{2M+2,\,j+M+1}(t\sqrt{L})} \subseteq \big\{(x,y)\in\Omega\times\Omega:\ |x-y|<t\big\}. \tag{3.53}$$

This, together with (3.32) and the fact that $\operatorname{supp} A_i \subset \widehat{B_i^\Omega}$, shows that

$$\operatorname{supp}\,(L^k b_{\ell,j,i}) \subseteq B_i^\Omega, \quad \text{for every} \quad k\in\{0,1,\dots,M\}. \tag{3.54}$$

To continue, for each ball B_i^Ω consider $h\in L^2(B_i^\Omega)$ such that $\|h\|_{L^2(B_j^\Omega)}=1$. Then for every $k = 0, 1, \dots, M$ there holds

$$
\begin{aligned}
&\Big| \int_\Omega (r_{B_i}^2 L)^k b_{\ell,j,i}(x) h(x)\, dx \Big| \\
&\quad = \Big| \int_{\Omega\times(0,\infty)} t^{2M} (r_{B_i}^2 L)^k \Psi^{(\ell)}_{2M+2,\, j+M+1}\big(t\sqrt{L}\big)\big(A_i(\cdot,t)\big)(x) h(x)\, \frac{dx\, dt}{t} \Big| \\
&\quad \le C r_{B_i}^{2M} \int_{\Omega\times(0,\infty)} \Big| A_i(x,t) (t^2 L)^k \Psi^{(\ell)}_{2M+2,\, j+M+1}\big(t\sqrt{L}\big)(h)(x) \Big| \, \frac{dx\, dt}{t} \\
&\quad \le C r_{B_i}^{2M} \Big(\iint_{\Omega\times(0,\infty)} \big|A_i(x,t)\big|^2 \frac{dx\, dt}{t} \Big)^{1/2} \\
&\qquad \times \Big(\iint_{\Omega\times(0,\infty)} \big| (t^2 L)^k \Psi^{(\ell)}_{2M+2,\, j+M+1}\big(t\sqrt{L}\big)(h)(x) \big|^2 \frac{dx\, dt}{t} \Big)^{1/2} \\
&\quad \le C r_{B_i}^{2M} |B_i^\Omega|^{\frac12 - \frac1p} \|h\|_{L^2(\Omega)} \le C r_{B_i}^{2M} |B_i^\Omega|^{\frac12 - \frac1p}. \qquad (3.55)
\end{aligned}
$$

Above, the first inequality is obtained by using the condition $0 < t < r_{B_i}$, the fact that A_i is a T_2^p-atom supported in $\widehat{B_i^\Omega}$ and property (3.48). Since h was arbitrary, this estimate entails $\|(r_{B_i}^2 L)^k b_{\ell,j,i}\|_{L^2(\Omega)} \le C r_{B_i}^{2M} |B_i^\Omega|^{\frac12 - \frac1p}$. Together with (3.54) this shows that each $a_{\ell,j,i}$ is, up to a fixed multiplicative constant, a $(p, 2, M)$-atom for L. This proves our earlier claim.

We now deal with the term f_2 in (3.45). Let Q_0 be the smallest cube containing Ω. We split Q_0 into subcubes, say

$$
Q_0 = \bigcup_{i=1}^{2^{Kn}} Q_0^i \qquad (3.56)
$$

such that each Q_0^i has side-length $\ell(Q_0^i) = \frac{1}{2\sqrt{n}}$ (when K is sufficiently large) and Q_0^i and Q_0^j are disjoint for any $i \ne j$. Suppose $Q_0^1, \cdots, Q_0^N$ are all cubes which intersect the domain Ω; then $\Omega \subset \cup_{i=1}^N Q_0^i$. Let $\{\eta_i\}_{i=1}^N$ be a family of smooth functions satisfying $\eta_i = 1$ in Q_0^i, and $\eta_i = 0$ outside $2Q_0^i$ for each i, and such that $\sum_{i=1}^N \eta_i = 1$ in $\widetilde{\Omega} = \{x : \operatorname{dist}(x, \Omega) < 1/2\}$. For each cube Q_0^i, we take a point $x_{Q_0^i} \in Q_0^i \cap \Omega$. Then f_2 can be further decomposed as

$$
\begin{aligned}
f_2 &= \sum_{m=0}^{2M+1} \sum_{\ell=0}^{2M+1-m} \sum_{j=0}^{\left[\frac{2M+1-m-\ell}{2}\right]} \sum_{i=1}^{N} c_2(M+1, m, \ell, j) \Psi^{(\ell)}_{2M+1-m, j}(L^{1/2})\big(\eta_i\, e^{-L} f\big) \\
&= \sum_{m=0}^{2M+1} \sum_{\ell=0}^{2M+1-m} \sum_{j=0}^{\left[\frac{2M+1-m-\ell}{2}\right]} \sum_{i=1}^{N} \gamma_i\, a_{m,\ell,j,i}, \qquad (3.57)
\end{aligned}
$$

where

$$
\gamma_i := \big| B^\Omega(x_{Q_0^i}, 2) \big|^{\frac1p - \frac12} \sup_{y \in 2Q_0^i} \big| \eta_i e^{-L} f(y) \big| \qquad (3.58)
$$

and

$$a_{m,\ell,j,i} := \gamma_i^{-1} c_2(M+1,m,\ell,j)\Psi^{(\ell)}_{2M+1-m,j}(L^{1/2})\big(\eta_i\, e^{-L}f\big). \tag{3.59}$$

It follows by the spectral theorem that

$$\big\|a_{m,\ell,j,i}\big\|_{L^2(\Omega)} \le C\gamma_i^{-1}\big\|\eta_i\, e^{-L}f\big\|_{L^2(\Omega)} \le C\big|B^\Omega(x_{Q_0^i},2)\big|^{\frac12-\frac1p},$$

since each Q_0^i has side-length $\ell(Q_0^i)=\frac{1}{2\sqrt{n}}$. Then it follows from an argument as in (3.54) that up to a multiplication by a harmless constant, each $a_{m,\ell,j,i}$ is a $(p,2,M)$ atom with $\operatorname{supp} a_{m,\ell,j,i}\subseteq B^\Omega(x_{Q_0^i},2)$. Thus, it remains to check that $\sum_i|\gamma_i|^p<\infty$. Note that

$$\sup_{y\in 2Q_0^i\cap\Omega}\big|e^{-L}f(y)\big| \le \min_{x\in 2Q_0^i\cap\Omega}\ \sup_{y\in\Omega:\,|x-y|<1}\big|e^{-L}f(y)\big|. \tag{3.60}$$

This implies that $|\gamma_i|\le C\min_{x\in 2Q_0^i\cap\Omega}N_{\mathrm{loc,\,h}}f(x)$ and, hence, there holds $|\gamma_i|\le C\,|2Q_0^i\cap\Omega|^{1/p}N_{\mathrm{loc,\,h}}f(x)$ for all $x\in 2Q_0^i\cap\Omega$. Consequently, we have $|\gamma_i|^p\le C\int_{2Q_0^i\cap\Omega}\big|N_{\mathrm{loc,\,h}}f(x)\big|^p\,dx$. We may write

$$\sum_{i=1}^N|\gamma_i|^p \le C\sum_{i=1}^N\int_{2Q_0^i\cap\Omega}\big|N_{\mathrm{loc,\,h}}f(x)\big|^p\,dx \le C\int_\Omega\big|N_{\mathrm{loc,\,h}}f(x)\big|^p\,dx. \tag{3.61}$$

Altogether, this shows that the inclusion $h^p_{L,N_{\mathrm{loc,\,h}}}(\Omega)\subseteq h^p_{L,at,M}(\Omega)$ is well defined and continuous, thus completing the proof of the theorem. $\square$

As a consequence of the theorem just proved, we may write $h^p_{L,at}(\Omega)$ in place of $h^p_{L,at,M}(\Omega)$ for every integer $M\in\mathbb{N}$ with $M>[\frac{n(1-p)}{2p}]$. In fact, Theorem 3.5 suggests making the following definition:

Definition 3.10. Let $\Omega\subseteq\mathbb{R}^n$ be a bounded Lipschitz domain. Let L be one of operators Δ_D and Δ_N on Ω, and assume that $0<p\le 1$. The Hardy space $h^p_L(\Omega)$ is then defined as

$$h^p_L(\Omega) := h^p_{L,at}(\Omega) = h^p_{L,at,M}(\Omega), \quad \forall\, M > \Big[\frac{n(1-p)}{2p}\Big]. \tag{3.62}$$

Finally, we state the following theorem. Its proof is similar to the case $p=1$ treated in Theorem 2.7 of [39] (cf. [41] for a proof in the case $p<1$ in a slightly different context, which may also be adapted to the present setting).

Theorem 3.11. *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded domain satisfying the doubling property (2.7). If $0<p<1$ and $\alpha=1/p-1$, then for every integer $M\in\mathbb{N}$ with $M>\big[\frac{n(1-p)}{2p}\big]$, $\Lambda^{\alpha,M}_L(\Omega)$ is the dual space of $h^p_{L,at,M}(\Omega)$. That is, each continuous linear functional on $h^p_{L,at,M}(\Omega)$ is a mapping of the form $h\mapsto\sum_{j=1}^\infty\lambda_j\int_\Omega a_j\ell\,dx$, where $\ell\in\Lambda^{\alpha,M}_L(\Omega)$ and $h=\sum_{j=1}^\infty\lambda_j a_j\in h^p_{L,at,M}(\Omega)$.*

If $p = 1$, then $\mathrm{bmo}_L(\Omega)$ is the dual space of $h^1_{L,at,M}(\Omega)$. More precisely, if $h = \sum_{j=1}^{\infty} \lambda_j a_j \in h^1_{L,at,M}(\Omega)$, then

$$h^1_{L,at,M}(\Omega) \ni h \mapsto \sum_{j=1}^{\infty} \lambda_j a_j := \lim_{k\to\infty} \sum_{j=1}^{k} \lambda_j \int_\Omega a_j \ell \, dx \tag{3.63}$$

is a well defined continuous linear functional for each $\ell \in \mathrm{bmo}_L(\Omega)$, whose norm is equivalent to $\|\ell\|_{\mathrm{bmo}_L(\Omega)}$. Moreover, each continuous linear functional on $h^1_{L,at,M}(\Omega)$ has this form.

4. Regularity of the inhomogeneous Dirichlet and Neumann problems in the context of Hardy spaces adapted to the Dirichlet and Neumann Laplacians

In this section, we assume that Ω is a bounded semiconvex domain in $\mathbb{R}^n$ (a concept defined in §4.2). The aim is to study the regularity of the inhomogeneous Dirichlet and Neumann problems in the context of the Hardy spaces $h^p_{\Delta_D}(\Omega)$ and $h^p_{\Delta_N}(\Omega)$, $0 < p \leq 1$, respectively.

4.1. Main results

Given an open, bounded subset Ω of $\mathbb{R}^n$ and $f \in C^\infty(\overline{\Omega})$, we denote by $\mathbb{G}_D(f)$ the unique solution in $W^{1,2}_0(\Omega)$ of the inhomogeneous Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

and refer to $\mathbb{G}_D$ as the Dirichlet Green operator.

Theorem 4.1. *Let Ω be a bounded, simply connected, semiconvex domain in $\mathbb{R}^n$. Let $\mathbb{G}_D$ be the Dirichlet Green operator for the problem (4.1). Then the operators*

$$\frac{\partial^2 \mathbb{G}_D}{\partial x_i \partial x_j}, \quad i,j = 1,\dots,n, \tag{4.2}$$

originally defined on $L^2(\Omega) \cap h^p_{\Delta_D}(\Omega)$, can be extended to bounded operators from $h^p_{\Delta_D}(\Omega)$ to $L^p(\Omega)$ for all $0 < p \leq 1$.

If $p = 1$, then the operators $\frac{\partial^2 \mathbb{G}_D}{\partial x_i \partial x_j}$, $i,j = 1,\dots,n$, are also of weak type $(1,1)$. Hence by interpolation, they can be extended as operators from $L^2(\Omega) \cap L^p(\Omega)$ to bounded operators on $L^p(\Omega)$ for $1 < p \leq 2$.

Define the Neumann Green operator $\mathbb{G}_N$ as the solution operator, mapping $f \in C^\infty(\overline{\Omega})$ to $u = \mathbb{G}_N(f) \in W^{1,2}(\Omega)$, for the Neumann problem

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

where it is also assumed that $\int_\Omega f = 0$ and the solution is normalized by requiring that $\int_\Omega u = 0$. Above, $\nu(x)$ denotes the outward unit normal to $\partial\Omega$ at $x \in \partial\Omega$, and $\partial_\nu = \nabla \cdot \nu$ stands for the normal derivative.

Theorem 4.2. *Let Ω be a bounded, simply connected, semiconvex domain in $\mathbb{R}^n$. Let $\mathbb{G}_N$ be the Neumann Green operator for the problem (4.3). Then the operators*

$$\frac{\partial^2 \mathbb{G}_N}{\partial x_i \partial x_j}, \quad i, j = 1, \dots, n, \tag{4.4}$$

originally defined on $\{f \in L^2(\Omega) \cap h^p_{\Delta_N}(\Omega) : \int_\Omega f\,dx = 0\}$, can be extended to bounded operators from $h^p_{\Delta_N}(\Omega)$ into $L^p(\Omega)$ for $0 < p \leq 1$.

If $p = 1$, then the operators $\frac{\partial^2 \mathbb{G}_N}{\partial x_i \partial x_j}$, $i, j = 1, \dots, n$, are also of weak type $(1, 1)$. Hence by interpolation, they can be extended from $L^p(\Omega) \cap L^p(\Omega)$ to bounded operators on $L^2(\Omega)$ for $1 < p \leq 2$.

The general strategy employed in the proofs of the above theorems is as follows. To begin with, if Ω is a bounded semiconvex domain, then we use the fact that for $i, j = 1, \dots, n$, the operators $\frac{\partial^2 \mathbb{G}_D}{\partial x_i \partial x_j}$ extend to bounded operators on $L^2(\Omega)$. In particular, this allows us to define $\frac{\partial^2 \mathbb{G}_D}{\partial x_i \partial x_j}$ on individual local $(p, 2, M)$ atoms associated with the Dirichlet Laplacian Δ_D, and we then proceed to extend the action of these operators to $h^p_{\Delta_D}(\Omega)$ for all $0 < p \leq 1$ by establishing appropriate bounds. The strategy for two derivatives on the Neumann Green operator is similar.

Executing this plan requires that we develop a number of auxiliary tools, many of which have independent interest. More specifically, we first review geometric preliminaries, and coercive estimates in the L^2-setting. Second, we set up machinery capable of treating operators which fall beyond the scope of the classical Calderón–Zygmund theory, along the lines of work in [26], [19] and [5]. This is essential for establishing L^p-boundedness results in the range $1 < p \leq 2$. Third, estimates for the range $0 < p \leq 1$ are established by making use of the properties of heat semigroups and the fact that elements of the Hardy spaces $h^p_{\Delta_D}(\Omega)$ and $h^p_{\Delta_N}(\Omega)$ have atomic decompositions in which the atoms have cancellation properties adapted to the operators in question (Dirichlet and Neumann Laplacians).

We wish to emphasize that our approach for estimating singular integrals with non-smooth kernels is rather flexible since, in principle, it does not differentiate between Dirichlet and Neumann boundary conditions. In this paper, we could only obtain regularity of the Neumann Green operators on convex domains instead of semiconvex domains as in the case of Dirichlet condition, is due to the fact that for Neumann condition, the gradient estimate for the heat kernels in Lemma 2.8 is known only for convex domains, while for the Dirichlet condition, the gradient estimate for the resolvents in Lemma 2.5 is known for more general domains satisfying a uniform exterior ball condition. If one can prove Lemma 2.8 for semiconvex domains, then the proofs in this paper should give regularity result for the Neumann Green operators on semiconvex domains.

4.2. Coercive estimates in semiconvex domains

Here we review a series of definitions and basic results.

Definition 4.3. Let $\mathcal{O}$ be an open set in $\mathbb{R}^n$. The collection of semiconvex functions on $\mathcal{O}$ consists of continuous functions $u : \mathcal{O} \to \mathbb{R}$ with the property that there exists $C > 0$ such that

$$2u(x) - u(x+h) - u(x-h) \leq C|h|^2, \ \forall\, x, h \in \mathbb{R}^n \text{ with } [x-h, x+h] \subseteq \mathcal{O}.$$

The best constant C above is referred to as the semiconvexity constant of u.

Some of the most basic properties of the class of semiconvex functions are collected in the next two propositions below. Proofs can be found in, e.g., [11].

Proposition 4.4. *Assume that $\mathcal{O}$ is an open, convex subset of $\mathbb{R}^n$. Given a function $u : \mathcal{O} \to \mathbb{R}$ and a finite constant $C > 0$, the following conditions are equivalent:*

(i) *u is semiconvex with semiconvexity constant C;*

(ii) *u satisfies*

$$u(\lambda x + (1-\lambda)y) - \lambda u(x) - (1-\lambda)u(y) \leq C\,\frac{\lambda(1-\lambda)}{2}|x-y|^2, \tag{4.5}$$

for all $x, y \in \mathcal{O}$ and all $\lambda \in [0,1]$;

(iii) *the function $\mathcal{O} \ni x \mapsto u(x) + C|x|^2/2 \in \mathbb{R}$ is convex in $\mathcal{O}$;*

(iv) *there exist two functions, $u_1, u_2 : \mathcal{O} \to \mathbb{R}$ such that $u = u_1 + u_2$, u_1 is convex, $u_2 \in C^2(\mathcal{O})$ and $\|\nabla^2 u_2\|_{L^\infty(\mathcal{O})} \leq C$;*

(v) *for any $v \in S^{n-1}$, the (distributional) second order directional derivative of u along v, i.e., $D_v^2 u$, satisfies $D_v^2 u \geq C$ in $\mathcal{O}$, in the sense that*

$$\int_{\mathcal{O}} u(x)\,\big(\mathrm{Hess}_\varphi(x) v\big)\cdot v\,dx \geq C\int_{\mathcal{O}} \varphi(x)\,dx, \ \ \forall\, \varphi \in C_0^\infty(\mathcal{O}), \ \varphi \geq 0, \tag{4.6}$$

where

$$\mathrm{Hess}_\varphi := \Big(\frac{\partial^2\varphi}{\partial x_j \partial x_k}\Big)_{1\leq j,k\leq n}$$

is the Hessian matrix of the function φ;

(vi) *the function u can be represented as $u(x) = \sup_{i\in I} u_i(x)$, $x \in \mathcal{O}$, where $\{u_i\}_{i\in I}$ is a family of functions in $C^2(\mathcal{O})$ with the property that $\|\nabla^2 u_i\|_{L^\infty(\mathcal{O})} \leq C$ for every $i \in I$;*

(vii) *the same as* (vi) *above except that, this time, each function u_i is of the form $u_i(x) = a_i + w_i \cdot x + C|x|^2/2$, for some number $a_i \in \mathbb{R}$ and vector $w_i \in \mathbb{R}^n$.*

We also have:

Proposition 4.5. *Suppose that $\mathcal{O}$ is an open subset of $\mathbb{R}^n$ and that the mapping $u : \mathcal{O} \to \mathbb{R}$ is a semiconvex function. Then the following assertions hold:*

(1) *The function u is locally Lipschitz in $\mathcal{O}$.*

(2) *The gradient of* u *(which, by Rademacher's theorem exists a.e. in* $\mathcal{O}$*), belongs to* $\mathrm{BV}_{loc}(\mathcal{O}, \mathbb{R}^n)$*.*

(3) *The function* u *is twice differentiable a.e. in* $\mathcal{O}$ *(Alexandroff's theorem). More concretely, for a.e. point* x_0 *in* $\mathcal{O}$ *there exists an* $n \times n$ *symmetric matrix* $H_u(x_0)$ *with the property that*

$$\lim_{x \to x_0} \frac{u(x) - u(x_0) - (x - x_0) \cdot \nabla u(x_0) + 2^{-1}\big(H_u(x_0)(x - x_0)\big) \cdot (x - x_0)}{|x - x_0|^2} \tag{4.7}$$

equals zero.

Definition 4.6. A nonempty, proper, bounded open subset Ω of $\mathbb{R}^n$ is called semiconvex provided there exist $b, c > 0$ with the property that for every $x_0 \in \partial\Omega$ there exist an $(n-1)$-dimensional affine variety $H \subset \mathbb{R}^n$ passing through x_0, a choice N of the unit normal to H, and cylinder $\mathcal{C}$ as in (3.10) and some semiconvex function $\varphi : H \to \mathbb{R}$ satisfying (3.11)–(3.12).

It is then clear from Proposition 4.5, Definition 3.4 and Definition 4.6 that bounded semiconvex domains form a subclass of the class of bounded Lipschitz domains. The key features which distinguish the former from the latter are described in the theorem below, which was proved in [53].

Theorem 4.7. *Let* $\Omega \subseteq \mathbb{R}^n$ *be a nonempty, bounded, open set. Then the following conditions are equivalent:*

(i) Ω *is a Lipschitz domain satisfying a* UEBC*;*

(ii) Ω *is a semiconvex domain;*

(iii) Ω *satisfies a* UEBC *and* $\partial\Omega$ *is weakly nontangentially accessible. The latter condition signifies that*

$$\forall\, x \in \partial\Omega, \qquad \exists\, \alpha > 0 \text{ such that } x \in \overline{\gamma_{\Omega,\alpha}(x)}, \tag{4.8}$$

where

$$\gamma_{\Omega,\alpha}(x) := \{y \in \Omega : \ |y - x| < (1 + \alpha)\, \mathrm{dist}\,(y, \partial\Omega)\}. \tag{4.9}$$

The following coercive estimates in semiconvex domains play a basic role in subsequent developments.

Theorem 4.8. *Let* $\Omega \subseteq \mathbb{R}^n$ *be a bounded semiconvex domain with outward unit normal* ν*. If the scalar function* $u \in W^{1,2}(\Omega)$ *is such that* $\Delta u \in L^2(\Omega)$ *and either* $u = 0$ *on* $\partial\Omega$*, or* $\partial_\nu u = 0$ *on* $\partial\Omega$*, then actually* $u \in W^{2,2}(\Omega)$ *and*

$$\|u\|_{W^{2,2}(\Omega)} \leq C\big(\|\Delta u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\big), \tag{4.10}$$

where $C > 0$ *depends only on the Lipschitz character and the uniform exterior ball constant of* Ω*. Moreover, in either case one also has*

$$\|\nabla^2 u\|_{L^2(\Omega)} \leq C\|\Delta u\|_{L^2(\Omega)}. \tag{4.11}$$

The case when u satisfies a homogeneous Dirichlet boundary condition has been known for a while (cf. [1], which builds on the work in [46]) but the case of the homogeneous Neumann boundary condition has only recently been dealt with in [54] (for the case of convex domains see [36]). In fact, in the paper just cited, a much more general result has been proved, in the setting of differential forms. More specifically, the following theorem appears in [54].

Theorem 4.9. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded semiconvex domain with outward unit normal $\nu = (\nu_1, \dots, \nu_n)$ identified with the 1-form $\nu_1 dx_1 + \cdots + \nu_n dx_n$. Fix $\ell \in \{0, 1, \dots, n\}$ and assume that $w \in L^2(\Omega, \Lambda^\ell)$ has the property that $dw \in L^2(\Omega, \Lambda^{\ell+1})$ and $\delta w \in L^2(\Omega, \Lambda^{\ell-1})$, in a distributional sense, where d and δ stand, respectively, for the exterior derivative operator and its formal adjoint. In addition, with "$\wedge$" denoting the exterior product of forms, suppose that $\nu \wedge w = 0$ in a variational sense, i.e., as a functional in the space*

$$W^{-1/2,2}(\partial\Omega, \Lambda^{\ell+1}) := \big(W^{1/2,2}(\partial\Omega, \Lambda^{\ell+1})\big)^*.$$

Then actually $w \in W^{1,2}(\Omega, \Lambda^\ell)$ and

$$\|w\|_{W^{1,2}(\Omega,\Lambda^\ell)} \leq C \left\{ \|dw\|_{L^2(\Omega,\Lambda^{\ell+1})} + \|\delta w\|_{L^2(\Omega,\Lambda^{\ell-1})} + \|w\|_{L^2(\Omega,\Lambda^\ell)} \right\}, \tag{4.12}$$

for some finite constant $C > 0$ which depends only on the Lipschitz character and the uniform exterior ball constant of Ω.

Furthermore, if in addition to the background hypotheses made on Ω so far, it is also assumed that $b_{n-\ell}(\Omega)$, the Betti number of order $n-\ell$, vanishes, then (4.12) can be strengthened to

$$\|\nabla w\|_{L^2(\Omega)} \leq C \left\{ \|dw\|_{L^2(\Omega,\Lambda^{\ell+1})} + \|\delta w\|_{L^2(\Omega,\Lambda^{\ell-1})} \right\}. \tag{4.13}$$

Theorem 4.8 is, essentially, obtained by specializing Theorem 4.9 to the case when $\ell = 0$ and $\ell = n - 1$ (in the latter scenario, an application of the Hodge star isomorphism is also required).

4.3. The tools: singular integrals and weighted estimates of the spatial derivative of the heat kernel

The following theorem is Theorem 1 in [26]:

Theorem 4.10. *Let T be bounded linear operator from $L^2(\Omega)$ into $L^2(\Omega)$ with Ω satisfying the condition (2.7). Assume that there exists a class of operators A_t, $t > 0$, defined on $L^2(\Omega)$ which is represented by kernels $a_t(x, y)$ in the sense that*

(a) *$A_t u(x) = \int_\Omega a_t(x, y) u(y) dy$ for any function $u \in L^2(\Omega) \cap L^1(\Omega)$, and the kernels $a_t(x, y)$ satisfy the following condition:*

$$|a_t(x, y)| \leq \frac{1}{|B^\Omega(y, \sqrt{t})| \big(1 + \frac{|x-y|}{\sqrt{t}}\big)^{n+\beta}}, \quad \forall\, x, y \in \Omega,\ \forall\, t > 0,\ \beta > 0. \tag{4.14}$$

(b) *The composite operator* $T(I - A_t)$ *has a measurable kernel* $\tilde{K}_t(x,y)$*, and there exist constants* C *and* $c > 0$ *such that*

$$\int_{x\in\Omega:|x-y|\geq c\sqrt{t}} |\tilde{K}_t(x,y)|\,dx \leq C, \quad y \in \Omega. \tag{4.15}$$

Moreover, the operator T *is of weak type* $(1,1)$*. Hence* T *can be extended from* $L^2(\Omega) \cap L^p(\Omega)$ *to a bounded operator on* $L^p(\Omega)$ *for all* $1 < p \leq 2$.

4.3.1. Singular integrals. Let Ω be a bounded semiconvex domain in $\mathbb{R}^n$ (hence, a bounded Lipschitz domain satisfying a UEBC) and let L be one of the operators Δ_D and Δ_N on Ω. Let ∇^2 denote a generic second order derivative. From Theorem 4.8 it follows that the operator $T = \nabla^2 L^{-1}$ is bounded on $L^2(\Omega)$. We use the formula

$$Tf = \int_0^\infty \nabla^2 e^{-tL} f\,dt, \tag{4.16}$$

in which the integral converges in the strong operator topology on the space of linear and bounded operators on $L^2(\Omega)$. Indeed, observe that

$$\int_0^\infty \nabla^2 e^{-tL} f\,dt = \nabla^2 L^{-1} \int_0^\infty L e^{-tL} f\,dt = \nabla^2 L^{-1} \int_0^\infty \frac{d}{dt}(-e^{-tL} f)\,dt, \tag{4.17}$$

then use the facts that $||e^{-tL} f - f||_{L^2(\Omega)} \to 0$ as $t \to 0$ and $||e^{-tL} f||_{L^2(\Omega)} \to 0$ as $t \to \infty$.

It follows that the associated kernel of T is given by

$$\int_0^\infty \nabla_x^2 p_{t,L}(x,y)\,dt\,, \tag{4.18}$$

where $p_{t,L}$ is the kernel of e^{-tL}. In the sequel, we assume the following condition (whose validity will be discussed later).

Working assumption: There exists $\gamma > 0$ such that the analytic semigroup e^{-tL} generated by L has kernels $p_{t,L}(x,y)$ satisfying the L^1-estimate

$$\int_{x\in\Omega:\,|x-y|\geq\sqrt{s}} \left|\nabla_x^2 p_{t,L}(x,y)\right| dx \leq Ct^{-1} e^{-\gamma s/t}, \quad \forall y \in \Omega,\ s,t > 0, \tag{4.19}$$

where $C > 0$ is a constant independent of $y \in \Omega$ and $s,t > 0$.

Then the following proposition holds:

Proposition 4.11. *Let* Ω *be a bounded semiconvex domain in* $\mathbb{R}^n$ *and suppose that* (4.19) *holds. Then the operator* $\nabla^2 L^{-1}$ *is bounded from* $h^p_L(\Omega)$ *into* $L^p(\Omega)$ *if* $0 < p \leq 1$ *and there exists a finite, positive constant* C_p *such that*

$$\left\|\nabla^2 L^{-1}(f)\right\|_{L^p(\Omega)} \leq C_p \left\|f\right\|_{h^p_L(\Omega)}, \quad \forall\, f \in h^p_L(\Omega). \tag{4.20}$$

Furthermore, if $p = 1$ *then* $\nabla^2 L^{-1}$ *is of weak type* $(1,1)$*. Hence, by interpolation, this operator can be extended to a bounded mapping on* $L^p(\Omega)$ *whenever* $1 < p \leq 2$.

Proof. Set $T := \nabla^2 L^{-1}$, assume that $0 < p \leq 1$, and fix an arbitrary $f \in h^p_L(\Omega)$. Relying on the atomic decomposition results established for the space $h^p_L(\Omega)$ we may write

$$f = \sum_{r_B \leq 1} \lambda_B a_B + \sum_{r_B > 1} \widetilde{\lambda}_B \widetilde{a}_B =: f_1 + f_2, \tag{4.21}$$

with

$$\sum_{r_B \leq 1} |\lambda_B|^p + \sum_{r_B > 1} |\widetilde{\lambda}_B|^p < +\infty. \tag{4.22}$$

Next, the L^2-theory (cf. the discussion at the beginning of this subsection) shows that for each atom $\widetilde{a}_B$ supported in a ball $B \subseteq \Omega$ with $r_B > 1$, we have

$$T(\widetilde{a}_B) \in L^2(\Omega) \quad \text{and} \quad \|T(\widetilde{a}_B)\|_{L^2(\Omega)} \leq C. \tag{4.23}$$

In particular, $T(f_2) \in L^q(\Omega)$ for all $0 < q \leq 2$, plus a natural estimate.

There remains to study the action of T on f_1 (in which case the fact that each atom supported in a ball B with $r_B \leq 1$ enjoys an "L-cancellation" property is going to be essential). By Lemma 3.6, it is therefore enough to show that for each $(p, 2, M)$ atom a, $0 < p \leq 1$, associated to a ball B as in Definition 3.1, one has $T(a) \in L^p(\Omega)$, and

$$\big\|T(a)\big\|_{L^p(\Omega)} \leq C \tag{4.24}$$

with C independent of a. The verification of (4.24) constitutes much of the bulk of the remaining of the proof. By Hölder's inequality, we can write

$$\big\|T(a)\big\|^p_{L^p(\Omega)} = \sum_{j=0}^{\infty} \Big\| |T(a)|^p \Big\|_{L^1(U_j^\Omega(B^\Omega))} \leq |B^\Omega|^{1-p} \sum_{j=0}^{\infty} 2^{jn(1-p)} \big\|T(a)\big\|^p_{L^1(U_j^\Omega(B^\Omega))}, \tag{4.25}$$

where the annuli $U_j^\Omega(B^\Omega)$ are defined in (3.2). Since the operator T is bounded on $L^2(\Omega)$, we have that for $j = 0, 1, 2$,

$$\big\|T(a)\big\|_{L^1(U_j^\Omega(B^\Omega))} \leq C|B^\Omega|^{1/2} \|a\|_{L^2(B^\Omega)} \leq C|B|^{1-\frac{1}{p}}. \tag{4.26}$$

Fix $j \geq 3$. Since

$$T(a) = \nabla^2 L^{-1}(a) = 2 \int_0^\infty \nabla^2 e^{-2tL}(a)\, dt, \tag{4.27}$$

one can write for each $j \geq 3$,

$$\begin{aligned} &\big\|T(a)\big\|_{L^1(U_j^\Omega(B^\Omega))} \\ &\leq 2 \Big\| \int_0^{r_B^2} \nabla^2 e^{-2tL}(a)\, dt \Big\|_{L^1(U_j^\Omega(B^\Omega))} + 2 \Big\| \int_{r_B^2}^{\infty} \nabla^2 e^{-2tL}(a) dt \Big\|_{L^1(U_j^\Omega(B^\Omega))} \qquad (4.28) \\ &=: \mathrm{I} + \mathrm{II}. \end{aligned}$$

We first estimate I. Since $j \geq 3$, we have that $\operatorname{dist}(U_j^\Omega(B^\Omega), B^\Omega) \geq 2^{j-2} r_B$. Thus, making use of (4.19), we obtain

$$\big\|\nabla^2 e^{-2tL}(a)\big\|_{L^1(U_j^\Omega(B^\Omega))} = \int_{U_j^\Omega(B^\Omega)} \Big| \int_B \nabla_x^2 p_{2t,L}(x,y) a(y)\, dy \Big|\, dx$$

$$\leq \|a\|_{L^1(B^\Omega)} \int_{x\in\Omega:\, |x-y|\geq 2^{j-2} r_B} \Big|\nabla_x^2 p_{2t,L}(x,y)\Big|\, dx \leq C|B|^{1-\frac{1}{p}} t^{-1} e^{-\gamma\frac{(2^j r_B)^2}{2t}}. \tag{4.29}$$

Therefore, for every $M_0 > 0$,

$$\begin{aligned} \mathrm{I} &\leq C|B|^{1-\frac{1}{p}} \int_0^{r_B^2} e^{-\gamma\frac{2^{2j} r_B^2}{2t}} \frac{dt}{t} \\ &\leq C|B|^{1-\frac{1}{p}} \int_0^{r_B^2} \Big(\frac{t}{2^{2j} r_B^2}\Big)^{M_0} \frac{dt}{t} \leq C(M_0) 2^{-2jM_0} |B|^{1-\frac{1}{p}}. \end{aligned} \tag{4.30}$$

To estimate II, we need the following lemma:

Lemma 4.12. *Retain the same background hypotheses as above; in particular, suppose* (4.19) *holds. Then for every $k \in \mathbb{N}$, there exists $\gamma_k > 0$ such that the k-th order time derivative $\frac{d^k}{dt^k} p_{t,L}(x,y)$ of the heat kernel $p_{t,L}$ satisfies*

$$\int_{x\in\Omega:\, |x-y|\geq\sqrt{s}} \Big|\nabla_x^2 \Big(\frac{d^k}{dt^k} p_{t,L}(x,y)\Big)\Big|\, dx \leq C t^{-(k+1)} e^{-\gamma_k s/t}, \tag{4.31}$$

for all $y \in \Omega$ and $s, t > 0$, where $C > 0$ is a constant independent of $y \in \Omega$ and $s, t > 0$.

Proof. We use the commutativity of the semigroup $\{e^{-tL}\}_{t>0}$ to obtain that for every $k \in \mathbb{N}$,

$$\frac{d^k}{dt^k} e^{-2tL} = (-2L)^k e^{-2tL} = 2^k e^{-tL}\Big(\frac{d^k}{dt^k} e^{-tL}\Big).$$

For each fixed $y \in \Omega$ this gives

$$\begin{aligned} &\int_{x\in\Omega:\, |x-y|\geq\sqrt{s}} \Big|\nabla_x^2\Big(\frac{d^k}{dt^k} p_{2t,L}(x,y)\Big)\Big|\, dx \\ &\quad = 2^k \int_{x\in\Omega:\, |x-y|\geq\sqrt{s}} \Big| \int_\Omega \nabla_x^2 p_{t,L}(x,w)\Big(\frac{d^k}{dt^k} p_{t,L}(w,y)\Big) dw\Big|\, dx \\ &\quad \leq C \int_{x\in\Omega:\, |x-y|\geq\sqrt{s}} \int_{w\in\Omega:\, |w-y|\geq\frac{\sqrt{s}}{2}} \Big|\nabla_x^2 p_{t,L}(x,w)\Big(\frac{d^k}{dt^k} p_{t,L}(w,y)\Big)\Big|\, dw\, dx \\ &\qquad + C \int_{x\in\Omega:\, |x-y|\geq\sqrt{s}} \int_{w\in\Omega:\, |w-y|\leq\frac{\sqrt{s}}{2}} \Big|\nabla_x^2 p_{t,L}(x,w)\Big(\frac{d^k}{dt^k} p_{t,L}(w,y)\Big)\Big|\, dw\, dx \\ &\quad =: J_{s,t}^{(1)}(y) + J_{s,t}^{(2)}(y). \end{aligned} \tag{4.32}$$

For the term $J^{(1)}_{s,t}(y)$, we invoke (4.19) and (2.10) in order to write

$$
\begin{aligned}
J^{(1)}_{s,t}(y) &\le C \int_{w\in\Omega:\,|w-y|\ge\frac{\sqrt{s}}{2}} \Big(\int_\Omega \Big|\nabla^2_x p_{t,L}(x,w)\Big|\,dx\Big)\Big|\frac{d^k}{dt^k}p_{t,L}(w,y)\Big|\,dw \\
&\le Ct^{-1}\int_{w\in\Omega:\,|w-y|\ge\frac{\sqrt{s}}{2}} \Big|\frac{d^k}{dt^k}p_{t,L}(w,y)\Big|\,dw \le Ct^{-(k+1)}e^{-\gamma_k s/t} \qquad (4.33)
\end{aligned}
$$

for some $\gamma_k > 0$. On the other hand, observe that if $|x-y|\ge\sqrt{s}$ and $|w-y|\le\sqrt{s}/2$, then $|x-w|\ge\sqrt{s}/2$. The same argument as above also gives that $J^{(2)}_{s,t}(y)\le Ct^{-(k+1)}e^{-\gamma_k s/t}$, and so the desired estimate readily follows. The proof of Lemma 4.12 is complete. □

Back to the proof of Proposition 4.11. Let us consider term II. Pick $M_0\in(\frac{n(1-p)}{2p}, M)$ where $M>\frac{n(1-p)}{2p}$. Since $a=L^Mb$, with the help of Lemma 4.12 we may write

$$
\begin{aligned}
\mathrm{II} &\le C\int_{r_B^2}^\infty \Big\|\nabla^2_x\Big(\frac{d^M}{dt^M}e^{-2tL}b\Big)\Big\|_{L^1(U_j^\Omega(B^\Omega))}\,dt \\
&\le C\,\|b\|_{L^1(B^\Omega)}\int_{r_B^2}^\infty\Big(\int_{|x-y|\ge 2^{j-2}r_B}\Big|\nabla^2_x\Big(\frac{d^M}{dt^M}p_{2t,L}(x,y)\Big|\,dx\Big)\,dt \\
&\le C\,\|b\|_{L^1(B^\Omega)}\int_{r_B^2}^\infty e^{-\gamma_M\frac{2^{2j}r_B^2}{2t}}\frac{dt}{t^{M+1}} \le C\,r_B^{2M}|B|^{1-\frac{1}{p}}\int_{r_B^2}^\infty\Big(\frac{t}{2^{2j}r_B^2}\Big)^{M_0}\frac{dt}{t^{M+1}} \\
&\le C\,2^{-2jM_0}|B|^{1-\frac{1}{p}}, \qquad (4.34)
\end{aligned}
$$

where in the last inequality in (4.34) we have used the condition $M>\frac{n(1-p)}{2p}$. Collecting the estimates obtained for terms I and II, we arrive at the conclusion that for every $M_0\in(\frac{n(1-p)}{2p}, M)$ and $j\ge 3$,

$$
\big\|T(a)\big\|_{L^1(U_j^\Omega(B^\Omega))}\le C\,2^{-2jM_0}|B|^{1-\frac{1}{p}}. \tag{4.35}
$$

In turn, when combined with the estimates for the case when $j\in\{0,1,2\}$, this gives

$$
\big\|T(a)\big\|^p_{L^p(\Omega)}\le C+C\sum_{j=3}^\infty 2^{j(n(1-p)-2M_0p)}\le C. \tag{4.36}
$$

This finishes the proof of (4.24). As a result, T is bounded from $h^p_L(\Omega)$ to $L^p(\Omega)$ for $0<p\le 1$.

We now turn to the proof of the fact that T is of weak type $(1,1)$. The idea is to make use of Theorem 4.10 above. Consider the composite operator $T(I-e^{-tL})$, $t>0$, whose associated integral kernel is denoted by $\tilde{K}_t(x,y)$. Since

$$
T=\nabla^2L^{-1}=\int_0^\infty\nabla^2e^{-sL}\,ds, \tag{4.37}
$$

it follows that

$$T(I-e^{-tL}) = \int_0^\infty \Big(\nabla^2 e^{-sL} - \nabla^2 e^{-(s+t)L}\Big)ds = \int_0^t \nabla^2 e^{-sL}\,ds \tag{4.38}$$

where, in the last step, we have decoupled the integrands and made a change of variables. Thus, by once again invoking (4.19),

$$\begin{aligned}\int_{x\in\Omega,\,|x-y|\geq\sqrt{t}} \big|\tilde{K}_t(x,y)\big|\,dx &\leq \int_{x\in\Omega,\,|x-y|\geq\sqrt{t}} \Big|\int_0^t \nabla^2 p_{s,L}(x,y)\,ds\Big|\,dx \\ &\leq \int_0^t \Big(\int_{x\in\Omega,\,|x-y|\geq\sqrt{t}} \big|\nabla^2 p_{s,L}(x,y)\big|dx\Big)\,ds \leq C\int_0^t s^{-1}e^{-\gamma t/s}\,ds \leq C' \end{aligned}\tag{4.39}$$

for some finite, positive constant C' which is independent of $y\in\Omega$ and $t>0$. With this in hand, and having already established the L^2-boundedness of T, Theorem 4.10 applies and yields that T is of weak type $(1,1)$. This completes the proof of Proposition 4.11. □

4.3.2. Weighted estimates of the spatial derivative of the heat kernel. A key element in the proof of Theorems 4.1 and 4.2 is to verify the "working assumption" (4.19) and in this section we give a condition on the kernel of T guaranteeing that (4.19) holds. We begin by stating the following lemma (see also Lemma 2.1 in [19]).

Lemma 4.13. *Suppose $\Omega\subseteq\mathbb{R}^n$ is a bounded open set satisfying the doubling condition (2.7). Then, for each $\gamma>0$ there exists a finite positive constant C_γ, which is allowed to depend on γ but which is independent of Ω and $y\in\Omega$, with the property that*

$$\int_{x\in\Omega,\,|x-y|\geq\sqrt{s}} e^{-2\gamma\frac{|x-y|^2}{t}}\,dx \leq C_\gamma\, e^{-\gamma s/t}\,|B^\Omega(y,\sqrt{t})|, \quad \forall\, s\geq 0, \quad \forall\, t>0. \tag{4.40}$$

Proof. We have

$$\begin{aligned}\int_{x\in\Omega,\,|x-y|\geq\sqrt{s}} e^{-2\gamma\frac{|x-y|^2}{t}}\,dx &\leq e^{-\gamma s/t}\int_\Omega e^{-\gamma\frac{|x-y|^2}{t}}\,dx \\ &\leq e^{-\gamma s/t}\int_{B^\Omega(y,\sqrt{t})} e^{-\gamma\frac{|x-y|^2}{t}}\,dx \\ &\qquad + e^{-\gamma s/t}\sum_{k=0}^\infty \int_{x\in\Omega,\,2^k\sqrt{t}\leq|x-y|<2^{k+1}\sqrt{t}} e^{-\gamma\frac{|x-y|^2}{t}}\,dx \\ &\leq e^{-\gamma s/t}|B^\Omega(y,\sqrt{t})| + e^{-\gamma s/t}\sum_{k=0}^\infty e^{-\gamma k}\,|B^\Omega(y,2^{k+1}\sqrt{t})| \\ &\leq C_\gamma\, e^{-\gamma s/t}|B^\Omega(y,\sqrt{t})|, \end{aligned}\tag{4.41}$$

by the doubling property of Ω in $\mathbb{R}^n$. □

Note that if L is one of the operators Δ_D and Δ_N on Ω, then for all $\gamma \in (0, 1/4)$ we can apply Lemma 4.13 with $s = 0$ in order to obtain that the kernel $p_{t,L}(x, y)$ of the semigroup e^{-tL} satisfies

$$\int_\Omega \big|p_{t,L}(x,y)\big|^2 e^{\gamma\frac{|x-y|^2}{t}}\,dx \le \frac{C_\gamma}{\big|B^\Omega(y,\sqrt{t})\big|}, \quad \forall\, y \in \Omega,\ \forall\, t > 0. \tag{4.42}$$

By Lemma 2.3 in [19], we also have the following weighted estimate for the first spatial derivative of the heat kernel.

Lemma 4.14. *Let Ω be a bounded open set in $\mathbb{R}^n$. If $L = \Delta_D$ and Ω satisfies the doubling property (2.7) or $L = \Delta_N$ and Ω has the extension property, then for all $\gamma \in (0, 1/4)$, the kernel $p_{t,L}(x, y)$ of the semigroup e^{-tL} satisfies*

$$\int_\Omega \big|\nabla_x p_{t,L}(x,y)\big|^2 e^{\gamma\frac{|x-y|^2}{t}}\,dx \le \frac{C_\gamma}{t\,\big|B^\Omega(y,\sqrt{t})\big|}, \quad \forall\, y \in \Omega,\ \forall\, t > 0, \tag{4.43}$$

with a constant $C_\gamma > 0$ dependent on γ, but independent of Ω or $y \in \Omega$.

We now prove the following proposition:

Proposition 4.15. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let L be one of the operators Δ_D and Δ_N. Suppose also that when $L = \Delta_D$ then Ω also satisfies the doubling property (2.7), while if $L = \Delta_N$ then Ω has the extension property. In addition, assume that for some $\gamma > 0$ there exists a positive constant C_γ, which is allowed to depend on γ but which is independent of Ω and $y \in \Omega$, with the property that the second order spatial derivatives of the heat kernel $p_{t,L}(x, y)$ of e^{-tL} satisfy the L^2-estimate*

$$\int_\Omega \Big|\nabla_x^2 p_{t,L}(x,y)\Big|^2 e^{\gamma\frac{|x-y|^2}{t}}\,dx \le C_\gamma\, t^{-2}|B^\Omega(y,\sqrt{t})|^{-1}, \quad \forall\, y \in \Omega, \forall\, t > 0. \tag{4.44}$$

Then (4.19) holds. More precisely, we have the L^1-estimate

$$\int_{|x-y|\ge\sqrt{s}} \big|\nabla_x^2 p_{t,L}(x,y)\big|\,dx \le C'_\gamma t^{-1} e^{-\gamma s/(4t)}, \quad \forall\, y \in \Omega,\ \forall\, s,t > 0, \tag{4.45}$$

for some positive constant C'_γ which may depend on γ but which is independent of Ω and $y \in \Omega$.

Proof. Fix $s, t > 0$ and $y \in \Omega$. Then using Cauchy–Schwarz's inequality and Lemma 4.13 we obtain

$$\begin{aligned}
&\int_{|x-y|\ge\sqrt{s}} \big|\nabla_x^2 p_{t,L}(x,y)\big|\,dx \\
&\quad\le \Big(\int_\Omega \big|\nabla_x^2 p_{t,L}(x,y)\big|^2 e^{\gamma\frac{|x-y|^2}{t}}\,dx\Big)^{1/2}\Big(\int_{|x-y|\ge\sqrt{s}} e^{-\gamma\frac{|x-y|^2}{t}}\,dx\Big)^{1/2} \\
&\quad\le C'_\gamma\, t^{-1}\, e^{-\gamma s/(4t)}. \qquad (4.46)
\end{aligned}$$

This concludes the proof of Proposition 4.15. ∎

4.4. Proofs of Theorems 4.1 and 4.2

In this section we shall employ Propositions 4.11 and 4.15 in order to prove Theorems 4.1 and 4.2. As already pointed out, $\nabla^2 L^{-1}$ is bounded on $L^2(\Omega)$ when L is one of the operators Δ_D and Δ_N. The strategy used in the proofs of Theorems 4.1 and 4.2 is to check that either (4.19) or (4.44) holds.

4.4.1. The inhomogeneous Dirichlet problem. Fix a bounded semiconvex domain $\Omega \subset \mathbb{R}^n$ and recall that $p_{t,\Delta_D}(x,y)$ denotes the heat kernel of the semigroup $e^{-t\Delta_D}$. Then $p_{t,\Delta_D}(\cdot,y)$ and their time derivatives $\frac{d}{dt}p_{t,\Delta_D}(\cdot,y)$ belong to the domain of Δ_D. In particular, $p_{t,\Delta_D}(\cdot,y) \in W^{1,2}(\Omega)$ and so $p_{t,\Delta_D}(\cdot,y) = 0$ on $\partial\Omega$ for every fixed $y \in \Omega$ and $\nabla p_{t,\Delta_D}(\cdot,y) \in L^2(\Omega)$. Hence, for each fixed $y \in \Omega$, the function $u := p_{t,\Delta_D}(\cdot,y) \in W_0^{1,2}(\Omega)$ is the unique solution of the inhomogeneous Dirichlet problem (4.1) with datum $f := -\frac{d}{dt}p_{t,\Delta_D}(\cdot,y)$, i.e.,

$$\left\{\begin{array}{ll} \Delta u = -\frac{d}{dt}p_{t,\Delta_D}(\cdot,y) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{4.47}$$

Given that Ω is a bounded semiconvex domain, it follows from Theorem 4.8 that $u \in W^{2,2}(\Omega)$, so that $\nabla_x^2 p_{t,\Delta_D}(x,y) \in L^2(\Omega)$. Moreover, there exists a constant C independent of $y \in \Omega$ such that, $\forall\, t > 0$,

$$\left\|\nabla_x^2 p_{t,\Delta_D}(\cdot,y)\right\|_{L^2(\Omega)} \le C\left\|\frac{d}{dt}p_{t,\Delta_D}(\cdot,y)\right\|_{L^2(\Omega)} \le Ct^{-1}\left|B^\Omega(y,\sqrt{t})\right|^{-1/2}, \tag{4.48}$$

where the last step uses the estimate

$$\left|\frac{d}{dt}p_{t,\Delta_D}(x,y)\right| \le \frac{C}{t\left|B^\Omega(y;\sqrt{t})\right|}\exp\Big(-\frac{|x-y|^2}{4t}\Big), \tag{4.49}$$

and (4.40). We next prove the following weighted version of estimate (4.48):

Proposition 4.16. *Let Ω be a semiconvex, bounded open subset of $\mathbb{R}^n$. Then for all $\gamma \in (0,1/8)$, there exists a positive constant C_γ, dependent on γ, but independent of Ω or $y \in \Omega$ such that*

$$\int_\Omega \left|\nabla_x^2 p_{t,\Delta_D}(x,y)\right|^2 e^{\gamma\frac{|x-y|^2}{t}}\,dx \le C_\gamma t^{-2}\left|B^\Omega(y,\sqrt{t})\right|^{-1} \quad \textit{for all } t > 0. \tag{4.50}$$

Before we prove Proposition 4.16, we first apply Theorem 4.9 to obtain the following result:

Lemma 4.17. *Let Ω be a bounded, simply connected, semiconvex, domain in $\mathbb{R}^n$, and assume that $f \in L^2(\Omega)$. Then the unique solution $u \in W_0^{1,2}(\Omega)$ of the partial differential equation $\Delta u = f$ in Ω belongs to $W^{2,2}(\Omega)$ and has the property that for any $\psi \in C^\infty(\mathbb{R}^n)$,*

$$\int_\Omega \psi^2\left|\nabla^2 u\right|^2 dx \le C\int_\Omega \left|\nabla\psi\right|^2\left|\nabla u\right|^2 dx + C\int_\Omega \psi^2 f^2\,dx, \tag{4.51}$$

for some finite constant $C > 0$ *independent of* f. *Moreover,*

$$\int_\Omega \big|\nabla^2 u\big|^2\,dx \le C\int_\Omega f^2\,dx. \tag{4.52}$$

Proof. Fix an arbitrary $f \in L^2(\Omega)$ along with some $\psi \in C^\infty(\mathbb{R}^n)$. The fact that the solution $u \in W_0^{1,2}$ of the problem $\Delta u = f$ in Ω belongs to $W^{2,2}(\Omega)$ is contained in Theorem 4.8. Having established that, we next invoke (the last part of) Theorem 4.9 for the vector field $w := \psi\nabla u$, canonically identified with a 1-form. Note that $w \in W^{1,2}(\Omega,\Lambda^1)$ and

$$\nu\wedge w = \psi\sum_{1\le j<k\le n}(\nu_j\partial_k - \nu_k\partial_j)u\,dx_j\wedge dx_k = 0, \tag{4.53}$$

since $u = 0$ on $\partial\Omega$ and $\nu_j\partial_k - \nu_k\partial_j$ is a tangential differential operator on $\partial\Omega$ for each j,k. Given that we have $|\psi\nabla^2 u| \le |\nabla\psi||\nabla u| + |\nabla w|$, this yields

$$\begin{aligned}\int_\Omega \psi^2\big|\nabla^2 u\big|^2\,dx &\le C\int_\Omega \big|\nabla\psi\big|^2\big|\nabla u\big|^2\,dx + C\int_\Omega |\nabla w|^2\,dx\\ &\le C\int_\Omega \big|\nabla\psi\big|^2\big|\nabla u\big|^2\,dx + C\int_\Omega |\delta w|^2\,dx + C\int_\Omega |dw|^2\,dx.\end{aligned} \tag{4.54}$$

On the other hand, since $d(\nabla u) = d^2u = 0$ and $\delta(\nabla u) = -\Delta u$, we have

$$|\delta w| \le |\psi\Delta u| + |\nabla\psi||\nabla u| \quad\text{and}\quad |dw| \le |\nabla\psi||\nabla u| \quad\text{in } \Omega. \tag{4.55}$$

When combined with (4.54) this readily yields (4.51), after recalling that $\Delta u = f$. Finally, (4.52) follows from (4.51) by taking $\psi \equiv 1$. □

We are ready to prove Proposition 4.16.

Proof of Proposition 4.16. Fix a $\gamma \in (0,1/8)$ and $y \in \Omega$. We apply Lemma 4.17 with

$$u := p_{t,\Delta_D}(\cdot,y),\quad f := \frac{d}{dt}p_{t,\Delta_D}(\cdot,y),\quad\text{and}\quad \psi := e^{\gamma\frac{|\cdot-y|^2}{2t}}, \tag{4.56}$$

to obtain (for some fixed $\gamma' \in (0,\gamma)$)

$$\begin{aligned}\int_\Omega \big|\nabla_x^2 p_{t,\Delta_D}(x,y)\big|^2 e^{\gamma\frac{|x-y|^2}{t}}\,dx &\le \frac{C}{t}\int_\Omega \big|\nabla_x p_{t,\Delta_D}(x,y)\big|^2 e^{\gamma'\frac{|x-y|^2}{t}}\,dx\\ &\quad + C\int_\Omega \Big|\frac{d}{dt}p_{t,\Delta_D}(x,y)\Big|^2 e^{\gamma\frac{|x-y|^2}{t}}\,dx\\ &=: J_t^1(y) + J_t^2(y).\end{aligned} \tag{4.57}$$

By Lemma 4.14, we have that $J_t^1(y) \le Ct^{-2}\big|B^\Omega(y;\sqrt{t})\big|^{-1}$. For the term $J_t^2(y)$, we use the estimate (4.49) in order to obtain, as in (4.42),

$$J_t^2(y) \le Ct^{-2}\big|B^\Omega(y;\sqrt{t})\big|^{-1}. \tag{4.58}$$

The desired estimate (4.50) now follows, proving Proposition 4.16. □

At this stage, it is routine to complete the

Proof of Theorem 4.1. In concert, Propositions 4.16, 4.15 and 4.11 justify Theorem 4.1. □

4.4.2. The inhomogeneous Neumann problem. The proof of Theorem 4.2 follows the lines of the proof of Theorem 4.1 so we will only indicate the main changes. First, we have the following result (for convex domains, this appears as Theorem 2.1 in [3]):

Lemma 4.18. *Let Ω be a bounded, simply connected, semiconvex domain in $\mathbb{R}^n$. Fix $f \in L^2(\Omega)$ with $\int_\Omega f\,dx = 0$ and let u be the unique function in $W^{1,2}(\Omega)$ satisfying $\Delta u = f$ in Ω, $\partial_\nu u = 0$ on $\partial\Omega$ and $\int_\Omega u\,dx = 0$. Then $u \in W^{2,2}(\Omega)$ and for any $\psi \in C^\infty(\mathbb{R}^n)$ there holds*

$$\int_\Omega \psi^2 |\nabla^2 u|^2 \leq C \int_\Omega |\nabla\psi|^2 |\nabla u|^2\,dx + C \int_\Omega \psi^2 f^2\,dx, \tag{4.59}$$

for some finite constant $C > 0$ independent of f. In addition, we also have

$$\int_\Omega |\nabla^2 u|^2 \leq C \int_\Omega f^2\,dx. \tag{4.60}$$

Proof. The fact that, under the hypotheses stipulated in the statement of the lemma, we have $u \in W^{2,2}(\Omega)$ follows from Theorem 4.8. Once this has been established, given a function $\psi \in C^\infty(\mathbb{R}^n)$, we consider the $(n-1)$-form $w := \psi\delta(u\,dx_1 \wedge \cdots \wedge dx_n) \in W^{1,2}(\Omega, \Lambda^{n-1})$ or, more explicitly,

$$w = \psi \sum_{j=1}^n (-1)^j \partial_j u\,dx_1 \wedge \cdots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \cdots \wedge dx_n. \tag{4.61}$$

Note that this choice of w ensures that $\nu \wedge w = \psi(\partial_\nu u)\,dx_1 \wedge \cdots \wedge dx_n = 0$ on $\partial\Omega$. Also, since $d\delta(u\,dx_1 \wedge \cdots \wedge dx_n) = (\Delta u)\,dx_1 \wedge \cdots \wedge dx_n = f\,dx_1 \wedge \cdots \wedge dx_n$ and $\delta^2 = 0$, we obtain

$$|\delta w| \leq |\nabla\psi||\nabla u| \quad \text{and} \quad |dw| \leq |\psi f| + |\nabla\psi||\nabla u| \quad \text{in } \Omega. \tag{4.62}$$

On the other hand, $|\psi\nabla^2 u| \leq |\nabla\psi||\nabla u| + |\nabla w|$ which, in concert with (4.13) and (4.62), allows us to estimate

$$\begin{aligned}
\int_\Omega \psi^2 |\nabla^2 u|^2\,dx &\leq C \int_\Omega |\nabla\psi|^2 |\nabla u|^2\,dx + C \int_\Omega |\nabla w|^2\,dx \\
&\leq C \int_\Omega |\nabla\psi|^2 |\nabla u|^2\,dx + C \int_\Omega |\delta w|^2\,dx + C \int_\Omega |dw|^2\,dx \\
&\leq C \int_\Omega |\nabla\psi|^2 |\nabla u|^2\,dx + C \int_\Omega \psi^2 f^2\,dx. \qquad (4.63)
\end{aligned}$$

This proves (4.59) and (4.60) follows by specializing this to the case when $\psi \equiv 1$. □

Let us now consider the task of estimating the second order spatial derivatives of the heat kernel associated with the Neumann Laplacian. First, it follows from the boundedness of the operators $\frac{d^k}{dt^k}e^{-t\Delta_N} = (-1)^k \Delta_N^k e^{-t\Delta_N}$, $k \in \{0,1\}$, on $L^2(\Omega)$ that the heat kernel $p_{t,\Delta_N}(\cdot,y)$ and its time derivative $\frac{d}{dt}p_{t,\Delta_N}(\cdot,y)$ belong to the domain of the operator Δ_N. In particular, $p_{t,\Delta_N}(\cdot,y) \in W^{1,2}(\Omega)$. When Ω is a bounded, simply connected, semiconvex domain, it follows from Theorem 4.8 that for each fixed $y \in \Omega$, the function $u := p_{t,\Delta_N}(\cdot,y) \in W^{2,2}(\Omega)$ is the unique solution of the Neumann problem (4.3) with $f := -\frac{d}{dt}p_{t,\Delta_N}(\cdot,y)$, i.e.,

$$\left\{ \begin{array}{ll} \Delta u = -\frac{d}{dt}p_{t,\Delta_N}(\cdot,y) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{4.64}$$

In particular, $\nabla_x^2 p_{t,\Delta_N}(x,y) \in L^2(\Omega)$. Granted Proposition 4.11, the key step in the proof of Theorem 4.2 is establishing the following result:

Proposition 4.19. *Let Ω be a bounded, simply connected, semiconvex domain in $\mathbb{R}^n$. Then there exists $\gamma > 0$ and a positive constant C_γ such that for each $y \in \Omega$*

$$\int_\Omega \big|\nabla_x^2 p_{t,\Delta_N}(x,y)\big|^2 e^{\gamma\frac{|x-y|^2}{t}}\,dx \le C_\gamma t^{-2}\big|B^\Omega(y;\sqrt{t})\big|^{-1}, \quad \textit{for every } t > 0. \tag{4.65}$$

Proof. The proof is similar to that of Proposition 4.16, using this time Lemma 4.18 in place of Lemma 4.17. We omit the details. □

We finally present the endgame in the

Proof of Theorem 4.2. This is a consequence of Proposition 4.19, Proposition 4.15, and Proposition 4.11. □

5. Regularity of the inhomogeneous Dirichlet and Neumann problems in the context of the standard Hardy spaces

For $0 < p \le 1$ we let $h^p(\mathbb{R}^n)$ denote the classical (local) Hardy space in $\mathbb{R}^n$. We consider two versions of this space adapted to an arbitrary open subset Ω of $\mathbb{R}^n$. Let $\mathscr{D}(\Omega)$ denote the space of C^∞ functions with compact support in Ω, and let $\mathscr{D}'(\Omega)$ denote its dual, the space of distributions on Ω. The first adaptation of the local Hardy space to Ω, denoted by $h^p_r(\Omega)$, consists of elements of $\mathscr{D}'(\Omega)$ which are the restrictions to Ω of elements of $h^p(\mathbb{R}^n)$. That is, for $0 < p \le 1$ we set

$$\begin{aligned} h^p_r(\Omega) &:= \big\{f \in \mathscr{S}'(\mathbb{R}^n) : \text{there exists } F \in h^p(\mathbb{R}^n) \text{ such that } F|_\Omega = f\big\} \\ &= h^p(\mathbb{R}^n)/\big\{F \in h^p(\mathbb{R}^n) : \ F = 0 \text{ in } \Omega\big\}, \end{aligned} \tag{5.1}$$

which is equipped with the quasi-norm

$$\|f\|_{h^p_r(\Omega)} := \inf\big\{\|F\|_{h^p(\mathbb{R}^n)} : \ F \in h^p(\mathbb{R}^n) \text{ such that } F|_\Omega = f\big\}. \tag{5.2}$$

The second adaptation of the local Hardy space to Ω, denoted by $h^p_z(\Omega)$, consists of distributions f in Ω with the property that (informally speaking) the extension of f by zero to $\mathbb{R}^n$ belongs to $h^p(\mathbb{R}^n)$.

More specifically, for $0 < p \leq 1$ we define

$$h^p_z(\Omega) := h^p(\mathbb{R}^n) \cap \{f \in h^p(\mathbb{R}^n) : f = 0 \text{ on } (\bar{\Omega})^c\} / \{f \in h^p(\mathbb{R}^n) : f = 0 \text{ on } \Omega\}. \tag{5.3}$$

We can identify $h^p_z(\Omega)$ with a set of distributions in $\mathscr{D}'(\Omega)$ which, when equipped with the natural quotient norm, becomes a subspace of $h^p_r(\Omega)$ (see [15], [13] and [56] for more details).

Definition 5.1. Let Ω be an open subset of $\mathbb{R}^n$ and assume that $0 < p \leq 1$. A bounded, measurable function $a : \Omega \to \mathbb{R}$ is called a type (a) local p-atom if it is supported on a cube $Q \subset \Omega$ with side-length $\ell_Q \leq 1$ and such that $4Q \cap \partial\Omega = \emptyset$, and which has the property that $\|a\|_{L^2} \leq |Q|^{1/2-1/p}$ and $\int_Q a(x)x^\alpha \, dx = 0$ for all multi-indices α with $|\alpha| \leq [n(1/p - 1)]$.

We call a measurable function $a : \Omega \to \mathbb{R}$ a type (b) local p-atom provided there exists a cube $Q \subseteq \mathbb{R}^n$ with the property that $a \equiv 0$ on $\Omega \setminus Q$ and for which either $\ell_Q > 1$, or $2Q \cap \partial\Omega = \emptyset$ and $4Q \cap \partial\Omega \neq \emptyset$ and the size condition $\|a\|_{L^2} \leq |Q|^{1/2-1/p}$ (but not necessarily the moment condition) is satisfied.

Various atomic decomposition of Hardy spaces in domains in $\mathbb{R}^n$ have been established in the literature; see [15], [13] and [56]. For example, in the case of $h^p_r(\Omega)$, the following result holds.

Proposition 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and assume that $n/(n+1) < p \leq 1$. Then the following are equivalent:*

(i) $f \in h^p_r(\Omega)$;

(ii) f *has an atomic decomposition*

$$f = \sum_{\text{type (a) atoms}} \lambda_Q a_Q + \sum_{\text{type (b) atoms}} \lambda_Q b_Q$$

with

$$\sum_{\text{type (a) atoms}} |\lambda_Q|^p + \sum_{\text{type (b) atoms}} |\lambda_Q|^p < \infty.$$

5.1. Relations between the Hardy spaces

Suppose that Ω is a bounded semiconvex domain in $\mathbb{R}^n$ and assume that $\frac{n}{n+1} < p \leq 1$. In this subsection we explore the interplay between the spaces $h^p_r(\Omega)$, $h^p_{\Delta_D}(\Omega)$, $h^p_z(\Omega)$ and $h^p_{\Delta_N}(\Omega)$. It is known that the operator Δ_N conserves probability, that is $e^{-t\Delta_N}1 = 1$ (see Chapter 4 of [57]). From Lemma 9.1 in [39], we have that for every $(p, 2, M)$-atom a associated with Δ_N, we have

$$\int_\Omega a(x)\, dx = 0. \tag{5.4}$$

As a consequence,

$$h^p_{\Delta_N}(\Omega) \subseteq h^p_z(\Omega) \subseteq h^p_r(\Omega) \quad \text{whenever} \quad \frac{n}{n+1} < p \leq 1. \tag{5.5}$$

We wish to augment this with the following result.

Proposition 5.3. (i) *Let Ω be a bounded convex domain in $\mathbb{R}^n$. Then $h^p_z(\Omega) = h^p_{\Delta_N}(\Omega)$ for $n/(n+1) < p \leq 1$.*

(ii) *Let Ω be a bounded semiconvex domain in $\mathbb{R}^n$. Then $h^p_r(\Omega) = h^p_{\Delta_D}(\Omega)$ for $n/(n+1) < p \leq 1$.*

Proof. First, based on the regularity of $p_{t,\Delta_N}(x,y)$ discussed in Lemma 2.8 and a standard argument (as in, e.g., [60]), whenever $n/(n+1) < p \leq 1$ we have the inclusion $h^p_z(\Omega) \subseteq h^p_{\Delta_N}(\Omega)$ on a bounded convex domain Ω in $\mathbb{R}^n$. In concert with (5.5) this shows that $h^p_z(\Omega) = h^p_{\Delta_N}(\Omega)$ if $n/(n+1) < p \leq 1$.

Let us now prove the inclusion $h^p_r(\Omega) \subseteq h^p_{\Delta_D}(\Omega)$ for $n/(n+1) < p \leq 1$. From the atomic decomposition in Proposition 5.2 it suffices to prove that if $a : \Omega \to \mathbb{R}$ is either a type (a) local p-atom or a type (b) local p-atom in $h^p_r(\Omega)$ with $n/(n+1) < p \leq 1$ (cf. Definition 5.1) then

$$N_{\mathrm{loc,\,h}}a(x) := \sup_{y\in\Omega,\,|y-x|<t\leq 1} |e^{-t\Delta_D}a(y)| \in L^p(\Omega) \quad \text{and} \quad \|N_{\mathrm{loc,\,h}}a\|_{L^p(\Omega)} \leq C, \tag{5.6}$$

for some finite constant $C > 0$ independent of the atom in question. We shall do so by considering two separate cases.

Case 1. Assume that $a : \Omega \to \mathbb{R}$ is a type (a) local p-atom. In particular, this function is supported in a cube $Q \subset \Omega$, of center x_Q and side-length $\ell_Q \leq 1$, such that $4Q \cap \partial\Omega = \emptyset$, and which satisfies $\|a\|_{L^2} \leq |Q|^{1/2-1/p}$ and $\int_Q a(x)\,dx = 0$. The latter condition allows us to write for each fixed $x \notin 4Q$

$$\begin{aligned} &\sup_{y\in\Omega,\,|y-x|<t\leq 1} |e^{-t\Delta_D}a(y)| \\ &\qquad = \sup_{y\in\Omega,\,|y-x|<t\leq 1} \left| \int_\Omega \big(p_{t,\Delta_D}(y,z) - p_{t,\Delta_D}(y,x_Q)\big)a(z)\,dz \right|. \end{aligned} \tag{5.7}$$

Given that $n/(n+1) < p \leq 1$, we may choose the parameter α from Lemma 2.7 so that $\alpha \in (n(\frac{1}{p}-1), 1)$. Using the regularity of $p_{t,\Delta_D}(x,y)$ established in Lemma 2.7 we obtain from (5.7) that for each $x \notin 4Q$,

$$\sup_{y\in\Omega,\,|y-x|<t\leq 1} |e^{-t\Delta_D}a(y)| \leq C\,\frac{\ell_Q^\alpha}{|x-x_Q|^{n+\alpha}}. \tag{5.8}$$

In turn, this decay estimate gives that $\|N_{\mathrm{loc,\,h}}a\|_{L^p(\Omega\setminus 4Q)} \leq C$, so this piece is of the right order. As far as the contribution from $4Q$ is concerned, we use Hölder's inequality and the L^2 theory to estimate it to the same effect. See for example, Chapter 3 in [60]. This completes the treatment in *Case* 1.

Case 2. Suppose $a : \Omega \to \mathbb{R}$ is a type (b) local p-atom. Hence, there exists a cube $Q \subseteq \mathbb{R}^n$, with center x_Q and side-length ℓ_Q, having the property that the atom in question is supported in $Q \cap \overline{\Omega}$ and for which either $\ell_Q > 1$ or $2Q \cap \partial\Omega = \emptyset$ and $4Q \cap \partial\Omega \neq \emptyset$. In the latter scenario, pick $z_Q \in 4Q \cap \partial\Omega$. Instead of using the moment condition of the atom (as we did before) we now make use of the Dirichlet boundary condition in the form of $p_{t,\Delta_D}(y, z_Q) = 0$, since $z_Q \in \partial\Omega$. Hence,

$$e^{-t\Delta_D}a(y) = \int_\Omega \big(p_{t,\Delta_D}(y,z) - p_{t,\Delta_D}(y,z_Q)\big)a(z)\,dz. \tag{5.9}$$

It follows from the regularity of $p_{t,\Delta_D}(x,y)$ proved in Lemma 2.7 that (5.8) holds. Then, once again, (5.6) follows. Finally, in the case when $\ell_Q > 1$, a direct, crude estimate shows that (5.8) is valid. This completes the proof of the inclusion $h^p_r(\Omega) \subseteq h^p_{\Delta_D}(\Omega)$ when $n/(n+1) < p \le 1$.

The identification between the spaces $h^1_r(\Omega)$ and $h^1_{\Delta_D}(\Omega)$ was proved in Theorem 1 of [9]. Following a suggestion of the referee, we now prove the inclusion $h^p_{\Delta_D}(\Omega) \subseteq h^p_r(\Omega)$ for $n/(n+1) < p < 1$. Let us take an atom a as in Definition 3.1 with support in $Q \cap \overline{\Omega}$ with a cube Q centered in Ω.

Case 1. In case (i), a is immediately a type (b) local p-atom.

Case 2. In case (ii), one can take $M = 1$ and write $a = \Delta_D b$ for some function $b \in \mathcal{D}(L)$. If $4Q \subset \Omega$, then it is a type (a) local p-atom. Indeed, pick $\varphi \in C_0^\infty(\Omega)$ such that $\varphi = 1$ on $2Q$ with support in Ω. From the condition $b \in \mathcal{D}(\Delta_D)$, we have

$$\int_Q a(y)dy = \int_\Omega a(y)\varphi(y)dy = -\int_\Omega \nabla b(y)\,\nabla\varphi(y)dy = 0.$$

If $4Q \cap \partial\Omega \neq \emptyset$, then we may decompose a into a series of type (b) local p-atoms via a Whitney decomposition as in the proof of Proposition 1.5 in [15]; we omit the details here. Hence, Theorem 3.11 and Proposition 5.2 may be used in order to complete the proof of the inclusion $h^p_{\Delta_D}(\Omega) \subseteq h^p_r(\Omega)$ for $n/(n+1) < p < 1$. The proof of Proposition 5.3 is therefore finished. $\square$

Remark 5.4. It should be noted that the proof of the case $p \in \left(\frac{n}{n+1}, 1\right)$ in part (ii) of Proposition 5.3 is also valid for $p = 1$ and, as such, it simplifies the corresponding argument in [9]. (We owe this observation to the referee.)

5.2. Main results

The main goal of this section is to establish regularity results for the Green operators associated with the inhomogeneous Dirichlet and Neumann problems in the context of the standard Hardy spaces $h^p_r(\Omega)$ and $h^p_z(\Omega)$ when $n/(n+1) < p \le 1$.

Theorem 5.5. *Let Ω be a bounded, simply connected, semiconvex domain in $\mathbb{R}^n$ and recall that $\mathbb{G}_D$ stands for the Green operator associated with the inhomogeneous Dirichlet problem* (4.1)*. Then the operators*

$$\frac{\partial^2 \mathbb{G}_D}{\partial x_i \partial x_j}, \quad i,j = 1,\dots,n, \tag{5.10}$$

originally defined on $L^2(\Omega) \cap h^p_{\Delta_D}(\Omega)$, extend as bounded linear mappings from $h^p_r(\Omega)$ to $h^p_r(\Omega)$ whenever $n/(n+1) < p \le 1$.

Now we state the corresponding result for the Neumann problem.

Theorem 5.6. *Let Ω be a bounded, simply connected, convex domain in $\mathbb{R}^n$ and recall that $\mathbb{G}_N$ stands for the Green operator associated with the inhomogeneous*

Neumann problem (4.3). *Then the operators*

$$\frac{\partial^2 \mathbb{G}_N}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n, \tag{5.11}$$

originally defined on $\{f \in L^2(\Omega) \cap h^p_{\Delta_N}(\Omega) \colon \int_\Omega f\,dx = 0\}$, *extend as bounded linear mappings from* $h^p_z(\Omega)$ *to* $h^p_r(\Omega)$ *whenever* $n/(n+1) < p \leq 1$.

5.2.1. The regularity of the inhomogeneous Dirichlet problem in $h^p_r(\Omega)$ on a bounded semiconvex domain. Let $\phi \in C^\infty_0(B(0,1))$ be a nonnegative radial function with the property that $\int_{\mathbb{R}^n} \phi\,dx = 1$. Given an open set $\Omega \subseteq \mathbb{R}^n$, for every point $x \in \Omega$ we denote $d(x) := \operatorname{dist}(x, \Omega^c)$, where $\Omega^c := \mathbb{R}^n \backslash \Omega$. As a preamble to the proof of Theorem 5.5, we first record the following characterization of the membership to the space $h^p_r(\Omega)$ from [56].

Proposition 5.7. *Assume that* $\Omega \subseteq \mathbb{R}^n$ *is open. A distribution* f *is in the Hardy space* $h^p_r(\Omega)$, $0 < p \leq 1$, *if and only if the radial maximal function*

$$f^+(x) := \sup_{0<t<d(x)/2} \big| f * \phi_t(x) \big| \in L^p(\Omega) \tag{5.12}$$

belongs to $L^p(\Omega)$.

Proof of Theorem 5.5. The first part of the proof largely follows [62] and we include it here primarily for the reader's convenience. Recall that the standard (radial) fundamental solution the Laplace operator $\Delta = \sum_{j=1}^n \partial_j^2$ in $\mathbb{R}^n$ is given by

$$\Gamma(x) := \begin{cases} \frac{1}{2\pi} \ln |x| & \text{if } n = 2, \\ \frac{c_n}{|x|^{n-2}} & \text{if } n \geq 3, \end{cases} \tag{5.13}$$

where $c_n := \big[(2-n)\omega_n\big]^{-1}$, and ω_n denotes the area of the unit sphere in $\mathbb{R}^n$. This allows us to solve the Poisson problem for the Laplacian in the whole space via integral operators. Indeed, as is well-known, the Newtonian potential

$$E(f)(x) := \int_\Omega \Gamma(x-y) f(y)\,dy, \quad x \in \Omega, \tag{5.14}$$

satisfies $\Delta\big(E(f)\big) = f$ in Ω, at least if f is reasonably well-behaved.

Next, let Ω be a bounded semiconvex domain in $\mathbb{R}^n$. For each $y \in \Omega$, we let $U(\cdot, y)$ be the solution of the Dirichlet problem

$$\begin{cases} \Delta U(\cdot, y) = 0 & \text{in } \Omega, \\ U(x,y) = \Gamma(y-x) & \text{for } x \in \partial\Omega. \end{cases} \tag{5.15}$$

Then the Green function for Δ_D on Ω (which is the integral kernel of the Dirichlet Green potential $\mathbb{G}_D$) can be expressed as

$$G_D(x,y) = \Gamma(x-y) - U(x,y), \quad x, y \in \Omega, \ x \neq y. \tag{5.16}$$

As a consequence, the solution of the inhomogeneous Dirichlet problem (4.1) is given by the formula

$$\begin{aligned}\mathbb{G}_D(f)(x) &= \int_\Omega G_D(x,y)f(y)\,dy = \int_\Omega \Gamma(x-y)f(y)\,dy - \int_\Omega U(x,y)f(y)\,dy \\ &= E(f)(x) - U(f)(x), \end{aligned} \tag{5.17}$$

where we have set

$$U(f)(x) := \int_\Omega U(x,y)f(y)\,dy, \quad x \in \Omega. \tag{5.18}$$

To prove Theorem 5.5, by Proposition 5.2 it therefore suffices to show that each atom $a \in h^p_r(\Omega)$ (cf. Definition 5.1) satisfies the estimate

$$\Big\| \frac{\partial^2 \mathbb{G}_D(a)}{\partial x_i \partial x_j} \Big\|_{h^p_r(\Omega)} \leq C(p,n), \quad i,j = 1,\dots,n, \tag{5.19}$$

with a constant independent of the actual atom. Hence, the discussion naturally branches out into two separate cases.

Case 1. Assume that $a : \Omega \to \mathbb{R}$ is a type (b) atom supported in a cube $Q \subset \mathbb{R}^n$. On the one hand, when $\ell_Q \geq 1$, it follows from the L^2 theory that (5.19) holds. On the other hand, if $2Q \cap \partial\Omega = \emptyset$ and $4Q \cap \partial\Omega \neq \emptyset$, let x_Q be the center of Q and let ℓ_Q be the side-length of the cube Q. In particular, $3\ell_Q/2 \leq d(x_Q) \leq \sqrt{n}\ell_Q/2$. Consider a family of the dyadic cubes $\{Q_k\}_k$ which make up a Whitney decomposition of Ω (cf., e.g., [60]). It follows that for any $x \in Q_k$, we have

$$1 + \frac{|x - x_Q|}{\ell_Q} \approx 1 + \frac{|x_{Q_k} - x_Q|}{\ell_Q}. \tag{5.20}$$

We claim that if $n(\frac{1}{p} - \frac{1}{2}) < s < \frac{n+2}{2}$, then

$$\int_\Omega \big|\nabla_x^2 \mathbb{G}_D(a)(x)\big|^2 \Big(1 + \frac{|x - x_Q|}{\ell_Q}\Big)^{2s} dx \leq C|Q|^{1-\frac{2}{p}}. \tag{5.21}$$

Let us temporarily assume this claim and show how this is used to obtain (5.19). The details are as follows. Denoting by $\mathbf{1}_k$ be the characteristic function of Q_k, we have

$$\nabla_x^2 \mathbb{G}_D(a) = \sum_k \nabla_x^2 \mathbb{G}_D(a)\mathbf{1}_k = \sum_k \lambda_k a_k, \tag{5.22}$$

where, for each k, we have set

$$\lambda_k := |Q_k|^{\frac{1}{p}-\frac{1}{2}} \big\|\nabla_x^2 \mathbb{G}_D(a)\mathbf{1}_k\big\|_{L^2(\Omega)}, \quad a_k := \lambda_k^{-1} \cdot \frac{\nabla_x^2 \mathbb{G}_D(a)\mathbf{1}_k}{\|\nabla_x^2 \mathbb{G}_D(a)\mathbf{1}_k\|_{L^2(\Omega)}}. \tag{5.23}$$

Obviously, a_k is a type (b) atom. Note that $s \in \big(n(\frac{1}{p} - \frac{1}{2}), \frac{n+2}{2}\big)$ and thus $\frac{2sp}{2-p} > n$.

Based on this, (5.22) and Hölder's inequality, we then obtain

$$\begin{aligned}
\sum_k |\lambda_k|^p &= \sum_k \Big(\int_\Omega \big|\nabla_x^2 \mathbb{G}_D(a)(x)\mathbf{1}_k(x)\big|^2\, dx\Big)^{p/2} |Q_k|^{1-\frac{p}{2}} \\
&\le C \sum_k \Big\{ \int_{Q_k} \big|\nabla_x^2 \mathbb{G}_D(a)(x)\big|^2 \Big(1+\frac{|x-x_Q|}{\ell_Q}\Big)^{2s} dx\Big\}^{p/2} \times \\
&\qquad\qquad \times \Big(1+\frac{|x_{Q_k}-x_Q|}{\ell_Q}\Big)^{-ps} |Q_k|^{1-\frac{p}{2}} \\
&\le C \Big\{ \int_\Omega \big|\nabla_x^2 \mathbb{G}_D(a)(x)\big|^2 \Big(1+\frac{|x-x_Q|}{\ell_Q}\Big)^{2s} dx\Big\}^{p/2} \\
&\qquad\qquad \times \Big\{ \int_\Omega \Big(1+\frac{|x-x_Q|}{\ell_Q}\Big)^{-\frac{2sp}{2-p}} dx\Big\}^{1-\frac{p}{2}} \\
&\le C|Q|^{\frac{p}{2}-1}|Q|^{1-\frac{p}{2}} < +\infty, \qquad (5.24)
\end{aligned}$$

as desired.

There remains to prove our claim in (5.21). Set $E_0 := 2Q \cap \Omega$ and, for every $j \ge 1$, introduce

$$E_j := \big\{x \in \Omega : \ 2^{j-1} d(x_Q) \le |x - x_Q| \le 2^j d(x_Q)\big\}. \qquad (5.25)$$

Next, pick $\varphi \in C_0^\infty(\mathbb{R}^+)$ satisfying $\varphi(t) = 0$ if $t \le 1/2$ or $t \ge 4$, and $\varphi \equiv 1$ if $1 \le t \le 2$. Going further, for $j \ge 1$, set $\varphi_j(x) := \varphi\big(\frac{|x-x_Q|}{2^j \ell_Q}\big)$ for each $x \in \mathbb{R}^n$. Hence, based on support considerations,

$$\varphi_j a = 0 \quad \text{for each} \ \ j \ge 3. \qquad (5.26)$$

Also, a calculation gives that for each $j \ge 1$ we have

$$\begin{aligned}
\Delta\big(\varphi_j \mathbb{G}_D(a)\big) &= (\Delta\varphi_j)\mathbb{G}_D(a) + 2\nabla\varphi_j \nabla(\mathbb{G}_D(a)) + \varphi_j \Delta(\mathbb{G}_D(a)) \\
&= (\Delta\varphi_j)\mathbb{G}_D(a) + 2\nabla\varphi_j \nabla(\mathbb{G}_D(a)) + \varphi_j a. \qquad (5.27)
\end{aligned}$$

Recall from Theorem 4.7 that our domain Ω satisfies a UEBC. Granted this condition, Grüter and Widman have proved (see [38]) that the Green function $G_D(x,y)$ associated with the operator Δ_D obeys the following estimate

$$\big|G_D(x,y)\big| + \big|\nabla_x G_D(x,y)\big||x-y| \le C\frac{d(y)}{|x-y|^{n-1}}, \quad \forall\, x,y \in \Omega. \qquad (5.28)$$

For each $j \ge 3$ we may then estimate

$$\begin{aligned}
\int_{E_j} \big|\nabla_x^2 \mathbb{G}_D(a)(x)\big|^2 dx &\le \int_\Omega \Big|\nabla_x^2 \mathbb{G}_D\Big(\Delta\big(\varphi_j \mathbb{G}_D(a)\big)\Big)(x)\Big|^2 dx \qquad (5.29)\\
\le C\int_\Omega \big|\Delta\big(\varphi_j \mathbb{G}_D(a)\big)(x)\big|^2 dx &\le C\int_\Omega \Big(\big|\mathbb{G}_D(a)(\Delta\varphi_j)\big|^2 + \big|\nabla\varphi_j \nabla\mathbb{G}_D(a)\big|^2\Big) dx.
\end{aligned}$$

The first inequality uses the fact that $\varphi_j \mathbb{G}_D(a) = \mathbb{G}_D\big(\Delta\big(\varphi_j \mathbb{G}_D(a)\big)\big)$ and $\varphi_j \equiv 1$ on E_j. The second inequality is a consequence of (4.11), while the third inequality relies on (5.27) and (5.26).

Next, consider

$$\mathbb{G}_D(a)(x) = \int_Q G_D(x,y)a(y)\,dy \quad \text{for } x \in \operatorname{supp}\varphi_j. \tag{5.30}$$

For each $x \in \operatorname{supp}\varphi_j$ and $y \in Q$ we have $|x-y| \approx 2^j \ell_q$ and $d(y) \leq C\ell_Q$, which when used in combination with (5.28) and the normalization of the atom yields

$$|\mathbb{G}_D(a)(x)| \leq C(2^j\ell_Q)^{1-n}\ell_Q^{n+1-\frac{n}{p}}, \quad \text{for } x \in \operatorname{supp}\varphi_j. \tag{5.31}$$

Similarly,

$$|\nabla\mathbb{G}_D(a)(x)| \leq (2^j\ell_Q)^{-n}\ell_Q^{n+1-\frac{n}{p}}, \quad \text{for } x \in \operatorname{supp}\varphi_j. \tag{5.32}$$

Given that

$$|\nabla\varphi_j| \leq C(2^j\ell_Q)^{-1} \quad \text{and} \quad |\Delta\varphi_j| \leq C(2^j\ell_Q)^{-2}, \tag{5.33}$$

we may further combine (5.31), (5.32) and (5.33) in order to further estimate the last term in (5.29) to obtain

$$\begin{aligned}\int_{E_j} \big|\nabla_x^2\mathbb{G}_D(a)(x)\big|^2\,dx &\leq C\big(2^j\ell_Q\big)^n|Q|^{2(1-\frac{1}{p})}\left\{\frac{\ell_Q^2(2^j\ell_Q)^{2(1-n)}}{\big(2^j\ell_Q\big)^4} + \frac{\ell_Q^2(2^j\ell_Q)^{-2n}}{\big(2^j\ell_Q\big)^2}\right\}\\ &= C2^{-j(2+n)}|Q|^{1-\frac{2}{p}}.\end{aligned} \tag{5.34}$$

It can be verified relying on (4.11) and the normalization of the atom that the above estimate also holds for $j = 0, 1, 2$. Ultimately, this permits us to compute

$$\begin{aligned}\int_\Omega \big|\nabla_x^2\mathbb{G}_D(a)(x)\big|^2\Big(1+\frac{|x-x_Q|}{\ell_Q}\Big)^{2s}dx &= \sum_{j=0}^\infty \int_{E_j}\big|\nabla_x^2\mathbb{G}_D(a)(x)\big|^2\Big(1+\frac{|x-x_Q|}{\ell_Q}\Big)^{2s}dx\\ &\leq \sum_{j=0}^\infty 2^{j(2s-2-n)}|Q|^{1-\frac{2}{p}} = C|Q|^{1-\frac{2}{p}},\end{aligned} \tag{5.35}$$

which gives estimate (5.21). This completes the proof of (5.19) in *Case* 1.

Case 2. Assume that $a : \Omega \to \mathbb{R}^n$ is a type (a) atom. It is then clear from (5.5) that, when extended by zero outside of its support, this function satisfies $\|a\|_{h^p(\mathbb{R}^n)} \leq C$. Given that $n/(n+1) < p \leq 1$, we therefore obtain

$$\Big\|\Big(\frac{\partial^2E(a)}{\partial x_i\partial x_j}\Big)\Big|_\Omega\Big\|_{h_r^p(\Omega)} \leq \Big\|\frac{\partial^2E(a)}{\partial x_i\partial x_j}\Big\|_{h^p(\mathbb{R}^n)} \leq C_{p,n}\|a\|_{h^p(\mathbb{R}^n)} \leq C'_{p,n} \tag{5.36}$$

by classical results in [32] (cf. also [35]). Hence, the proof of (5.19) has been reduced to showing that for $i, j = 1, \dots, n$,

$$\|H_{ij}(a)\|_{h_r^p(\Omega)} \leq C_{p,n}, \quad \text{where} \quad H_{ij}(f) := \frac{\partial^2U(f)}{\partial x_i\partial x_j}. \tag{5.37}$$

From (5.15) and (5.18) we know that $H_{ij}(a)$ is a harmonic function in Ω. Consequently, if we now employ Proposition 5.7 in which we take the function ϕ to be radial, the Mean Value Theorem for harmonic functions gives that

$$\big(H_{ij}(a)\big)^+(x) = \sup_{0<t<d(x)/2} \Big| \int_\Omega H_{ij}(a)(y)\phi_t(x-y)\,dy \Big| = \big|H_{ij}(a)(x)\big|. \tag{5.38}$$

This, in combination with Proposition 5.7, shows that

$$\begin{aligned} \big\|H_{ij}(a)\big\|_{h^p_r(\Omega)} &= \Big\|\big(H_{ij}(a)\big)^+\Big\|_{L^p(\Omega)} = \|H_{ij}(a)\|_{L^p(\Omega)} = \Big\|\frac{\partial^2 U(a)}{\partial x_i \partial x_j}\Big\|_{L^p(\Omega)} \\ &\le \Big\|\frac{\partial^2 E(a)}{\partial x_i \partial x_j}\Big\|_{L^p(\Omega)} + \Big\|\frac{\partial^2 \mathbb{G}_D(a)}{\partial x_i \partial x_j}\Big\|_{L^p(\Omega)}. \end{aligned} \tag{5.39}$$

For the first term in the last line of (5.39), we once again make use of the results of [32] in order obtain (recall that $\frac{n}{n+1} < p \le 1$)

$$\Big\|\frac{\partial^2 E(a)}{\partial x_i \partial x_j}\Big\|_{L^p(\Omega)} \le \Big\|\frac{\partial^2 E(a)}{\partial x_i \partial x_j}\Big\|_{L^p(\mathbb{R}^n)} \le C_{p,n}. \tag{5.40}$$

Consider the second term in the last line of (5.39). Since $\frac{n}{n+1} < p \le 1$, Proposition 5.3 applies and gives that $a \in h^p_r(\Omega) \subseteq h^p_{\Delta_D}(\Omega)$ as well as the estimate $\|a\|_{h^p_{\Delta_D}(\Omega)} \le \|a\|_{h^p_r(\Omega)} \le c$, for some $c = c(\Omega, n, p) > 0$, finite constant. By Theorem 4.1, we therefore see that for all $i, j = 1, 2, \dots, n$, and $\frac{n}{n+1} < p \le 1$,

$$\Big\|\frac{\partial^2 \mathbb{G}_D(a)}{\partial x_i \partial x_j}\Big\|_{L^p(\Omega)} \le C_{p,n}\|a\|_{h^p_{\Delta_D}(\Omega)} \le C'_{p,n}, \tag{5.41}$$

which suits our purpose. The desired estimate (5.19) now readily follows. Hence, the proof of Theorem 5.5 is complete. □

5.2.2. The regularity of the inhomogeneous Neumann problem in $h^p_z(\Omega)$ on a bounded convex domain. We first prove the following result:

Theorem 5.8. *Assume that Ω is a bounded, simply connected, semiconvex domain in $\mathbb{R}^n$. Then the operators (5.11), originally defined on the space $\{f \in L^2(\Omega) \cap h^p_{\Delta_N}(\Omega) : \int_\Omega f\,dx = 0\}$, extend as bounded linear mappings from $h^p_{\Delta_N}(\Omega)$ into $h^p_r(\Omega)$ whenever $n/(n+1) < p \le 1$.*

Proof. For each $y \in \Omega$, we let $V(\cdot, y)$ be the solution of the Neumann problem

$$\begin{cases} \Delta V(\cdot, y) = |\Omega|^{-1} & \text{in} \quad \Omega, \\ \partial_{\nu(x)}[V(x,y)] = \partial_{\nu(x)}[\Gamma(x-y)] & \text{for} \quad x \in \partial\Omega. \end{cases} \tag{5.42}$$

Then a convenient way of expressing the Green function for Δ_N in Ω (i.e., the integral kernel of the Neumann Green potential $\mathbb{G}_N$) is

$$G_N(x,y) = \Gamma(x-y) - V(x,y), \quad x, y \in \Omega, \;\; x \ne y. \tag{5.43}$$

The Neumann problem (4.3) has a unique solution, up to an additive constant, given by the formula

$$\mathbb{G}_N(f)(x) = \int_\Omega G_N(x,y)f(y)\,dy = \int_\Omega \Gamma(x-y)f(y)\,dy - \int_\Omega V(x,y)f(y)\,dy$$
$$= E(f)(x) - V(f)(x), \tag{5.44}$$

where $E(f)$ is the Newtonian potential introduced in (5.14) and we have set

$$V(f)(x) := \int_\Omega V(x,y)f(y)\,dy, \quad x \in \Omega. \tag{5.45}$$

To prove Theorem 5.8, by Lemma 3.6 it suffices to show that for every $(p,2,M)$-atom a associated with Δ_N (cf. Definition 3.1) satisfies

$$\Big\|\frac{\partial^2 \mathbb{G}_N(a)}{\partial x_i \partial x_j}\Big\|_{h^p_r(\Omega)} \le C(p,n), \quad i,j = 1,\dots,n, \tag{5.46}$$

with a constant independent of the actual atom. With this in mind, we proceed as in *Case* 2 of the proof of Theorem 5.5 and obtain

$$\Big\|\frac{\partial^2 \mathbb{G}_N(a)}{\partial x_i \partial x_j}\Big\|_{h^p_r(\Omega)} \le \Big\|\frac{\partial^2 E(a)}{\partial x_i \partial x_j}\Big\|_{h^p_r(\Omega)} + \Big\|\frac{\partial^2 V(a)}{\partial x_i \partial x_j}\Big\|_{h^p_r(\Omega)} \le C'_{p,n} + \Big\|\frac{\partial^2 V(a)}{\partial x_i \partial x_j}\Big\|_{h^p_r(\Omega)}. \tag{5.47}$$

Recall from (5.42) that $\frac{\partial^2 V(a)}{\partial x_i \partial x_j}$ is harmonic in Ω. Hence, an application of Proposition 5.7 in which we take the function ϕ to be radial yields, on account of the Mean Value Property for harmonic functions, that

$$\Big(\frac{\partial^2 V(a)}{\partial x_i \partial x_j}\Big)^+(x) = \sup_{0<t<d(x)/2}\Big|\int_\Omega \frac{\partial^2 V(a)}{\partial y_i \partial y_j}(y)\phi_t(x-y)dy\Big| = \Big|\frac{\partial^2 V(a)}{\partial x_i \partial x_j}(x)\Big|. \tag{5.48}$$

This, in combination with Proposition 5.7, shows that

$$\Big\|\frac{\partial^2 V(a)}{\partial x_i \partial x_j}\Big\|_{h^p_r(\Omega)} = \Big\|\Big(\frac{\partial^2 V(a)}{\partial x_i \partial x_j}\Big)^+\Big\|_{L^p(\Omega)} = \Big\|\frac{\partial^2 V(a)}{\partial x_i \partial x_j}\Big\|_{L^p(\Omega)}$$
$$\le \Big\|\frac{\partial^2 E(a)}{\partial x_i \partial x_j}\Big\|_{L^p(\Omega)} + \Big\|\frac{\partial^2 \mathbb{G}_N(a)}{\partial x_i \partial x_j}\Big\|_{L^p(\Omega)}. \tag{5.49}$$

Given that $\frac{n}{n+1} < p \le 1$, (5.40) holds and this takes care of the first term. Consider the second term. It follows from Theorem 4.2 that

$$\Big\|\frac{\partial^2 \mathbb{G}_N(a)}{\partial x_i \partial x_j}\Big\|_{L^p(\Omega)} \le C_{p,n}, \quad 0 < p \le 1, \tag{5.50}$$

which is of the right order. The desired estimate, (5.46), therefore follows and, hence, the proof of Theorem 5.8 is finished. □

Finally, it is now easy to complete the

Proof of Theorem 5.6. From (i) of Proposition 5.3 and Theorem 5.8, Theorem 5.6 readily follows. □

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