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Geometric-arithmetic averaging of dyadic weights

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Abstract

The theory of Muckenhoupt's weight functions arises in many areas of analysis, for example in connection with bounds for singular integrals and maximal functions on weighted spaces. We prove that a certain averaging process gives a method for constructing A_p weights from a measurably varying family of dyadic A_p weights. This averaging process is suggested by the relationship between the A_p weight class and the space of functions of bounded mean oscillation. The same averaging process also constructs weights satisfying reverse Hölder (RH_p) conditions from families of dyadic RH_p weights, and extends to the polydisc as well.

1. Introduction

Several classes of functions are defined in terms of a property that the function must satisfy on each interval, with a uniform constant. Well known examples from harmonic analysis and complex analysis include Muckenhoupt's A_p weights, the reverse-Hölder weight classes RH_p, the class of doubling weights, and the space BMO of functions of bounded mean oscillation. Such classes have strictly larger dyadic analogues, where the defining property is required only on dyadic intervals. Certain types of averaging provide a bridge between these dyadic counterparts and the original function classes. Specifically, these averages convert each suitable family of functions in the dyadic class to a single function in the smaller, nondyadic class. We can think of averaging as an improving operation, in this sense.

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An easily stated example is the following. If all translates of a function f defined on the unit circle $\mathbb{T} = [0, 1]$ are in dyadic BMO, or equivalently if f is in dyadic BMO on every translated grid of dyadic intervals on the circle, then the function f itself is in true BMO. This result is a special case of a theorem in [5], applied to the identity

$$f(x) = \int_0^1 \tau_t f(x+t)\, dt,$$

where for $t \in \mathbb{R}$ the translation operator τ_t is defined by $\tau_t f(\cdot) := f(\cdot - t)$, and $x + t$ is to be interpreted as $x + t \mod 1$.

Now, what if a function f on $\mathbb{T}$ can be written as the *translation-average*

$$f(x) := \int_0^1 f^t(x+t)\, dt$$

of dyadic BMO functions $\{f^t\}_{t\in[0,1]}$ that are *not* identical translates of each other? If they satisfy the hypotheses of [5], then still f is in true BMO. However, the analogous statements can fail for A_p weights, for RH_p weights, and for doubling weights [16].

In this paper we show that a different type of averaging works for both A_p and RH_p (Theorems 1 and 2). This is the *geometric-arithmetic average* defined by

$$\Omega(x) := \exp\Big\{ \int_0^1 \log \omega^t(x+t)\, dt \Big\},$$

where $\{\omega^t\}_{t\in[0,1]}$ is a suitable family of weights in A_p^d or RH_p^d.

We also observe that translation-averaging does work for A_p and also for RH_p, under the additional assumption that the functions ω^t are doubling weights, not just dyadic doubling weights (Theorem 3). Equivalently, the translation-average of A_p weights is an A_p weight, and similarly for RH_p.

All these results generalize to the polydisc (Theorems 4 and 5).

The paper is organized as follows. In Section 2, we state our geometric-arithmetic averaging results on the circle. In Section 3, we collect the definitions and background results used in the paper. Also, Lemma 1 in that section gives a unified characterization of weights in A_p for $1 \le p \le \infty$, RH_p for $1 < p \le \infty$, and their dyadic counterparts, in terms of conditions on the oscillation of their logarithms. We take some care throughout in tracing the dependence of the various constants, although some of our bounds on the constants may not be sharp. In Section 4, we prove geometric-arithmetic averaging for A_p^d and RH_p^d weights (Theorems 1 and 2). In Section 5, we prove translation-averaging for A_p^d and RH_p^d weights that are doubling (Theorem 3). In Section 6, we generalize our results to the polydisc (Theorems 4 and 5).

2. The geometric-arithmetic average on the circle

In [16], examples are constructed to show that, for $\{\omega^t\}_{t\in[0,1]}$ a measurably varying family of dyadic A_p^d weights (for arbitrary p with $1 \leq p \leq \infty$) or dyadic RH_p^d weights (for arbitrary p with $1 < p < \infty$) on the circle $\mathbb{T} = [0,1]$, with uniformly bounded dyadic A_p^d or RH_p^d constants, the *translation-average* $\omega(x) := \int_0^1 \omega^t(x+t)\,dt$ is not necessarily a doubling weight. Therefore ω need not be in true A_p, nor in true RH_p.

The main result of the current paper is that, by contrast, the *geometric-arithmetic average* $\Omega(x) := \exp\{\int_0^1 \log \omega^t(x+t)\,dt\}$ always turns a measurably varying family of suitably normalized dyadic A_p^d weights into an A_p weight (for arbitrary p with $1 \leq p \leq \infty$), and a measurably varying family of suitably normalized dyadic RH_p^d weights into an RH_p weight (for arbitrary p with $1 < p \leq \infty$).

Theorem 1. *Fix p with $1 \leq p \leq \infty$. Let $\{\omega^t\}_{t\in[0,1]}$ be a family of dyadic A_p weights on the circle $\mathbb{T}$, $\omega^t \in A_p^d(\mathbb{T})$, such that*

(i) the mapping $t \mapsto \omega^t$ is measurable,

(ii) an appropriate average of the logarithms of the weights ω^t is finite:

$$\int_0^1\int_0^1 \big|\log \omega^t(x)\big|\,dx\,dt < \infty,$$

and

(iii) the A_p^d constants $A_p^d(\omega^t)$ are uniformly bounded, independent of $t \in [0,1]$.

Then the geometric-arithmetic average

$$\Omega(x) = \exp\Big\{\int_0^1 \log \omega^t(x+t)\,dt\Big\}$$

of the dyadic weights ω^t belongs to A_p on $\mathbb{T}$. Moreover, the A_p constant of Ω depends only on p and on the bound on the A_p^d constants of the ω^t.

Remark. A simple heuristic motivation for this result is as follows. The weights ω^t belong to A_p^d, so their logarithms $\log \omega^t$ are in BMO_d. Therefore, as shown in [5] (and later [13] for the one- and two-parameter settings and [15] for the general multiparameter setting), the translation-average $\log \Omega$ of the functions $\log \omega^t$ is in BMO, and so by the John–Nirenberg Theorem [8] sufficiently small powers Ω^δ of Ω are in A_p. It remains to show that Ω itself is in A_p.

Hypothesis (ii) of the theorem is merely a normalization condition: the fact that each $\omega^t(x)$ belongs to A_p^d already implies that $\log \omega^t(x)$ belongs to $L^1(dx)$.

The analogous result to Theorem 1 holds for reverse-Hölder weights.

Theorem 2. *Fix p with $1 < p \leq \infty$. Let $\{\omega^t\}_{t\in[0,1]}$ be a family of dyadic RH_p weights on the circle $\mathbb{T}$, $\omega^t \in RH_p^d(\mathbb{T})$, such that hypotheses (i) and (ii) of* Theorem 1 *hold, and*

(iii′) the RH_p^d constants $RH_p^d(\omega^t)$ are uniformly bounded, independent of $t \in [0,1]$.

Then the geometric-arithmetic average $\Omega(x) = \exp\{\int_0^1 \log \omega^t(x+t)\,dt\}$ of the dyadic weights ω^t lies in RH_p on $\mathbb{T}$. The RH_p constant of Ω depends only on p and on the bound on the RH_p^d constants of the ω^t.

These results also hold on $\mathbb{T}^k$, with constants that depend on the dimension k. In addition, they hold in the setting of the polydisc; see Section 6.

Remark. It is not necessary for the integral in t to be taken over the whole interval $[0,1]$. The proofs below go through without change when the integral is taken over an arbitrary subset $E \subset [0,1]$ of positive measure.

3. Definitions and tools

In this section we collect useful material about doubling weights, the weight classes A_p and RH_p, and their relationship to BMO. For fuller accounts of the theory of A_p and RH_p weights, see for example [3], [4], [6], and [2].

Let $\mathbb{T}$ denote the unit circle, obtained by identifying the endpoints of the interval $[0,1]$. In the definitions of our averages Ω and ω, $x+t$ is to be interpreted as $x+t \mod 1$.

Denote the collection of dyadic subintervals I of the circle $\mathbb{T}$ by $\mathcal{D}=\mathcal{D}[0,1]$:

$$\mathcal{D} := \{[0,1]\} \cup \left\{ I = \left[\frac{j}{2^k}, \frac{j+1}{2^k}\right) \Big| k \in \mathbb{N}, j \in \{0,1,\ldots,2^k-1\} \right\}.$$

Throughout the paper, Q denotes a general subinterval of $\mathbb{T}$, while I, J, K and L denote dyadic subintervals of $\mathbb{T}$.

The functions we consider are real-valued.

We use the symbol $|E|$ to denote the Lebesgue measure of a set E, the symbol $⨍_E$ to denote $\frac{1}{|E|}\int_E$, and the symbol f_E for the average value $⨍_E f$ of a function f on a set E. The notation $E \subset F$ includes the possibility $E = F$.

Definition 1. Let $\omega(x)$ be a nonnegative locally integrable function on the circle $\mathbb{T}$. We say ω is a *doubling weight* with *doubling constant* C if for all intervals $Q \subset \mathbb{T}$

$$\int_{\widetilde{Q}} \omega(x)\,dx \leq C \int_Q \omega(x)\,dx,$$

where $\widetilde{Q}$ is the *double* of Q; that is, $\widetilde{Q}$ is the interval with the same midpoint as Q and twice the length of Q. We say ω is a *dyadic doubling weight* with *dyadic doubling constant* C if the analogous inequality holds for all dyadic intervals $I \subset \mathbb{T}$, where $\widetilde{I}$ is the *dyadic double* of I: that is, $\widetilde{I}$ is unique dyadic interval of length $|\widetilde{I}| = 2|I|$ that contains I.

The A_p weights were identified by Muckenhoupt [12] as the weights ω for which the Hardy–Littlewood maximal function is bounded from $L^p(d\mu)$ to itself, where $d\mu = \omega(x)\,dx$. Here we give the definitions of the classes A_p and RH_p on the circle $\mathbb{T}$; the analogous definitions hold on $\mathbb{R}$ and on (one-parameter) $\mathbb{R}^k$. We delay the corresponding definitions for the polydisc setting until Section 6.

Definition 2. Let $\omega(x)$ be a nonnegative locally integrable function on the circle $\mathbb{T}$. For real p with $1 < p < \infty$, we say ω is an A_p *weight*, written $\omega \in A_p$, if

$$A_p(\omega) := \sup_Q \Big(\fint_Q \omega \Big) \Big(\fint_Q \Big(\frac{1}{\omega}\Big)^{1/(p-1)} \Big)^{p-1} < \infty.$$

For $p = 1$, we say ω is an A_1 *weight*, written $\omega \in A_1$, if

$$A_1(\omega) := \sup_Q \Big(\fint_Q \omega \Big) \Big(\frac{1}{\operatorname{ess\,inf}_{x\in Q} \omega(x)} \Big) < \infty.$$

For $p = \infty$, we say ω is an *A-infinity weight*, written $\omega \in A_\infty$, if

$$A_\infty(\omega) := \sup_Q \Big(\fint_Q \omega \Big) \exp \Big(\fint_Q \log \Big(\frac{1}{\omega}\Big) \Big) < \infty.$$

Here the suprema are taken over all intervals $Q \subset \mathbb{T}$. The quantity $A_p(\omega)$ is called the A_p *constant of* ω.

The *dyadic* A_p *classes* A_p^d for $1 \le p \le \infty$ are defined analogously, with the suprema $A_p^d(\omega)$ being taken over only the dyadic intervals $I \subset \mathbb{T}$.

Definition 3. Let $\omega(x)$ be a nonnegative locally integrable function on the circle $\mathbb{T}$. For real p with $1 < p < \infty$, we say ω is a *reverse-Hölder-p weight*, written $\omega \in RH_p$ or $\omega \in B_p$, if

$$RH_p(\omega) := \sup_Q \Big(\fint_Q \omega^p \Big)^{1/p} \Big(\fint_Q \omega \Big)^{-1} < \infty.$$

For $p = \infty$, we say ω is a *reverse-Hölder-infinity weight*, written $\omega \in RH_\infty$ or $\omega \in B_\infty$, if

$$RH_\infty(\omega) := \sup_Q \Big(\operatorname*{ess\,sup}_{x\in Q} \omega(x) \Big) \Big(\fint_Q \omega \Big)^{-1} < \infty.$$

Here the suprema are taken over all intervals $Q \subset \mathbb{T}$. The quantity $RH_p(\omega)$ is called the RH_p *constant* of ω.

For $1 < p \le \infty$, we say ω is a *dyadic reverse-Hölder-p weight*, written $\omega \in RH_p^d$ or $\omega \in B_p^d$, if the analogous condition

$$\sup_{I\in\mathcal{D}} \Big(\fint_I \omega^p \Big)^{1/p} \Big(\fint_I \omega \Big)^{-1} < \infty \quad \text{or} \quad \sup_{I\in\mathcal{D}} \Big(\operatorname{ess\,sup}_{x\in I} \omega(x) \Big) \Big(\fint_I \omega \Big)^{-1} < \infty$$

holds with the supremum being taken over only the dyadic intervals $I \subset \mathbb{T}$, and if in addition ω is a dyadic doubling weight. We define the RH_p^d *constant* $RH_p^d(\omega)$ of ω to be the larger of this dyadic supremum and the dyadic doubling constant.

The A_p inequality (or the RH_p inequality) implies that the weight ω is doubling, and the dyadic A_p inequality implies that ω is dyadic doubling. However, the dyadic RH_p inequality does not imply that ω is dyadic doubling, which is why the dyadic doubling assumption is needed in the definition of RH_p^d.

The A_p classes are nested and increasing with p, while the RH_p classes are nested and decreasing with p. Moreover,

$$A_q(\omega) \le A_p(\omega), \qquad \text{for } 1 \le p < q \le \infty,$$

and

$$RH_p(\omega) \le RH_q(\omega), \qquad \text{for } 1 < p < q \le \infty.$$

Also

$$A_\infty = \bigcup_{p\ge 1} A_p = \bigcup_{q>1} RH_q, \qquad A_1 \subsetneq \bigcap_{p>1} A_p, \qquad RH_\infty \subsetneq \bigcap_{p>1} RH_p.$$

The dyadic versions of the assertions in this paragraph also hold.

The example $w(x) = [\log(1/|x|)]^{-1}$ (for x near zero) cited in [9] shows that the inclusion of A_1 in $\bigcap_{p>1} A_p$ is proper.

The example $\omega(x) = \max\{\log(1/|x|), 1\}$ given in [2] shows that RH_∞ is a proper subset of $\cap_{p>1} RH_p$.

However, as noted in [2], if a weight ω is in A_p for each $p > 1$ and if the constants $A_p(\omega)$ are uniformly bounded, then $\omega \in A_1$; and the corresponding statement holds for RH_p and RH_∞.

As noted above, for a nonnegative locally integrable function ω,

$$\begin{aligned} \omega \text{ is in } A_\infty &\iff \omega \text{ is in } A_p \text{ for some } p \in [1,\infty) \\ &\iff \omega \text{ is in } RH_q \text{ for some } q \in (1,\infty). \end{aligned}$$

In the first equivalence the A_∞ constant depends only on the A_p constant and on p, which in turn depend only on the A_∞ constant. Similarly, the A_∞

constant depends only on the RH_q constant and on q, which depend only on the A_∞ constant. See for example [6, Theorem 9.3.3] where the constants in these and other characterizations of A_∞ are carefully analyzed. The analogous statements hold for the dyadic classes A^d_∞, A^d_p, and RH^d_p.

The classes of A_p and RH_p weights can be characterized by conditions on the oscillation of the logarithm of the weight, as follows.

Lemma 1. *Let ω be a nonnegative locally integrable function on $\mathbb{T}$. Let $\varphi := \log \omega$. Then the following five statements hold.*

(*a*) *ω is in A_∞ if and only if*

$$\sup_Q \fint_Q \exp\{\varphi(x) - \varphi_Q\}\, dx < \infty. \tag{3.1}$$

(*b*) *For $1 < p < \infty$, ω is in A_p if and only if inequality* (3.1) *holds and also*

$$\sup_Q \fint_Q \exp\left\{\frac{-(\varphi(x) - \varphi_Q)}{p-1}\right\} dx < \infty. \tag{3.2}$$

(*c*) *ω is in A_1 if and only if inequality* (3.1) *holds and also*

$$\sup_Q \left[\varphi_Q - \operatorname*{ess\,inf}_{x\in Q} \varphi(x)\right] < \infty. \tag{3.3}$$

(*d*) *ω is in RH_∞ if and only if*

$$\sup_Q \left[\operatorname*{ess\,sup}_{x\in Q} \varphi(x) - \varphi_Q\right] < \infty. \tag{3.4}$$

(*e*) *For $1 < p < \infty$, ω is in RH_p if and only if*

$$\sup_Q \fint_Q \exp\{p(\varphi(x) - \varphi_Q)\}\, dx < \infty. \tag{3.5}$$

In each part, the value of $A_p(\omega)$ or $RH_p(\omega)$ depends on the value(*s*) *of the supremum* (*suprema*) *in the characterization given and, when $1 < p < \infty$, also on p. Conversely, the value*(*s*) *of the supremum* (*suprema*) *depend on the value of $A_p(\omega)$ or $RH_p(\omega)$ and, when $1 < p < \infty$, also on p.*

Taking the suprema in inequalities (3.1)–(3.5) *over only dyadic intervals $I \subset \mathbb{T}$, the dyadic analogues of parts* (*a*)–(*e*) *hold for the dyadic classes A^d_∞, A^d_p, A^d_1, RH^d_∞, and RH^d_p, except that in parts* (*d*) *and* (*e*) *one needs in addition to inequality* (3.4) *or* (3.5) *the extra hypothesis that ω is dyadic doubling. The dependence of the constants in the dyadic case is the same as in the continuous case.*

Parts (b) and (c) appear in [4], [6], and [3], for example, and part (d) is in [2, Cor 4.6]. Inequality (3.3) says that an A_∞ weight ω lies in A_1 when its logarithm φ belongs to the space BLO of functions of *bounded lower oscillation*, while inequality (3.4) says that ω is in RH_∞ when $-\varphi$ belongs to BLO.

Proof. Let C_1, C_2, C_3, C_4, and C_5 be the suprema in inequalities (3.1), (3.2), (3.3), (3.4), and (3.5) respectively.

(a) It is immediate that $A_\infty(\omega) = C_1$, since for each interval Q the A_∞ quantity is

$$\Big(⨍_Q \omega(x)\,dx\Big) \exp\Big\{ ⨍_Q \log \frac{1}{w(x)}\,dx\Big\} = ⨍_Q \exp\{\varphi(x) - \varphi_Q\}\,dx.$$

(b) We show that $C_1 \le A_p(\omega)$, $C_2 \le A_p(\omega)^{1/(p-1)}$, and $A_p(\omega) \le C_1 C_2^{p-1}$. Let

$$\psi := \log\left[\Big(\frac{1}{\omega}\Big)^{1/(p-1)}\right] = \frac{-\varphi}{p-1}.$$

Let Q be an interval in $\mathbb{T}$. Then by Jensen's inequality,

$$\begin{aligned} &⨍_Q \exp\{\varphi(x) - \varphi_Q\}\,dx = \Big(⨍_Q w\Big) \exp\Big((p-1) ⨍_Q \psi(x)\,dx\Big) \\ &\le \Big(⨍_Q w\Big)\Big[⨍_Q \exp\psi(x)\,dx\Big]^{p-1} = \Big(⨍_Q w\Big)\Big[⨍_Q \Big(\frac{1}{\omega}\Big)^{1/(p-1)}\Big]^{p-1} \le A_p(\omega). \end{aligned}$$

Thus inequality (3.1) holds with $C_1 \le A_p(\omega)$. Similarly, by Jensen's inequality,

$$\begin{aligned} ⨍_Q \exp\Big\{\frac{-(\varphi(x)-\varphi_Q)}{p-1}\Big\}\,dx &= \Big(⨍_Q \Big(\frac{1}{\omega}\Big)^{1/(p-1)}\Big)\Big[\exp\Big(⨍_Q \varphi\Big)\Big]^{1/(p-1)} \\ &\le \Big[\Big(⨍_Q \Big(\frac{1}{\omega}\Big)^{1/(p-1)}\Big)^{p-1}\Big(⨍_Q \omega\Big)\Big]^{1/(p-1)} \le A_p(\omega)^{1/(p-1)}. \end{aligned}$$

Thus inequality (3.3) holds with $C_2 \le A_p(\omega)^{1/(p-1)}$.

For the converse,

$$\begin{aligned} &\Big(⨍_Q \omega\Big)\Big(⨍_Q \Big(\frac{1}{\omega}\Big)^{1/(p-1)}\Big)^{p-1} = \\ &\qquad = \Big(⨍_Q \exp\varphi(x)\,dx\Big)\Big(⨍_Q \exp\psi(x)\,dx\Big)^{p-1} e^{-\varphi_Q} e^{-(p-1)\psi_Q} \\ &\qquad = \Big(⨍_Q \exp\{\varphi(x)-\varphi_Q\}\,dx\Big)\Big(⨍_Q \exp\{\psi(x)-\psi_Q\}\,dx\Big)^{p-1} \le C_1 C_2^{p-1}, \end{aligned}$$

and thus $A_p(\omega) \le C_1 C_2^{p-1}$.

(*c*) We show that $C_1 \le A_1(\omega)$, $C_3 \le \log A_1(\omega)$, and $A_1(\omega) \le C_1 \exp C_3$. If ω is in A_1, then for each interval Q in $\mathbb{T}$ we have

$$\fint_Q e^{\varphi(x)}\,dx = \fint_Q w(x)\,dx \le A_1(\omega) \operatorname*{ess\,inf}_{x\in Q} w(x) \le A_1(\omega) e^{\varphi_Q}.$$

It follows that

$$\fint_Q e^{\varphi(x)-\varphi_Q}\,dx \le A_1(\omega).$$

Thus φ satisfies inequality (3.1) with constant $C_1 \le A_1(\omega)$.

By Jensen's inequality and the A_1 property,

$$e^{\varphi_Q} \le \fint_Q w(x)\,dx \le A_1(\omega) \exp\left\{ \operatorname*{ess\,inf}_{x\in Q} \varphi(x) \right\}. \tag{3.6}$$

Therefore

$$\varphi_Q \le \log A_1(\omega) + \operatorname*{ess\,inf}_{x\in Q} \varphi(x).$$

Thus φ satisfies inequality (3.3) with constant $C_3 \le \log A_1(\omega)$.

Now suppose that φ satisfies inequalities (3.1) and (3.3). Then for each interval Q,

$$\begin{aligned}\fint_Q w(x)\,dx = \fint_Q e^{\varphi(x)}\,dx &\le C_1\, e^{\varphi_Q} \\ &\le C_1 \exp\left\{C_3 + \operatorname*{ess\,inf}_{x\in Q} \varphi(x)\right\} = C_1 e^{C_3} \operatorname*{ess\,inf}_{x\in Q} w(x).\end{aligned}$$

Thus ω satisfies the A_1 property with constant $A_1(\omega) \le C_1 e^{C_3}$.

(*d*) We show that $C_4 \le \log(RH_\infty(\omega)A_\infty(\omega))$ and $RH_\infty(\omega) \le e^{C_4}$, and that the bound on C_4 depends only on $RH_\infty(\omega)$.

Suppose ω is in RH_∞. Then ω is in A_∞, so inequality (3.1) holds with $C_1 = A_\infty(\omega)$. Further, ω is in every RH_p for $p \in (1,\infty)$, and $A_\infty(\omega)$ depends only on $RH_p(\omega)$, while $RH_p(\omega) \le RH_\infty(\omega)$. Thus $A_\infty(\omega)$ depends only on $RH_\infty(\omega)$. Now for each interval Q in $\mathbb{T}$, inequality (3.1) implies that

$$\operatorname*{ess\,sup}_{x\in Q} \omega(x) \le RH_\infty(\omega) \fint_Q e^{\varphi(x)}\,dx \le RH_\infty(\omega) A_\infty(\omega) e^{\varphi_Q}.$$

Taking logarithms, we see that

$$\operatorname*{ess\,sup}_{x\in Q} \varphi(x) \le \varphi_Q + \log(RH_\infty(\omega)A_\infty(\omega)),$$

and so inequality (3.4) holds with $C_4 \le \log(RH_\infty(\omega)A_\infty(\omega))$.

Conversely, if inequality (3.4) holds, then by Jensen's inequality

$$\operatorname*{ess\,sup}_{x\in Q}\omega(x)\le \exp\{C_4+\varphi_Q\}\le e^{C_4}\fint_Q \exp\varphi(x)\,dx = e^{C_4}\fint_Q \omega.$$

Thus ω satisfies the RH_∞ property with constant $RH_\infty(\omega)\le e^{C_4}$.

(*e*) We show that $C_5\le RH_p(\omega)A_\infty(\omega)$ and $RH_p(\omega)\le C_5^{1/p}$, and that this bound on C_5 depends only on p and on $RH_p(\omega)$. In terms of φ, the RH_p expression for a given interval Q is

$$\Big(\fint_Q \omega^p\Big)^{1/p}\Big(\fint_Q \omega\Big)^{-1} = \Big(\fint_Q e^{p(\varphi(x)-\varphi_Q)}\,dx\Big)^{1/p}\Big(\fint_Q e^{\varphi(x)-\varphi_Q}\,dx\Big)^{-1}.$$

Also, if ω is in RH_p, then ω is in A_∞ and $A_\infty(\omega)$ depends only on p and on $RH_p(\omega)$. It follows that

$$\Big(\fint_Q \exp\{p(\varphi(x)-\varphi_Q)\}\,dx\Big)^{1/p}\le$$

$$\le RH_p(\omega)\fint_Q \exp\{\varphi(x)-\varphi_Q\}\,dx \le RH_p(\omega)A_\infty(\omega).$$

Thus inequality (3.5) holds with $C_5\le RH_p(\omega)A_\infty(\omega)$, and this bound depends only on p and on $RH_p(\omega)$.

By Jensen's inequality, $\fint_Q \exp\{\varphi(x)-\varphi_Q\}\,dx\ge 1$. Thus if inequality (3.5) holds, then

$$\Big(\fint_Q \exp\{p(\varphi(x)-\varphi_Q)\}\,dx\Big)^{1/p}\Big(\fint_Q \exp\{\varphi(x)-\varphi_Q\}\,dx\Big)^{-1}\le C_5^{1/p}.$$

Thus ω is in RH_p and $RH_p(\omega)\le C_5^{1/p}$.

The same arguments go through for the dyadic classes A_p^d and RH_p^d. ■

Muckenhoupt's A_p weights are closely related to functions of bounded mean oscillation.

Definition 4. A real-valued function $f\in L^1(\mathbb{T})$ lies in the space BMO($\mathbb{T}$) of functions of *bounded mean oscillation* on the circle if its BMO norm is finite:

$$\|f\|_* := \sup_{Q\subset\mathbb{T}}\fint_Q |f(x)-f_Q|\,dx<\infty.$$

Dyadic BMO of the circle, written $\mathrm{BMO}_d(\mathbb{T})$, is the space of functions that satisfy the corresponding estimate where the supremum is taken over all dyadic subintervals $I\in\mathcal{D}$ of $[0,1]$. The dyadic BMO norm of f is denoted by $\|f\|_d$.

Elements of BMO, or of BMO_d, that differ only by an additive constant are equivalent; thus BMO and BMO_d are subspaces of $L^1_{\mathrm{loc}}/\mathbb{R}$.

For $1 \le p \le \infty$, if ω is in A_p then $\varphi := \log \omega$ is in BMO, with BMO norm depending only on the A_p constant $A_p(\omega)$. See for example [3, p. 409]. The same is true for RH_p weights. Specifically, we have the following result; we omit the proof.

Lemma 2. *Suppose* $1 \le p \le \infty$. *If* ω *is in* A_p *then* $\varphi := \log \omega$ *is in* BMO. *For* $1 < p < \infty$,

$$||\varphi||_* \le A_p(\omega) + (p-1)A_p(\omega)^{1/(p-1)}.$$

For $p = 1$, $||\varphi||_* \le 2A_1(\omega)$. *For* $p = \infty$, $||\varphi||_*$ *depends only on* $A_\infty(\omega)$. *For* $1 < p \le \infty$, *if* ω *is in* RH_p *then* $\varphi := \log \omega$ *is in* BMO, *with* $||\varphi||_*$ *depending only on* $RH_p(\omega)$ *and on* p. *The analogous statements hold in the dyadic setting.*

We use a characterization of the dyadic BMO functions on the circle in terms of the size of Haar coefficients. The Haar function h_I associated with the dyadic interval I is given by $h_I(x) = |I|^{-1/2}$ if x is in the left half of I, $h_I(x) = -|I|^{-1/2}$ if x is in the right half of I, and $h_I = 0$ otherwise. The Haar coefficient over I of f is $(f, h_I) := \int_I f(x)h_I(x)\,dx$. The Haar series for f is

$$f(x) := \sum_{I \in \mathcal{D}} (f, h_I)\, h_I(x),$$

and the L^2-norm of f is given in terms of the Haar coefficients by

$$\|f\|_2 = \left[\sum_{J \in \mathcal{D}} (f, h_J)^2 \right]^{1/2}.$$

The John–Nirenberg Theorem [8] implies that for each $p \ge 1$ and for each f in $L^1(\mathbb{T})$, the expression

$$\|f\|_{d,p} := \sup_{I \in \mathcal{D}} \left(\fint_I |f(x) - f_I|^p \, dx \right)^{1/p}$$

is comparable to the dyadic BMO norm $\|f\|_d$.

A function $f \in L^1(\mathbb{T})$ of mean value zero is in $\mathrm{BMO}_d(\mathbb{T})$ if and only if there is a constant C such that for all $I \in \mathcal{D}$,

$$\sum_{J \subset I, J \in \mathcal{D}} (f, h_J)^2 \le C|I|. \tag{3.7}$$

The smallest such constant C is equal to $\|f\|^2_{d,2}$. Since the sum in inequality (3.7) ranges over only dyadic intervals J, there is no need to restrict the interval I itself to be dyadic.

4. Proofs of Theorems 1 and 2

We begin this section with three lemmas, which we then use to prove the geometric-arithmetic averaging result for both A_p and RH_p.

Lemma 3 below gives an estimate on Haar expansions of BMO_d functions. Lemmas 4 and 5, which rely on the estimates (4.1) and (4.2) from Lemma 3, will allow us to pass from the dyadic versions to the non-dyadic versions of the inequalities that characterize A_p and RH_p.

Throughout this section we use the following notation. Let $\mathcal{D}_n := \{I \in \mathcal{D} \mid |I| = 2^{-n}\}$ be the collection of dyadic intervals of length 2^{-n}, for $n = 0$, 1, 2, ... Expanding each φ^t in Haar series, we have

$$\begin{aligned}\varphi(x) &= \int_0^1 \sum_{J\in\mathcal{D}} (\varphi^t, h_J) h_J(x+t)\, dt \\ &= \sum_{n=0}^{\infty} \int_0^1 \sum_{J\in\mathcal{D}_n} (\varphi^t, h_J)\, h_J(x+t)\, dt = \sum_{n=0}^{\infty} \varphi_n(x),\end{aligned}$$

so that φ_n is the translation-average over t of the slices at scale 2^{-n} of the Haar expansions for the functions φ^t.

Fix an interval $Q \subset \mathbb{T}$; this Q need not necessarily be dyadic. Split the sum for $\varphi(x)$, at the scale of $|Q|$, into two parts φ_A and φ_B in which the dyadic intervals J are respectively small and large compared with Q:

$$\varphi = \varphi_A + \varphi_B, \qquad \varphi_A(x) := \sum_{n:2^{-n}<|Q|} \varphi_n(x), \qquad \varphi_B(x) := \sum_{n:2^{-n}\geq|Q|} \varphi_n(x).$$

The following result is proved in the course of the proof of Theorem 2 of [13].

Lemma 3. *Suppose that* $\{\varphi^t\}_{t\in[0,1]}$ *is a family of dyadic* BMO *functions on* $\mathbb{T}$, $\varphi^t \in \mathrm{BMO}_d(\mathbb{T})$, *such that*

(*i*) *the mapping* $t \mapsto \varphi^t$ *is measurable,*

(*ii*) *the* BMO_d *constants* $||\varphi^t||_d$ *are uniformly bounded, independent of* $t\in[0,1]$, *and*

(*iii*) *for each* $t \in [0,1]$, *the function* φ^t *has mean value zero on* $\mathbb{T}$.

Then there are constants C_A *and* C_B *depending on the bound on the* BMO_d *constants* $\|\varphi^t\|_d$, *and independent of* Q, *such that for each interval* $Q \subset \mathbb{T}$ *and for each point* $x_0 \in Q$,

$$\frac{1}{|Q|}\int_Q |\varphi_A(x)|^2\, dx \leq C_A, \tag{4.1}$$

$$\frac{1}{|Q|}\int_Q |\varphi_B(x) - \varphi_B(x_0)|\, dx \leq C_B. \tag{4.2}$$

Lemma 4. *Let β be a real number. Suppose $\{\omega^t\}_{t\in[0,1]}$ is a family of nonnegative locally integrable functions on $\mathbb{T}$ such that for all $\varphi^t := \log \omega^t$, hypotheses (i)–(iii) of* Lemma 3 *hold. Let $\varphi(x) = \log \Omega(x) := \int_0^1 \log \omega^t(x+t)\,dt$. Suppose there is a constant $C^d(\beta)$ such that for all $t \in [0,1]$ and for all dyadic intervals $I \subset \mathbb{T}$,*

$$\fint_I \exp\left[\beta(\varphi^t(x) - \varphi^t_I)\right] dx \le C^d(\beta). \tag{4.3}$$

Then there is a constant $C(\beta)$, depending only on $C^d(\beta)$, such that for all intervals $Q \subset \mathbb{T}$,

$$\fint_Q \exp\left[\beta(\varphi(x) - \varphi_Q)\right] dx \le C(\beta). \tag{4.4}$$

In fact, for many choices of β, hypothesis (*ii*) of Lemma 3 is implied by inequality (4.3), together with Lemmas 1 and 2. This point is made clear in the proof of Theorems 1 and 2.

Proof. We first establish an inequality that controls the exponentials of the Haar expansions of the φ^t. By inequality (4.3), for each dyadic interval $I \subset \mathbb{T}$ we have

$$\begin{aligned} &\fint_I \exp\Big[\beta \sum_{J\subset I, J\in\mathcal{D}} (\varphi^t, h_J)h_J(x)\Big]\,dx = \\ &\qquad = \fint_I \exp\Big[\beta \sum_{J\in\mathcal{D}} (\varphi^t, h_J)h_J(x) - \beta \sum_{J\supsetneq I, J\in\mathcal{D}} (\varphi^t, h_J)h_J(x) \\ &\qquad\quad - \beta \sum_{J\cap I=\emptyset, J\in\mathcal{D}} (\varphi^t, h_J)h_J(x)\Big]\,dx \\ &\qquad = \fint_I \exp\left[\beta(\varphi^t(x) - \varphi^t_I)\right] dx \le C^d(\beta). \end{aligned} \tag{4.5}$$

We have used the fact that the average f_I of a function $f \in L^1(\mathbb{T})$ over an interval I containing the point x can be written as

$$f_I = \fint_I \sum_{J\in\mathcal{D}} (f, h_J)h_J(s)\,ds = \sum_{J\supsetneq I, J\in\mathcal{D}} (f, h_J)h_J(x),$$

and the observation that $\exp[-\beta \sum_{J\cap I=\emptyset, J\in\mathcal{D}} (\varphi^t, h_J)h_J(x)] = 1$ for $x \in I$.

Fix an interval $Q \subset \mathbb{T}$, not necessarily dyadic. For each $x \in Q$ we have

$$\begin{aligned} \left|\varphi_B(x) - \fint_Q \varphi(s)\,ds\right| &\le \fint_Q |\varphi_B(x) - \varphi(s)|\,ds \\ &\le \fint_Q |\varphi_A(s)|\,ds + \fint_Q |\varphi_B(x) - \varphi_B(s)|\,ds \le \sqrt{C_A} + C_B, \end{aligned}$$

by Cauchy–Schwarz and the estimates (4.1) and (4.2) from Lemma 3.

Therefore

$$\begin{aligned}\fint_Q \exp\left[\beta(\varphi(x)-\varphi_Q)\right] dx &= \fint_Q \exp\left[\beta\varphi_A(x)\right]\, \exp\left[\beta(\varphi_B(x)-\varphi_Q)\right] dx\\ &\le \exp\left[\beta(\sqrt{C_A}+C_B)\right] \fint_Q \exp\left[\beta\varphi_A(x)\right] dx.\end{aligned}$$

Thus it suffices to bound the quantity

$$\begin{aligned}\fint_Q \exp\left[\beta\varphi_A(x)\right] dx &= \fint_Q \exp\left[\beta \sum_{2^{-n}<|Q|} \int_0^1 \sum_{J\in\mathcal{D}_n} (\varphi^t, h_J)h_J(x+t)\, dt\right] dx\\ &= \fint_Q \exp\left[\int_0^1 \beta \sum_{|J|<|Q|, J\in\mathcal{D}} (\varphi^t, h_J)h_J(x+t)\, dt\right] dx\\ &\le \int_0^1 \fint_Q \exp\left[\beta \sum_{|J|<|Q|, J\in\mathcal{D}} (\varphi^t, h_J)h_J(x+t)\right] dx\, dt\\ &= \int_0^1 \fint_{Q+t} \exp\left[\beta \sum_{|J|<|Q|, J\in\mathcal{D}} (\varphi^t, h_J)h_J(x)\right] dx\, dt.\end{aligned}$$

We have used Jensen's inequality and Tonelli's Theorem.

In order to apply inequality (4.5), we want to replace the interval Q in the preceding expression by appropriate dyadic intervals. Fix t. There are two adjacent dyadic intervals K_t and L_t such that $Q+t\subset K_t\cup L_t$ and $|Q|\le |K_t|=|L_t|<2|Q|$. Then

$$\begin{aligned}\fint_Q \exp[\beta\varphi_A(x)]\, dx &\le \int_0^1 \frac{1}{|Q|}\int_{Q+t} \exp\left[\beta \sum_{|J|<|Q|, J\in\mathcal{D}} (\varphi^t, h_J)h_J(x)\right] dx\, dt\\ &\le \int_0^1 \frac{2}{|K_t|}\int_{K_t\cup L_t} \exp\left[\beta \sum_{|J|<|K_t|, J\in\mathcal{D}} (\varphi^t, h_J)h_J(x)\right] dx\, dt\\ &= 2\int_0^1 \left\{\frac{1}{|K_t|}\int_{K_t} \exp\left[\beta \sum_{|J|<|K_t|, J\in\mathcal{D}} (\varphi^t, h_J)h_J(x)\right] dx\right.\\ &\qquad \left. + \frac{1}{|L_t|}\int_{L_t} \exp\left[\beta \sum_{|J|<|L_t|, J\in\mathcal{D}} (\varphi^t, h_J)h_J(x)\right] dx\right\} dt\\ &\le 2\int_0^1 C^d(\beta)+C^d(\beta)\, dt = 4\, C^d(\beta),\end{aligned}$$

by inequality (4.5). Thus

$$\fint_Q \exp\left[\beta(\varphi(x)-\varphi_Q)\right] dx \le 4\, C^d(\beta) \exp\left[\beta\left(\sqrt{C_A}+C_B\right)\right].$$

Taking the supremum over all intervals $Q \subset \mathbb{T}$, we see that inequality (4.4) holds for $\varphi = \log \Omega$, with constant $C(\beta) = 4\, C^d(\beta) \exp\left[\beta\left(\sqrt{C_A} + C_B\right)\right]$. ■

Lemma 5. *Suppose* $\{\omega^t\}_{t\in[0,1]}$ *is a family of nonnegative locally integrable functions on* $\mathbb{T}$ *such that for all* $\varphi^t := \log \omega^t$, *hypotheses* (i) *and* (iii) *of* Lemma 3 *hold. As before, let* $\varphi(x) = \log \Omega(x) := \int_0^1 \log \omega^t(x+t)\, dt$.

(a) *Suppose there is a constant* C_3^d *such that for all* $t \in [0,1]$ *and for all dyadic intervals* $I \subset \mathbb{T}$,

$$\left[\varphi_I^t - \operatorname*{ess\,inf}_{x\in I} \varphi^t(x)\right] \le C_3^d.$$

Then there is a constant C_3 *depending only on* C_3^d *such that for all intervals* $Q \subset \mathbb{T}$,

$$\left[\varphi_Q - \operatorname*{ess\,inf}_{x\in Q} \varphi(x)\right] \le C_3.$$

(b) *Similarly, if there is a constant* C_4^d *such that for all* $t \in [0,1]$ *and for all dyadic intervals* $I \subset \mathbb{T}$,

$$\left[\operatorname*{ess\,sup}_{x\in I} \varphi^t(x) - \varphi_I^t\right] \le C_4^d,$$

then there is a constant C_4 *depending only on* C_4^d *such that for all intervals* $Q \subset \mathbb{T}$,

$$\left[\operatorname*{ess\,sup}_{x\in Q} \varphi(x) - \varphi_Q\right] \le C_4.$$

Proof. Observe that hypothesis (ii) of Lemma 3 follows from the assumption in part (a) or the assumption in part (b), together with the dyadic versions of Lemmas 1 and 2. In particular, the BMO_d constants $\|\varphi^t\|_d$ depend only on C_3^d or C_4^d.

(a) For each dyadic interval $I \subset \mathbb{T}$ and for a.e. $x \in I$, we have

$$C_3^d \ge \varphi_I^t - \varphi^t(x) = -\sum_{J\subset I, J\in\mathcal{D}} (\varphi^t, h_J) h_J(x), \tag{4.6}$$

using the observation that $\sum_{J\cap I=\emptyset, J\in\mathcal{D}} (\varphi^t, h_J) h_J(x) = 0$ for $x \in I$.

Fix an interval $Q \subset \mathbb{T}$, not necessarily dyadic. For $x \in Q$ consider the quantity

$$\begin{aligned}\varphi_Q - \varphi(x) &= \fint_Q \left[\varphi_A(s) + \varphi_B(s)\right] ds - \left[\varphi_A(x) + \varphi_B(x)\right] \\ &= \underbrace{\fint_Q \varphi_A(s)\, ds}_{\text{(I)}} + \underbrace{\left[\fint_Q \varphi_B(s)\, ds - \varphi_B(x)\right]}_{\text{(II)}} + \underbrace{\left[-\varphi_A(x)\right]}_{\text{(III)}}.\end{aligned}$$

We bound the terms (I), (II), and (III) separately. By Lemma 3,

$$\text{(I)} \le \Big[⨍_Q |\varphi_A(s)|^2 \, ds \Big]^{1/2} \le \sqrt{C_A} \quad \text{and} \quad \text{(II)} \le ⨍_Q |\varphi_B(s) - \varphi_B(x)| \, ds \le C_B$$

for all $x \in Q$. Also, given $x \in Q$ and $t \in [0,1]$, there is a unique dyadic interval $I_{x,t}$ such that $x+t \in I_{x,t}$ and $|Q|/2 \le |I_{x,t}| < |Q|$. Then $h_J(x+t) = 0$ if $J \not\subset I_{x,t}$. It follows that

$$\begin{aligned} -\varphi_A(x) &= -\sum_{n:2^{-n}<|Q|} \int_0^1 \sum_{J \in \mathcal{D}_n} (\varphi^t, h_J) h_J(x+t) \, dt \\ &= -\int_0^1 \sum_{|J|<|Q|, J \in \mathcal{D}} (\varphi^t, h_J) h_J(x+t) \, dt \\ &= -\int_0^1 \sum_{J \subset I_{x,t}, J \in \mathcal{D}} (\varphi^t, h_J) h_J(x+t) \, dt \le C_3^d \end{aligned}$$

for a.e. $x \in Q$, by inequality (4.6). So

$$\varphi_Q - \operatorname*{ess\,inf}_{x \in Q} \varphi(x) \le \sqrt{C_A} + C_B + C_3^d.$$

Thus inequality (3.3) holds with constant $C_3 = \sqrt{C_A} + C_B + C_3^d$.

(*b*) In the case of RH_∞, the same argument shows that inequality (3.4) holds with constant $C_4 = C_4^d + C_B + \sqrt{C_A}$. ■

Proof of Theorems 1 and 2. We prove Theorems 1 and 2 together. Let K denote any one of the classes A_p with $1 \le p \le \infty$, or RH_p with $1 < p \le \infty$. For ω in K let $K(\omega)$ denote the corresponding A_p constant or RH_p constant. Let K^d denote the corresponding dyadic class, and for ω^t in K^d let $K^d(\omega^t)$ denote the corresponding constant.

It follows from the definitions of A_p and RH_p that if ω is in K then for each constant $\lambda > 0$ the weight $\lambda\omega$ is also in K, and $K(\lambda\omega) = K(\omega)$.

For each $t \in [0,1]$ let ω^t be a weight in K^d, and let $\varphi^t := \log \omega^t$. Let $\varphi := \log \Omega$. By hypothesis (*iii*) of Theorem 1 or (*iii′*) of Theorem 2, the constants $K^d(\omega^t)$ are bounded independently of $t \in [0,1]$.

Without loss of generality, we may assume that for a.e. $t \in [0,1]$ the functions φ^t have mean value zero. To see this, note that the mean value $\varphi^t_{\mathbb{T}} := ⨍_{\mathbb{T}} \varphi^t(x) \, dx$ is finite for a.e. t by hypothesis (*ii*). Let $\widetilde{\varphi}^t := \varphi^t - \varphi^t_{\mathbb{T}}$. Then for $x \in \mathbb{T}$,

$$\begin{aligned} \Omega(x) &= \exp\left\{ \int_0^1 \varphi^t(x+t) \, dt \right\} = \exp\left\{ \int_0^1 \varphi^t_{\mathbb{T}} \, dt \right\} \exp\left\{ \int_0^1 \widetilde{\varphi}^t(x+t) \, dt \right\} \\ &= \exp\left\{ \int_0^1 \int_0^1 \log \omega^t(x) \, dx \, dt \right\} \exp\left\{ \int_0^1 \widetilde{\varphi}^t(x+t) \, dt \right\}. \end{aligned}$$

Denote the second term on the right-hand side by $\widetilde{\Omega}(x)$. The first term on the right-hand side is finite by hypothesis (ii) of Theorems 1 and 2, and is positive. Thus Ω is in K if and only if $\widetilde{\Omega}$ is in K. Moreover, $K(\Omega) = K(\widetilde{\Omega})$.

We show that hypotheses (i)–(iii) of Lemma 3 hold here. The mapping $t \mapsto \widetilde{\varphi}^t = (\log \omega^t) - \varphi^t_{\mathbb{T}}$ is measurable, since $t \mapsto \omega^t$ is measurable by hypothesis. By the dyadic version of Lemma 2, the BMO_d norms $||\widetilde{\varphi}^t||_d = ||\varphi^t||_d$ are uniformly bounded, by a constant depending only on p and on the bound on the K^d constants of the dyadic weights ω^t. Each $\widetilde{\varphi}^t$ has mean value zero, by construction.

For convenience we now drop the tildes, writing φ^t for $\widetilde{\varphi}^t$ and Ω for $\widetilde{\Omega}$ from here on.

As an aside, we note that $\log \Omega$ is in BMO (see [5]), and so by the John–Nirenberg Theorem the function Ω^δ is in A_p for $\delta > 0$ sufficiently small. We now prove that Ω itself is in A_p.

Case 1: $K = A_\infty$. By the dyadic version of Lemma 1(a), there is a constant C^d_1 depending only on the bound on the A^d_∞ constants of the ω^t such that each φ^t satisfies inequality (4.3) with $\beta = 1$:

$$\fint_I \exp\{\varphi^t(x) - \varphi^t_I\}\, dx \le C^d_1.$$

Take $C^d(1) = C^d_1$. Then by Lemma 4, $\varphi = \log \Omega$ satisfies inequality (4.4) with $\beta = 1$: there is a constant $C(1)$ depending only on $C^d(1)$ such that

$$\fint_Q \exp\{\varphi(x) - \varphi_Q\}\, dx \le C(1).$$

Take $C_1 = C(1)$. Lemma 1(a) now implies that $\Omega \in A_\infty$, with A_∞ constant bounded by C_1. The dependence of the constants is illustrated in the upper row of Figure 1 (taking $p = \infty$ there). We see that $A_\infty(\Omega)$ depends only on the bound on the A^d_∞ constants of the weights ω^t.

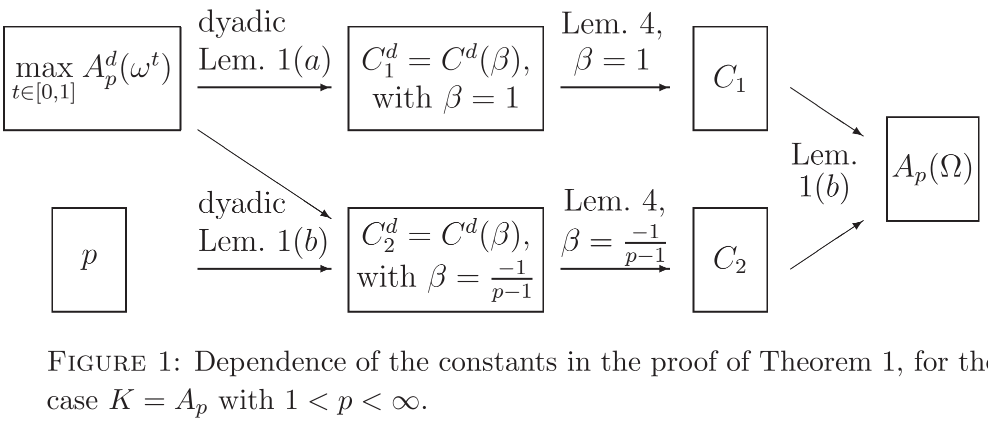

FIGURE 1: Dependence of the constants in the proof of Theorem 1, for the case $K = A_p$ with $1 < p < \infty$.

Case 2: $K = A_p$, $1 < p < \infty$. As for case 1, using Lemma 1(b) and Lemma 4 with both $\beta = 1$ and $\beta = -1/(p-1)$. Figure 1 illustrates the dependence of the constants. We find that $A_p(\Omega)$ depends only on p and on the bound on the A_p^d constants of the ω^t.

Case 3: $K = A_1$. As for case 1, using Lemma 1(c), Lemma 4 with $\beta = 1$, and Lemma 5. The constant $A_1(\Omega)$ depends only on the bound on the A_1^d constants of the ω^t.

Case 4: $K = RH_\infty$. As for case 1, using Lemma 1(d) and Lemma 5. The constant $RH_\infty(\Omega)$ depends only on the bound on the RH_∞^d constants of the ω^t.

Case 5: $K = RH_p$. As for case 1, using Lemma 1(e) and Lemma 4 with $\beta = p$. We find that $RH_p(\Omega)$ depends only on p and on the bound on the RH_p^d constants of the ω^t.

This completes the proof of Theorems 1 and 2. ■

Remark. An alternative proof of Theorem 1 for $K = A_p$, with $1 < p \leq \infty$, can be obtained as follows from the result for $K = A_1$, using factorization of A_p weights [10]. Suppose $1 \leq p < \infty$. If ω_1 and ω_2 are A_1 weights, then $\omega = \omega_1\omega_2^{1-p}$ is an A_p weight, with constant $A_p(\omega) \leq A_1(\omega_1)A_1(\omega_2)^{p-1}$. Conversely, if $\omega \in A_p$, then there exist ω_1 and ω_2 in A_1 such that $\omega = \omega_1\omega_2^{1-p}$. The A_1 constants of ω_1 and ω_2 depend only on p and on the A_p constants of ω, as noted in [6, p. 717]. The analogous results hold in the dyadic setting A_p^d.

Lemma 6. *If* Theorem 1 *holds for* A_1, *then it holds for each* A_p, $1 \leq p \leq \infty$.

Proof. By the (dyadic) factorization theorem, for each $t \in [0,1]$ there exist ω_1^t, $\omega_2^t \in A_1^d$ such that $\omega^t = \omega_1^t(\omega_2^t)^{1-p}$. Furthermore $A_1(\omega_1^t)$ and $A_1(\omega_2^t)$ are uniformly bounded, independent of $t \in [0,1]$, by a constant depending on p and on the bound for the constants $A_p(\omega^t)$. Then

$$\begin{aligned}
\Omega(x) &= \exp\left\{\int_0^1 \log \omega^t(x+t)\,dt\right\} \\
&= \exp\left\{\int_0^1 \log\left[\omega_1^t(x+t)\big(\omega_1^t(x+t)\big)^{1-p}\right]\,dt\right\} \\
&= \exp\left\{\int_0^1 \log \omega_1^t(x+t)\,dt + (1-p)\int_0^1 \big(\omega_1^t(x+t)\big)\,dt\right\} \\
&= \left[\exp\left\{\int_0^1 \log \omega_1^t(x+t)\,dt\right\}\right]\left[\exp\left\{\int_0^1 \log \omega_2^t(x+t)\,dt\right\}\right]^{1-p}.
\end{aligned}$$

By hypothesis, both of the expressions in square brackets are in A_1. Therefore Ω is in A_p as required. Furthermore, the A_p constant of Ω depends only on the A_p^d constants of the weights ω^t.

The result for A_∞ follows immediately from the A_p case, $1 \le p < \infty$, using the observations on the dependence of the constants made before Lemma 1 above. ■

5. The translation-average of doubling weights

It appears that the obstacle to the translation-average $\omega(x) = \int_0^1 \omega^t(x+t)\,dt$ of A_p^d weights ω^t being in A_p is that the assumption of the ω^t being dyadic doubling is insufficient to guarantee that the translation-average is actually doubling [16]. A natural conjecture is that if the translation-average of a given family of A_p^d weights is in fact doubling, then this doubling weight also belongs to true A_p. As a step in this direction, we show that the presumably stronger assumption that each of the A_p^d weights ω^t is doubling (in other words, that ω^t is in A_p) does imply that their translation-average ω is in A_p.

Theorem 3. *Suppose $\{\omega^t\}_{t\in[0,1]}$ is a family of doubling weights on $\mathbb{T}$, with doubling constants bounded by a constant C_{dbl} independent of $t \in [0,1]$. Suppose the mapping $t \mapsto \omega^t$ is measurable. Fix p with $1 \le p \le \infty$, and suppose each ω^t is in A_p^d, with A_p^d constant bounded by a constant $V_{p,d}$ independent of $t \in [0,1]$. Then the translation-average*

$$\omega(x) := \int_0^1 \omega^t(x+t)\,dt$$

belongs to A_p on $\mathbb{T}$, with A_p constant depending only on p, $V_{p,d}$, and C_{dbl}.

Similarly, for p with $1 < p \le \infty$, if each ω^t is in RH_p^d, with the supremum in the defining inequality for RH_p^d (Definition 3) *bounded by a constant $V_{p,d}$ independent of $t \in [0,1]$, then the translation-average $\omega(x)$ belongs to RH_p on $\mathbb{T}$, with $RH_p(\omega)$ depending on p, $V_{p,d}$, and C_{dbl}.*

Proof. A doubling weight assigns comparable mass, with a constant depending only on the doubling constant, to any given interval Q and to each of the dyadic intervals at scale $|Q|$ that intersect Q. (This observation can fail if the weight is dyadic doubling but not doubling.)

As a consequence, for fixed p with $1 \le p \le \infty$, an A_p^d weight is doubling if and only if it is actually in A_p. Moreover the A_p constant depends only on the A_p^d constant and the doubling constant, which in turn depend only on the A_p constant. Similarly, for fixed p with $1 < p \le \infty$, an RH_p^d weight is doubling if and only if it is actually in RH_p. For example, one finds that for a doubling A_p^d weight ω^t with $1 < p < \infty$, the A_p constant of ω^t is bounded by $V_p := 2^{2p-1}\,V_{p,d}\,C_{\mathrm{dbl}}^2$, while for a doubling RH_p^d weight ω^t with $1 < p < \infty$, the RH_p constant of ω^t is bounded by $V_p := 2^{2/p}\,V_{p,d}\,C_{\mathrm{dbl}}^2$.

Theorem 3 now follows easily for A_p, $1 < p < \infty$, using Muckenhoupt's original identification of A_p in terms of the boundedness of the Hardy–Littlewood maximal function M, defined by $Mf(x) := \sup_{Q \ni x} ⨍_Q |f(y)|\, dy$. In particular, for these p, a nonnegative locally integrable function ω is in A_p if and only if there is a constant C such that for all locally integrable functions f,

$$\int_{\mathbb{R}} |Mf(x)|^p \,\omega(x)\, dx \leq C \int_{\mathbb{R}} |f(x)|^p \,\omega(x)\, dx. \tag{5.1}$$

Moreover, if inequality (5.1) holds then $\omega \in A_p$ and $A_p(\omega) \leq C$, while if $\omega \in A_p$ then inequality (5.1) holds with C depending on p and $A_p(\omega)$.

For RH_p, $1 < p < \infty$, Theorem 3 follows from Minkowski's Integral Inequality and the observation above on comparable mass. The cases of A_1, A_∞, and RH_∞ are also straightforward, and we omit the proofs. ∎

6. Generalizations to the polydisc

We extend the above results for $A_p(\mathbb{T})$ and $RH_p(\mathbb{T})$ to the setting of the polydisc. For ease of notation, the statements and proofs given below are expressed for the bidisc. However, they generalize immediately to the polydisc with arbitrarily many factors.

The theory of product weights was developed by K.-C. Lin in his thesis [11], while the dyadic theory was developed in Buckley's paper [1]. The product A_p and RH_p weights and the product doubling weights, and their dyadic analogues, are defined exactly as in Definitions 1–3 in Section 3, with intervals in $\mathbb{T}$ being replaced by rectangles in $\mathbb{T} \otimes \mathbb{T}$. It follows that a product weight belongs to $A_p(\mathbb{T} \otimes \mathbb{T})$ if and only if it belongs to $A_p(\mathbb{T})$ in each variable separately.

To be precise, $\omega \in A_p(\mathbb{T} \otimes \mathbb{T})$ if and only if $\omega(\cdot, y) \in A_p(\mathbb{T})$ uniformly for a.e. $y \in \mathbb{T}$ and $\omega(x, \cdot) \in A_p(\mathbb{T})$ uniformly for a.e. $x \in \mathbb{T}$. In one direction this is a consequence of the Lebesgue Differentiation Theorem, letting one side of the rectangle shrink to a point. The converse uses the equivalence between $\omega \in A_p(\mathbb{T} \otimes \mathbb{T})$ and inequality (5.1) with M replaced by the strong maximal function [14, p.83]. Further, the $A_p(\mathbb{T} \otimes \mathbb{T})$ constant depends only on the two $A_p(\mathbb{T})$ constants, and vice versa. The analogous characterizations in terms of the separate variables hold for product RH_p weights and for product doubling weights, and for the dyadic product A_p, RH_p, and doubling weights.

Theorem 4. *Fix p with $1 \leq p \leq \infty$. Let $\{\omega^{s,t}\}_{s,t\in[0,1]}$ be a family of dyadic A_p weights on the boundary of the bidisc, $\omega^{s,t} \in A_p^d(\mathbb{T} \otimes \mathbb{T})$, such that*

(i) *the mapping $(s,t) \mapsto \omega^{s,t}$ is measurable,* (*ii*) *an appropriate average of the logarithms of the weights* $\omega^{s,t}$ *is finite:*

$$\int_0^1\int_0^1\int_0^1\int_0^1 \big|\log \omega^{s,t}(x,y)\big|\,dx\,dy\,ds\,dt < \infty,$$

and

(*iii*) *the* $A_p^d(\mathbb{T}\otimes\mathbb{T})$ *constants* $A_p^d(\omega^{s,t})$ *are uniformly bounded, independent of* s, $t \in [0,1]$.

Then the product geometric-arithmetic average

$$\Omega(x,y) := \exp\left\{\int_0^1\int_0^1 \log \omega^{s,t}(x+s,y+t)\,ds\,dt\right\}$$

of the dyadic weights $\omega^{s,t}$ *belongs to* $A_p(\mathbb{T}\otimes\mathbb{T})$, *with* A_p *constant depending on* p *and on the bound on the* A_p^d *constants of the weights* $\omega^{s,t}$.

Similarly, for fixed p *with* $1 < p \leq \infty$, *if each* $\omega^{s,t}$ *is in* $RH_p^d(\mathbb{T}\otimes\mathbb{T})$, *if hypotheses* (*i*) *and* (*ii*) *hold, and if*

(*iii*′) *the* $RH_p^d(\mathbb{T}\otimes\mathbb{T})$ *constants* $RH_p^d(\omega^{s,t})$ *are uniformly bounded, independent of* s, $t \in [0,1]$,

then $\Omega(x,y)$ *lies in* $RH_p(\mathbb{T}\otimes\mathbb{T})$, *with* RH_p *constant depending on* p *and on the bound on the* RH_p^d *constants of the* $\omega^{s,t}$.

Proof. The proof is by iteration of the one-parameter argument, relying on Lemma 1 and Theorems 1 and 2. We give the argument for $K = A_p$, $1 < p < \infty$. The other cases follow from similar iteration arguments; we omit the proofs.

We show that, for a.e. fixed y, $\Omega(x,y)$ belongs to $A_p(\mathbb{T})$ in the variable x, with A_p constant independent of y. The hypotheses of Theorem 1 follow immediately from our assumptions. In particular, $s \mapsto \omega^{s,t}(\cdot,y)$ is measurable for each t and a.e. y, and $\omega^{s,t}(\cdot,y)$ belongs to $A_p^d(\mathbb{T})$ in the first variable for all s, t and for a.e. y, with constants independent of s, t, and y. Fix such a y; for emphasis we'll denote it by y^*.

Applying Theorem 1, we see that the function

$$\Omega_1(x,y^*) := \exp\left\{\int_0^1 \log \omega^{s,t}(x+s,y^*)\,ds\right\}$$

belongs to $A_p(\mathbb{T})$ in x, with constant independent of y^*. Let

$$\varphi_1(x,y^*) := \log \Omega_1(x,y^*) = \int_0^1 \log \omega^{s,t}(x+s,y^*)\,ds.$$

By Lemma 1(b) there are constants C_1 and C_2, depending only on p and on the $A_p^d(\mathbb{T}\otimes\mathbb{T})$ constants of the $\omega^{s,t}$, such that

$$\sup_Q \fint_Q \exp\left\{\varphi_1(x,y^*) - \big(\varphi_1(\cdot,y^*)\big)_Q\right\} dx \le C_1, \tag{6.1}$$

$$\sup_Q \fint_Q \exp\left\{\frac{-1}{p-1}\Big[\varphi_1(x,y^*) - \big(\varphi_1(\cdot,y^*)\big)_Q\Big]\right\} dx \le C_2. \tag{6.2}$$

We show that the same inequalities hold when φ_1 is replaced by

$$\varphi(x,y^*) := \log\Omega(x,y^*) = \int_0^1\int_0^1 \log\omega^{s,t}(x+s,y^*+t)\,ds\,dt.$$

Fix an interval Q. An application of Fubini's Theorem shows that

$$\begin{aligned}\varphi(x,&y^*) - \big(\varphi(\cdot,y^*)\big)_Q = \\ &= \int_0^1\int_0^1 \varphi^{s,t}(x+s,y^*+t)\,ds\,dt - \fint_Q\int_0^1\int_0^1 \varphi^{s,t}(x'+s,y^*+t)\,ds\,dt\,dx' \\ &= \int_0^1 \Big[\varphi_1(x,y^*+t) - \big(\varphi_1(\cdot,y^*+t)\big)_Q\Big]\,dt.\end{aligned}$$

Then by Jensen's inequality and Tonelli's Theorem,

$$\begin{aligned}\fint_Q &\exp\left\{\varphi(x,y^*) - \big(\varphi(\cdot,y^*)\big)_Q\right\} dx = \\ &\le \fint_Q\int_0^1 \exp\left\{\varphi_1(x,y^*+t) - \big(\varphi_1(\cdot,y^*+t)\big)_Q\right\} dt\,dx \\ &= \int_0^1\fint_Q \exp\left\{\varphi_1(x,y^*+t) - \big(\varphi_1(\cdot,y^*+t)\big)_Q\right\} dx\,dt \le C_1,\end{aligned}$$

using inequality (6.1) with y^* replaced by y^*+t for a.e. t.

The same argument shows that inequality (6.2) holds for $\varphi(\cdot,y^*)$ for a.e. y^*. Lemma 1 then implies that $\Omega(x,y^*)$ belongs to $A_p(\mathbb{T})$ in x for a.e. y^*, with uniform constants.

In an identical fashion, we find that $\Omega(x^*,y)$ belongs to $A_p(\mathbb{T})$ in y for a.e. x^*, with uniform constants, which proves the theorem for $K = A_p$, $1<p<\infty$. ■

Remark. As in the one-parameter case, there is an alternative proof of the geometric-arithmetic averaging result (Theorem 4) for $A_p(\mathbb{T}\otimes\mathbb{T})$ where $1<p\le\infty$, relying on the $A_1(\mathbb{T}\otimes\mathbb{T})$ case and the generalization to the bidisc setting [7] of the A_p factorization theorem. Moreover, the product A_p result can also be derived from the one-parameter result using the maximal function characterization of this weight class.

We also have a product version of Theorem 3.

Theorem 5. *Let* $\{\omega^{s,t}\}_{s,t\in[0,1]}$ *be a family of doubling weights on* $\mathbb{T}\otimes\mathbb{T}$*, with doubling constants bounded by a constant* C_{dbl} *independent of* s*,* $t \in [0,1]$*. Suppose the mapping* $(s,t) \mapsto \omega^{s,t}$ *is measurable. Fix* p *with* $1 \le p \le \infty$*, and suppose each* $\omega^{s,t}$ *is in* $A_p^d(\mathbb{T}\otimes\mathbb{T})$ *with* $A_p^d(\mathbb{T}\otimes\mathbb{T})$ *constant bounded by a constant* $V_{p,d}$ *independent of* $s,t \in [0,1]$*. Then the translation-average*

$$\omega(x,y) := \int_0^1\int_0^1 \omega^{s,t}(x+s,y+t)\,ds\,dt$$

belongs to $A_p(\mathbb{T}\otimes\mathbb{T})$*, with an* A_p *constant depending only on* p*,* $V_{p,d}$*, and* C_{dbl}*.*

Similarly, for $1 < p \le \infty$*, if the* $A_p^d(\mathbb{T}\otimes\mathbb{T})$ *assumption above is replaced by the assumption that each* $\omega^{s,t}$ *is in* $RH_p^d(\mathbb{T}\otimes\mathbb{T})$ *with the supremum in the defining inequality for* $RH_p^d(\mathbb{T}\otimes\mathbb{T})$ *bounded by a constant* $V_{p,d}$ *independent of* $s,t\in[0,1]$*, then* $\omega(x,y)$ *belongs to* $RH_p(\mathbb{T}\otimes\mathbb{T})$*, with* $RH_p(\omega)$ *depending on* p*,* $V_{p,d}$*, and* C_{dbl}*.*

In brief, the geometric-arithmetic average of each suitable family of $A_p(\mathbb{T}\otimes\mathbb{T})$ weights is also an $A_p(\mathbb{T}\otimes\mathbb{T})$ weight, and similarly for $RH_p(\mathbb{T}\otimes\mathbb{T})$.

The proof is by iteration of the one-parameter argument given above for Theorem 3.

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