End-point estimates and multi-parameter paraproducts on higher dimensional tori

John T. Workman

Abstract

Analogues of multi-parameter multiplier operators on $\mathbb{R}^d$ are defined on the torus $\mathbb{T}^d$. It is shown that these operators satisfy the classical Coifman-Meyer theorem. In addition, $L(\log L)^n$ end-point estimates are proved.

1. Introduction

This article is, in part, a continuation of [13, 14]. It is also derived from the author’s dissertation, which can be found in full at [17].


$$\Lambda_m^{(1)}(f_1, \ldots, f_d)(x) = \int_{\mathbb{R}^d} m(t)\hat{f}_1(t_1)\cdots\hat{f}_d(t_d)e^{2\pi ix(t_1+\cdots+t_d)} dt,$$

for Schwartz functions $f_j$ and where $m$ satisfies a standard Marcinkiewicz-Mihlin-Hörmander type condition [12]. It is well known this operator maps $L^{p_1}\times\cdots\times L^{p_d} \to L^p$ for $1/p_1 + \cdots + 1/p_d = 1/p$ and $1 < p_j < \infty$. The case when $p \geq 1$ was originally shown by Coifman and Meyer. The general case $p > 1/d$ was settled later in [9, 11].

Led by natural questions in non-linear partial differential equations, extensions of this operator were considered by Muscalu et. al.: first the so-called bi-parameter multiplier [13], then multi-parameter multipliers [14]. In this setting, $m$ is allowed to belong to a much wider class of multipliers which behave like the product of standard multipliers. Special cases of these multiplier operators had been previously considered by Christ and Journé [4, 10]. In [13, 14], it is shown that these multiplier operators satisfy the same $L^{p_1}\times\cdots\times L^{p_d} \to L^p$ property.

2000 Mathematics Subject Classification: 42B15.

Keywords: Paraproduct, polydisc.
However, in the single-parameter case of Coifman and Meyer, more is known. We have “end-point” estimates corresponding to the case when any or all of the $p_j$ are equal to 1. Here, the result is $L^{p_1} \times \cdots \times L^{p_d} \rightarrow L^{p,\infty}$. In the multi-parameter setting, no such end-point estimates are known.

A natural candidate for such an estimate would involve $L^\log L$ spaces, because of how they arise in interpolation results. Naively, an operator which maps $L^1 \rightarrow L^{1,\infty}$, and also satisfies some $L^p$ result, is often thought to also satisfy some $L^\log L$ to $L^1$ property. Indeed, we recall the result of Stein [16], which states $Mf$ is locally integrable if and only if $f$ is locally in $L^\log L$; alternatively, C. Fefferman [6] showed the maximal double Hilbert transform maps $L^\log L([0,1]^2)$ to $L^{1,\infty}([0,1]^2)$.

That $L^\log L$ estimates can only be gained in the compact setting is a rather common obstacle. To avoid this, we instead consider analogues of multiplier operators defined on the torus $\mathbb{T}^d$. This also allows a departure from the classical definition of $L^\log L$ spaces to a more iterative approach which blends perfectly with our methods. Ultimately, we show that the $s$-parameter multiplier operator $\Lambda_m^{(s)}$ in this setting satisfies the classical Coifman-Meyer theorem, along with the desired end-point estimate: for $p_j = 1$ each $L^{p_j}$ is replaced by $L^{(\log L)^{s-1}}$.

The organization is as follows. In the next section, characterizations of $L^{(\log L)^n}$ are developed for any probability space, and several important results therein are proved. Section 3 details the connections between $L^{(\log L)^n}$ spaces and the Hardy-Littlewood maximal operator. Section 4 deals with the notion of adapted families and a particular square function of Littlewood-Paley type. Section 5 introduces hybrid square-max operators. In Section 6, bi-parameter multiplier operators are handled, while section 7 is a non-rigorous survey of the proof for multi-parameter multipliers.

A remark on the notation used: we will write $A \lesssim B$ whenever $A \leq C \cdot B$ with some universal constant $C$.

2. Zygmund spaces and $L^{(\log L)^n}$

Let $(X, \rho)$ be a probability space. For $f : (X, \rho) \rightarrow \mathbb{C}$, denote the decreasing rearrangement of $f$ by $f^*$.

**Definition.** For $t > 0$ and $f : (X, \rho) \rightarrow \mathbb{C}$, let $f^{(s,1)}(t) = f^*(t)$ and for integers $n \geq 2$, set $f^{(s,n)}(t) = \frac{1}{t} \int_0^t f^{(s,n-1)}(s) \, ds$.

On a probability space, $f^*$ is supported on $[0,1]$. It is advantageous to informally think of each $f^{(s,n)}$ as being defined only on $(0,1]$.

We can immediately verify the following properties: (1) $f^{(s,n)}$ is nonnegative, decreasing, and identically 0 if and only if $f = 0$ a.e. [$\rho$]; (2) $f^{(s,n)} \leq$
\( f^{(s,n+1)}; \) \( (\alpha f)^{(s,n)} = |\alpha| f^{(s,n)} \) \( |f| \leq |g| \) a.e.\([\rho]\) implies \( f^{(s,n)} \leq g^{(s,n)} \) pointwise; \( (5) \) \( |f_k| \uparrow |f| \) a.e.\([\rho]\) implies \( f_k^{(s,n)} \uparrow f^{(s,n)} \) pointwise.

We would also like to show \( (f+g)^{(s,n)}(t) \leq f^{(s,n)}(t) + g^{(s,n)}(t) \) for all \( t > 0 \) and \( n \geq 2; \) this property does not hold for \( n = 1. \) By induction, it suffices to prove the result for \( n = 2. \) However, this is an immediate consequence of the following technical result of Bennett and Sharpley [3]:

\[
t f^{(s,2)}(t) = \int_0^t f^*(s) \, ds = \inf_{g=h} \left\{ ||g||_1 + t \|h\|_\infty \right\}.
\]

**Definition.** For \( f : (X, \rho) \to \mathbb{C} \) and integers \( n \geq 0, \) define \( \|f\|_{L(\log L)^n} \) by

\[
\|f\|_{L(\log L)^n} = \int_0^1 f^{(s,n+1)}(t) \, dt.
\]

Define the Zygmund space \( L(\log L)^n(X) \) as the set of functions \( f \) with \( \|f\|_{L(\log L)^n} < \infty. \)

We note that \( L(\log L)^0(X) = L^1(X), \) which is a useful notational shortcut. Clearly, \( \| \cdot \|_{L(\log L)^n} \) is a norm with the additional properties that \( |f| \leq |g| \) a.e.\([\rho]\) implies \( \|f\|_{L(\log L)^n} \leq \|g\|_{L(\log L)^n} \) and \( |f_k| \uparrow |f| \) a.e.\([\rho]\) implies \( \|f_k\|_{L(\log L)^n} \uparrow \|f\|_{L(\log L)^n}. \) Further, this definition of \( L(\log L)^n \) coincides with the classical space.

**Theorem 2.1.** \( f \in L(\log L)^n(X) \) if and only if

\[
\int_X |f(x)| (\log^+ |f(x)|)^n \rho(dx) < \infty.
\]

The proof is fairly technical but straightforward and is left to the reader. Using Hardy’s inequality, it is also easy to establish the following.

**Theorem 2.2.** For any \( 1 < p \leq \infty \) and \( n \geq 0, \)

\[
L_p(X) \subseteq L(\log L)^{n+1}(X) \subseteq L(\log L)^n(X) \subseteq L^1(X),
\]

with \( \|f\|_1 \leq \|f\|_{L(\log L)^n} \leq \|f\|_{L(\log L)^{n+1}} \lesssim \|f\|_p. \)

The principal reason for defining \( L(\log L)^n \) as we have is the ease in which we gain interpolation results.

**Lemma 2.3.** Let \( T \) be a sublinear operator which maps \( L^1(X) \to L^{1,\infty}(X) \) and \( L^p(X) \to L^{q,\infty}(X), \) for some \( 1 < p, q < \infty. \) Then, for \( n \in \mathbb{N}, \)

\[
(Tf)^{(s,n)}(t) \lesssim \left[ \frac{1}{t} \int_0^t f^{(s,n)}(s) \, ds + t^{-1/q} \int_1^t s^{1/p-1} f^{(s,n)}(s) \, ds \right],
\]

where \( m = (\frac{1}{q} - 1)(\frac{1}{p} - 1)^{-1}. \)
Proof. We show this by induction. The \( n = 1 \) case is a technical result established in [3]. Assume it is true for \( n - 1 \). Then,
\[
(Tf)^{(s,n)}(t) = \frac{1}{t} \int_0^t T^{(s,n-1)}(s) \, ds \\
\leq \frac{1}{t} \int_0^t \frac{1}{s} \int_0^{s^n} f^{(s,n-1)}(u) \, du \, ds + \frac{1}{t} \int_0^t s^{-1/q} \int_{s^n}^1 u^{1/p-1} f^{(s,n-1)}(u) \, du \, ds \\
=: I + II.
\]

By the change of variables \( r = s^n \),
\[
I = \frac{1}{mt} \int_0^1 \frac{1}{r} \int_0^r f^{(s,n-1)}(u) \, du \, dr = \frac{1}{mt} \int_0^1 f^{(s,n)}(r) \, dr.
\]

On the other hand, changing the order of integration gives
\[
II = \frac{1}{t} \int_0^1 u^{1/p-1} f^{(s,n-1)}(u) \int_0^{u^{1/m}} s^{-1/q} \, ds \, du \\
+ \frac{1}{t} \int_0^1 u^{1/p-1} f^{(s,n-1)}(u) \int_0^t s^{-1/q} \, ds \, du \\
= \frac{1}{1 - 1/q} \left[ \frac{1}{t} \int_0^1 f^{(s,n)}(u) \, du + \frac{1}{1 - 1/q} t^{-1/q} \int_0^1 u^{1/p-1} f^{(s,n-1)}(u) \, du \right] \\
\leq \frac{1}{1 - 1/q} \left[ \frac{1}{t} \int_0^1 f^{(s,n)}(u) \, du + t^{-1/q} \int_0^1 u^{1/p-1} f^{(s,n-1)}(u) \, du \right] .
\]

\[\square\]

Theorem 2.4. Let \( T \) be a sublinear operator which maps \( L^1(X) \to L^{1,\infty}(X) \) and \( L^p(X) \to L^{p,\infty}(X) \), for some \( 1 < p, q < \infty \). Then, for all \( n \in \mathbb{N} \), \( T \) also maps \( L(\log L)^n(X) \to L(\log L)^{n-1}(X) \).

Proof. Set \( m = \left( \frac{1}{q} - 1 \right) \left( \frac{1}{p} - 1 \right)^{-1} \). Using Lemma 2.3 and the same change of variables and Fubini arguments,
\[
\|Tf\|_{L(\log L)^{n-1}} = \int_0^1 (Tf)^{(s,n)}(t) \, dt \\
\leq \int_0^1 \frac{1}{t} \int_0^t f^{(s,n)}(s) \, ds \, dt + \int_0^1 t^{-1/q} \int_0^1 s^{1/p-1} f^{(s,n)}(s) \, ds \, dt \\
= \frac{1}{m} \int_0^1 \frac{1}{u} \int_0^u f^{(s,n)}(s) \, ds \, du + \int_0^1 s^{1/p-1} f^{(s,n)}(s) \int_0^s t^{-1/q} \, dt \, ds \\
= \frac{1}{m} \int_0^1 f^{(s,n+1)}(u) \, du + \frac{1}{1 - 1/q} \int_0^1 f^{(s,n)}(s) \, ds \lesssim \|f\|_{L(\log L)^n}.
\]

\[\square\]
Corollary 2.5. Let $T$ be a sublinear operator. If for some $1 < p, r < \infty$
\[
\left\| \left( \sum_{k=1}^{\infty} |Tf_k|^r \right)^{1/r} \right\|_{1,\infty} \lesssim \left\| \left( \sum_{k=1}^{\infty} |f_k|^r \right)^{1/r} \right\|_1 \quad \text{and}
\]
\[
\left\| \left( \sum_{k=1}^{\infty} |Tf_k|^r \right)^{1/r} \right\|_p \lesssim \left\| \left( \sum_{k=1}^{\infty} |f_k|^r \right)^{1/r} \right\|_p,
\]
then for all $n \in \mathbb{N}$
\[
\left\| \left( \sum_{k=1}^{\infty} |Tf_k|^r \right)^{1/r} \right\|_{L(\log L)^n} \lesssim \left\| \left( \sum_{k=1}^{\infty} |f_k|^r \right)^{1/r} \right\|_{L(\log L)^n}.
\]

**Proof.** This only requires viewing the above theory through the wider scope of Banach space-valued functions $f : (X, \rho) \to (B, \| \cdot \|_B)$ (see [8]). If instead one defined the decreasing rearrangement $f^*$ for Banach space-valued functions, in the natural way, and repeated the definitions and arguments of this section, everything would still hold. In particular, the previous theorem is valid; if $T$ is sublinear operator mapping $L_B^p(X)$ to $L_B^{1,\infty}(X)$ and $L_B^p(X)$ to $L_B^{1,\infty}(X)$, then $T : L(\log L)^n_B(X) \to L(\log L)^n_B(X)$. But, simply by definition, $f^*(t) = (\|f\|_B^*)(t)$, where $(\|f\|_B^*)$ is understood as the decreasing rearrangement of the map $x \mapsto \|f(x)\|_B$. Thus,
\[
\|f\|_{L(\log L)^n_B} = \left\| \|f\|_B \right\|_{L(\log L)^n}.
\]

Let $B = \ell^r$ and $T(f) = (Tf_1, Tf_2, \ldots)$, so that $T : L_B^1(X) \to L_B^{1,\infty}(X)$ and $L_B^p(X) \to L_B^p(X)$. Thus, $T : L(\log L)^n_B(X) \to L(\log L)^n_B(X)$, which is what was promised. \hfill \blacksquare

### 3. Connections to Hardy-Littlewood

Let us turn our attention to the probability space $(\mathbb{T}, m)$. Let $Mf$ denote the standard Hardy-Littlewood maximal operator on $\mathbb{T}$. Of course, $M$ maps $L^1(\mathbb{T}) \to L^{1,\infty}(\mathbb{T})$ and $L^p(\mathbb{T}) \to L^p(\mathbb{T})$ for all $1 < p < \infty$. So, by the interpolation results of the previous section, $M : L(\log L)^n(\mathbb{T}) \to L(\log L)^{n-1}(\mathbb{T})$. Further, from Fefferman and Stein [7], we know
\[
\left\| \left( \sum_{k=1}^{\infty} |Mf_k|^r \right)^{1/r} \right\|_{1,\infty} \lesssim \left\| \left( \sum_{k=1}^{\infty} |f_k|^r \right)^{1/r} \right\|_1 \quad \text{and}
\]
\[
\left\| \left( \sum_{k=1}^{\infty} |Mf_k|^r \right)^{1/r} \right\|_p \lesssim \left\| \left( \sum_{k=1}^{\infty} |f_k|^r \right)^{1/r} \right\|_p,
\]
for all $1 < p, r < \infty$, and therefore Corollary 2.5 applies. However, much more can be said.
Theorem 3.1. \( f^{(x,n+1)}(t) \sim (Mf)^{(x,n)}(t) \), where the underlying constants do not depend on \( f \) or \( t \).

It clearly suffices, by induction, to prove \( f^{(x,2)}(t) \sim (Mf)^{(x)}(t) \). But, this is a well-known result; see [2, 3].

Corollary 3.2. \( f \in L(\log L)^{n+1}(T) \) if and only if \( Mf \in L(\log L)^{n}(T) \), and, in particular, \( \|f\|_{L(\log L)^{n+1}} \sim \|Mf\|_{L(\log L)^{n}} \).

4. Adapted families

Definition. A smooth function \( \phi : T \to \mathbb{C} \) is adapted to an interval \( I \) with constants \( C_m > 0, m \in \mathbb{N} \), if

\[
|\phi(x)| \leq C_m \left(1 + \frac{\text{dist}_T(x, I)}{|I|}\right)^{-m} \quad \text{for all} \quad x \in T, m \in \mathbb{N},
\]

\[
|\phi'(x)| \leq C_m \frac{1}{|I|} \left(1 + \frac{\text{dist}_T(x, I)}{|I|}\right)^{-m} \quad \text{for all} \quad x \in T, m \in \mathbb{N}.
\]

A family of smooth functions \( \phi_I : T \to \mathbb{C} \), indexed by the dyadic intervals, is called an adapted family if each \( \phi_I \) is adapted to \( I \) with the same universal constants. We say \( \{\phi_I\}_I \) is a 0-mean adapted family if it is an adapted family, with the additional property that \( \int_T \phi_I \, dm = 0 \) for all \( I \).

For an adapted family \( \phi_I \), define \( \phi_I = |I|^{-1/2} \phi_I \), where \( |I| \) denotes Lebesgue measure. Note \( \|\phi_I\|_2 \lesssim 1 \) for all \( I \). Often, \( \phi_I \) is called an \( L^2 \)-normalized family. Per our notation, \( \phi_I \) will always represent an adapted family, and \( \phi_I \) will always represent the \( L^2 \)-normalization.

Conceptually, we often think of functions which are adapted to an interval \( I \) as being “almost supported” in \( I \). The following theorem, which is a variation of a result in [14], gives some rigid meaning to this.

Theorem 4.1. Let \( \phi_I : T \to \mathbb{C} \) be adapted to a dyadic interval \( I \), with \( |I| = 2^{-N} \). Then, we can write

\[
\phi_I = \sum_{k=1}^{\infty} 2^{-10k} \varphi^k_I,
\]

where each \( \varphi^k_I \) is adapted to \( I \), uniformly in \( k \), with \( \text{supp}(\varphi^k_I) \subseteq 2^k I \) for \( 1 \leq k \leq N \) and \( \varphi^k_I = 0 \) otherwise. Further, if \( \phi_I \) has integral 0, each \( \varphi^k_I \) can be chosen to have integral 0.

To clarify the notation above, for an interval \( I \) and constant \( \alpha > 0 \), \( \alpha I \) is the interval concentric with \( I \) so that \( |\alpha I| = \alpha |I| \).
Given an adapted family $\varphi_I$, its normalization $\phi_I$, and $f : \mathbb{T} \to \mathbb{C}$, we will be interested in “averages” of $f$ with respect to the family. Let

$$M'f(x) = \sup_I \frac{1}{|I|^{1/2}} |\langle \phi_I, f \rangle| \chi_I(x).$$

where the supremum is over all dyadic intervals. For a 0-mean adapted family $\varphi_I$, define the Littlewood-Paley (discrete) square function by

$$Sf(x) = \left( \sum_I \frac{|\langle \phi_I, f \rangle|^2}{|I|} \chi_I(x) \right)^{1/2},$$

where the sum is over all dyadic intervals.

Using Theorem 4.1, it is easily shown that $M'f \lesssim Mf$, so that $M'$ satisfies the same properties as $M$. It is known that $S : L^1 \to L^{1,\infty}$ and $L^p \to L^p$ for $1 < p < \infty$ (see [17] for a new approach). We will need to establish Fefferman-Stein inequalities for $S$ as well, but the only the special case $r = 2$ will be necessary.

**Theorem 4.2.** For $1 < p < \infty$ and any sequence $f_1, f_2, \ldots$ of complex-valued functions on $\mathbb{T}$

$$\left\| \left( \sum_{k=1}^{\infty} |Sf_k|^2 \right)^{1/2} \right\|_p \lesssim \left( \sum_{k=1}^{\infty} |f_k|^2 \right)^{1/2},$$

$$\left\| \left( \sum_{k=1}^{\infty} |Sf_k|^2 \right)^{1/2} \right\|_1 \lesssim \left( \sum_{k=1}^{\infty} |f_k|^2 \right)^{1/2}.$$  

Only considering the $r = 2$ allows us to use Rademacher functions and Khinchine’s inequality to “linearize.” For the weak-$L^1$ inequality, an alternate characterization called the Kolmogorov condition is helpful (see [8]). For full details, see [17].

### 5. Hybrid operators

The definitions of the hybrid operators $MS$, $SM$, and $SS$, their properties, and their relevance in our context are borrowed from [13].

We say a set $R \subset \mathbb{T}^2$ is a dyadic rectangle if there exist dyadic intervals $I$ and $J$ so that $R = I \times J$. Given two (possibly distinct) adapted families $\varphi_I$ and $\varphi_J$, we will write $\varphi_R(x, y) = \varphi_I(x) \varphi_J(y)$. For $\varphi_R = \varphi_I \otimes \varphi_J$, set $\phi_R = |R|^{-1/2} \varphi_R = \phi_I \otimes \phi_J$. 

For functions $f : \mathbb{T}^2 \to \mathbb{C}$, define

$$MMf(x, y) = \sup_R \frac{1}{|R|^{1/2}} |\langle \phi_R, f \rangle| \chi_R(x, y).$$

If $\{\varphi_R\}$ is a family such that $\int_\mathbb{T} \varphi_J dm = 0$ for all $J$, then define

$$MSf(x, y) = \sup_I \frac{1}{|I|^{1/2}} \left( \sum_J \frac{|\langle \phi_R, f \rangle|^2}{|J|} \chi_J(y) \right)^{1/2} \chi_I(x),$$

Analogously, if $\int_\mathbb{T} \varphi_I dm = 0$ for all $I$, define

$$SMf(x, y) = \left( \sum_I \left( \sup_J \frac{1}{|J|^{1/2}} |\langle \phi_R, f \rangle| \chi_J(y) \right)^2 \chi_I(x) \right)^{1/2}.$$ 

Finally, if $\int_\mathbb{T} \varphi_I dm = \int_\mathbb{T} \varphi_J dm = 0$, set

$$SSf(x, y) = \left( \sum_R \frac{|\langle \phi_R, f \rangle|^2}{|R|} \chi_R(x, y) \right)^{1/2}.$$ 

**Theorem 5.1.** Each of $MM$, $MS$, $SM$, and $SS$ maps $L^p(\mathbb{T}^2) \to L^p(\mathbb{T}^2)$ for all $1 < p < \infty$, $L(\log L)^{n+2}(\mathbb{T}^2) \to L(\log L)^{n}(\mathbb{T}^2)$ for all $n \geq 0$, and $L \log L(\mathbb{T}^2) \to L^{1,\infty}(\mathbb{T}^2)$.

**Proof.** Let $M_S$ denote the strong maximal operator (that is, where the supremum is taken over all bi-parameter rectangles). Define the 1st and 2nd variables maximal operators $M_1$ and $M_2$ as follows. For $f : \mathbb{T}^2 \to \mathbb{C}$, let $M_1f(x_1, x_2) = M(f(\cdot, x_2))(x_1)$ and $M_2f(x_1, x_2) = M(f(x_1, \cdot))(x_2)$. It is clear that $M_1, M_2$ satisfy all the $L^p$ properties and Fefferman-Stein inequalities that $M$ does. Define $M'_1, M'_2, S_1, S_2$ similarly.

Using Theorem 4.1 as before, $MMf \lesssim M_Sf$. But, $M_Sf \leq M_1 \circ M_2 f$, so that

$$\| MMf \|_p \lesssim \| M_1 \circ M_2 f \|_p \lesssim \| M_2 f \|_p \lesssim \| f \|_p,$$

$$\| MMf \|_{L(\log L)^n} \lesssim \| M_1 \circ M_2 f \|_{L(\log L)^n} \lesssim \| M_2 f \|_{L(\log L)^{n+1}} \lesssim \| f \|_{L(\log L)^{n+2}},$$

$$\| MMf \|_{1,\infty} \lesssim \| M_1 \circ M_2 f \|_{1,\infty} \lesssim \| M_2 f \|_1 \lesssim \| f \|_{L \log L}.$$ 

We abuse notation slightly and write $\langle f, \phi_I \rangle$ to mean $\int_\mathbb{T} \phi_I(x)f(x, y)\, dx$, a function of the variable $y$. Thus, $\langle \phi_R, f \rangle = \langle \phi_J, \langle f, \phi_I \rangle \rangle$ makes sense. Also,
we can consider the two variable function $\langle f, \phi_I \rangle \chi_I$. In this manner,

\[
SMf(x, y) = \left( \sum_I \left( \frac{1}{|I|^{1/2}} \left| \frac{\langle \phi_I, f \rangle}{|I|^{1/2}} \chi_I(x) \right| \chi_I(y) \right)^2 \right)^{1/2} \\
= \left( \sum_I \left( \sum_J \frac{1}{|J|} \left| \langle \phi_J, \langle f, \phi_I \rangle |I|^{1/2} \chi_I(x) \rangle \chi_J(y) \right| \chi_J(y) \right)^2 \right)^{1/2} \\
= \left( \sum_I M'_2 \left( \frac{\langle f, \phi_I \rangle |I|^{1/2}}{|I|^{1/2} \chi_I} \right) (x, y)^2 \right)^{1/2}.
\]

By the Fefferman-Stein inequalities on $M'$ (or $M'_2$),

\[
\|SMf\|_p = \left\| \left( \sum_I M'_2 \left( \frac{\langle f, \phi_I \rangle |I|^{1/2}}{|I|^{1/2} \chi_I} \right)^2 \right) \right\|_p^{1/2} \\
\lesssim \left\| \left( \sum_I \left| \frac{\langle f, \phi_I \rangle}{|I|} \chi_I \right| \right)^{1/2} \right\|_p = \|S_1f\|_p \lesssim \|f\|_p,
\]

and

\[
\|SMf\|_{L(\log L)^n} = \left\| \left( \sum_I M'_2 \left( \frac{\langle f, \phi_I \rangle |I|^{1/2}}{|I|^{1/2} \chi_I} \right)^2 \right) \right\|_{L(\log L)^n}^{1/2} \\
\lesssim \left\| \left( \sum_I \left| \frac{\langle f, \phi_I \rangle}{|I|} \chi_I \right| \right)^{1/2} \right\|_{L(\log L)^{n+1}} = \|S_1f\|_{L(\log L)^{n+1}} \lesssim \|f\|_{L(\log L)^{n+2}},
\]

and

\[
\|SMf\|_{1,\infty} = \left\| \left( \sum_I M'_2 \left( \frac{\langle f, \phi_I \rangle |I|^{1/2}}{|I|^{1/2} \chi_I} \right)^2 \right) \right\|_{1,\infty}^{1/2} \\
\lesssim \left\| \left( \sum_I \left| \frac{\langle f, \phi_I \rangle}{|I|} \chi_I \right| \right)^{1/2} \right\|_1 = \|S_1f\|_1 \lesssim \|f\|_{L \log L}.
\]

On the other hand,

\[
Msf(x, y) = \sup_I \left( \frac{1}{|I|^{1/2}} \sum_J \frac{\left| \langle \phi_J, f \rangle \right|^2}{|J|} \chi_J(y) \right) \chi_I(x) \\
\leq \left( \sum_J \left( \frac{\left( \sup_I \left| \langle \phi_I, f \rangle \chi_I(x) \right| \right)^2}{|J|} \right) \chi_J(y) \right)^{1/2}.
\]
This is essentially SM with the roles of I and J reversed. The same arguments as above can now be applied.

Finally,

\[ SSf(x, y) = \left( \sum_R \frac{|\langle \phi_R, f \rangle|^2}{|R|} \chi_R(x, y) \right)^{1/2} \]

\[ = \left[ \sum_I \sum_J \frac{1}{|I|} \left| \langle \phi_I, \langle f, \phi_I \rangle \chi_I(x) \rangle \chi_J(y) \right|^2 \right]^{1/2} \]

\[ = \left[ \sum_I S_2 \left( \langle f, \phi_I \rangle \chi_I(x, y) \right)^2 \right]^{1/2}, \]

so that the same proof works.

\[ \square \]

6. Bi-parameter multipliers

Given a vector \( \vec{t} = (t_1, \ldots, t_{2d}) \in \mathbb{R}^{2d} \), denote \( \rho_1(\vec{t}) = (t_1, t_3, \ldots, t_{2d-1}) \) and \( \rho_2(\vec{t}) = (t_2, t_4, \ldots, t_{2d}) \), which are both vectors in \( \mathbb{R}^d \). For multi-indices of nonnegative integers \( \alpha \), we set \( |\rho_1(\alpha)| = \alpha_1 + \alpha_3 + \cdots + \alpha_{2d-1} \), and similarly for \( |\rho_2(\alpha)| \). Conversely, for \( 1 \leq j \leq d \), let \( \vec{t}_j = (t_{2j-1}, t_{2j}) \in \mathbb{R}^2 \), so that \( \vec{t} = (\vec{t}_1, \ldots, \vec{t}_d) \).

**Definition.** Let \( m : \mathbb{R}^{2d} \to \mathbb{C} \) be smooth away the origin and uniformly bounded. We say \( m \) is a bi-parameter multiplier if

\[ |\partial^\alpha m(\vec{t})| \lesssim \|\rho_1(\vec{t})\|^{-|\alpha_1(\alpha)|} \|\rho_2(\vec{t})\|^{-|\alpha_2(\alpha)|} \]

for all vectors \( \alpha \) with \( |\alpha| \leq 2d(d+3) \), where \( \| \cdot \| \) is the Euclidean norm on \( \mathbb{R}^d \).

Given such a multiplier \( m \) on \( \mathbb{R}^{2d} \) and \( L^1 \) functions \( f_1, \ldots, f_d : \mathbb{T}^2 \to \mathbb{C} \), we define the associated multiplier operator \( \Lambda_m^{(2)}(f_1, \ldots, f_d) : \mathbb{T}^2 \to \mathbb{C} \) as

\[ \Lambda_m^{(2)}(f_1, \ldots, f_d)(\vec{x}) = \sum_{\vec{t} \in \mathbb{Z}^{2d}} m(\vec{t}) \hat{f}_1(\vec{t}_1) \cdots \hat{f}_d(\vec{t}_d) e^{2\pi i \vec{x} \cdot (\vec{t}_1 + \cdots + \vec{t}_d)}. \]

Consider the following theorem.

**Theorem 6.1.** For any bi-parameter multiplier \( m \) on \( \mathbb{R}^{2d} \), it follows that \( \Lambda_m^{(2)} : L^{p_1} \times \cdots \times L^{p_d} \to L^p \) for \( 1 < p_j < \infty \) and \( 1/p_1 + \cdots + 1/p_d = 1/p \). If any or all of the \( p_j \) are equal to 1, this still holds with \( L^p \) replaced by \( L^{p\infty} \) and \( L^{p_j} \) replaced by \( L \log L \). In particular, \( \Lambda_m^{(2)} : L \log L \times \cdots \times L \log L \to L^{1/d,\infty} \).
We focus only the bi-linear $d = 2$ case, but this makes no substantiative
difference in the proof. Note that in this case, the bi-parameter multiplier condition can be stated
\[
|\langle \phi_R^\alpha, \psi_R^\beta \rangle| \lesssim \|f\|^{-\alpha_1 - \beta_1}\|g\|^{-\alpha_2 - \beta_2}
\]
for all two-dimensional indices $\alpha, \beta$ with $|\alpha|, |\beta| \leq 10$.

It is by now a well established fact (see [14, 15, 17]) that the study of
multiplier operators of various sorts can be reduced to the study of finitely
many discrete paraproducts. For $f, g : \mathbb{T}^2 \to \mathbb{C}$, the bi-parameter bi-linear
paraproducts are defined by
\[
T^{a,b}(f,g)(x,y) = \sum_{R} \frac{1}{|R|^{1/2}} \langle \phi_R^1, f \rangle \langle \phi_R^2, g \rangle \phi_R^3(x,y),
\]
for $a, b = 1, 2, 3$, where $\phi_R^1, \phi_R^2, \text{ and } \phi_R^3$ are each the tensor product of two
normalized adapted families, as in the previous section. The sum is over all
dyadic rectangles $R$. Further, if $\phi_R^1 = \phi_i^1 \otimes \phi_i^2$, then $\int_{\mathbb{T}} \phi_i^1 dx = 0$ for $i \neq a$
and $\int_{\mathbb{T}} \phi_i^2 dx = 0$ for $i \neq b$.

In order to establish Theorem 6.1, we need only prove each paraproduct satisfies
the same bounds. First, the following lemma is a well-known
characterization of weak-$L^p$. A proof is given in [1].

**Lemma 6.2.** Fix $0 < p < \infty$ and $f : \mathbb{T}^d \to \mathbb{C}$. Suppose that for every measure-
able set $|E| > 0$ in $\mathbb{T}^d$, we can choose a subset $E' \subseteq E$ with $|E'| > |E|/2$
and $\|f, \chi_{E'}\| \leq A|E|^{1-1/p}$. Then, $\|f\|_{p, \infty} \lesssim A$. Conversely, if $\|f\|_{p, \infty} \leq A$,
then for any measurable set $|E| > 0$ there exists $E' \subseteq E$ with $|E'| > |E|/2$
and $\|f, \chi_{E'}\| \lesssim A|E|^{1-1/p}$.

**Theorem 6.3.** $T^{a,b} : L^{p_1} \times L^{p_2} \to L^p$ for $1 < p_1, p_2 < \infty$ and $1/p_1 + 1/p_2 = 1/p$.
If $p_1$ or $p_2$ or both are equal to 1, this still holds with $L^p$ replaced by $L^{p, \infty}$
and $L^{p_1}$ replaced by $L \log L$.

**Proof.** We will assume $a = 1$ and $b = 2$, as the other cases will follow similarly.

First, suppose $p > 1$. Then, necessarily $p_1, p_2 > 1$ and $1 < p' < \infty$. Note,
$1/p_1 + 1/p_2 + 1/p' = 1$. Fix $h \in L^{p'}(\mathbb{T})$ with $\|h\|_{p'} \leq 1$. Then,
\[
|\langle T^{1,2}(f,g), h \rangle| \leq \sum_{R} \frac{1}{|R|^{1/2}} |\langle \phi_R^1, f \rangle| |\langle \phi_R^2, g \rangle| |\langle \phi_R^3, h \rangle| \\
= \int_{\mathbb{T}^2} \sum_{R} \frac{|\langle \phi_R^1, f \rangle|}{|R|^{1/2}} \frac{|\langle \phi_R^2, g \rangle|}{|R|^{1/2}} \frac{|\langle \phi_R^3, h \rangle|}{|R|^{1/2}} \chi_R(x,y) dx dy.
\]
Hence, by Lemma 6.2, we will be done if we can find 
\[ \frac{\langle \phi_R^1, f \rangle}{|R|^{1/2}} \frac{\langle \phi_R^2, g \rangle}{|R|^{1/2}} \frac{\langle \phi_R^3, h \rangle}{|R|^{1/2}} \chi_R(x, y) = \]
\[ = \sum_I \sum_J \frac{\langle \phi_R^1, f \rangle}{|R|^{1/2}} \frac{\langle \phi_R^2, g \rangle}{|R|^{1/2}} \frac{\langle \phi_R^3, h \rangle}{|R|^{1/2}} \chi_R(x, y) \]
\[ \leq \sum_I \left[ \left( \frac{1}{|I|^{1/2}} \chi_I(x) \sup_J \frac{\langle \phi_R^2, g \rangle}{|J|^{1/2}} \chi_J(y) \right) \cdot \left( \sum_J \frac{\langle \phi_R^1, f \rangle}{|R|^{1/2}} \frac{\langle \phi_R^3, h \rangle}{|R|^{1/2}} \chi_R(x, y) \right) \right]. \]

Applying Hölder’s inequality, the last term is bounded by
\[ SM(g)(x, y) \left( \sum_I \left( \sum_J \frac{\langle \phi_R^1, f \rangle}{|R|^{1/2}} \frac{\langle \phi_R^3, h \rangle}{|R|^{1/2}} \chi_R(x, y) \right)^2 \right)^{1/2}. \]

Applying Hölder to the inner sum,
\[ \left( \sum_I \left( \sum_J \frac{\langle \phi_R^1, f \rangle}{|R|^{1/2}} \frac{\langle \phi_R^3, h \rangle}{|R|^{1/2}} \chi_R(x, y) \right)^2 \right)^{1/2} \leq \]
\[ \leq \left( \sum_I \left( \sum_J \frac{|\langle \phi_R^1, f \rangle|^2}{|R|} \chi_R(x, y) \right) \left( \sum_J \frac{|\langle \phi_R^3, h \rangle|^2}{|R|} \chi_R(x, y) \right) \right)^{1/2} \]
\[ \leq \left( \sup_I \frac{1}{|I|} \chi_I(x) \sum_J \frac{|\langle \phi_R^1, f \rangle|^2}{|J|} \chi_J(y) \right)^{1/2} \left( \sum_I \sum_J \frac{|\langle \phi_R^3, h \rangle|^2}{|R|} \chi_R(x, y) \right)^{1/2} \]
\[ = MS(f)(x, y) SS(h)(x, y). \]

Hence,
\[ |\langle T^{1,2}(f, g), h \rangle| \leq \int_{T^2} MSf(x, y) SMg(x, y) SS(h, y) \, dx \, dy \]
\[ \leq \| MSf \|_{p_1} \| SMg \|_{p_2} \| SS(h) \|_{p'} \leq \| f \|_{p_1} \| g \|_{p_2}. \]

As $h$ in the unit ball of $L^{p'}$ is arbitrary, we have $\| T^{1,2}(f, g) \|_p \leq \| f \|_{p_1} \| g \|_{p_2}$.

Now assume $1/2 \leq p \leq 1$. By interpolation, it is sufficient to show $T^{1,2} : L^{p_1} \times L^{p_2} \to L^{p,\infty}$ for all $1 \leq p_1, p_2 < \infty$. Fix $\| f \|_{p_1} = 1$ if $p_1 > 1$ or $\| f \|_{L^p \log L} = 1$ if $p_1 = 1$. Similarly for $g$ and $p_2$. Let $E \subseteq T^2$ with $|E| > 0$. By Lemma 6.2, we will be done if we can find $E' \subseteq E$, $|E'| > |E|/2$ so that $|\langle T^{1,2}(f, g), \chi_{E'} \rangle| \leq 1 \leq |E|^{1-1/p}$.
For $\tilde{k} \in \mathbb{N}^2$ and $R = I \times J$ a dyadic interval, denote $2^{\tilde{k}}R = 2^{k_1}I \times 2^{k_2}J$, and $|\tilde{k}| = k_1 + k_2$. Use Theorem 4.1 to write

$$\phi_R^{3,\tilde{k}} = \sum_{\tilde{k} \in \mathbb{N}^2} 2^{-10|\tilde{k}|} \phi_R^{3,\tilde{k}}$$

where each $\phi_R^{3,\tilde{k}}$ is the normalization of the tensor product of two 0-mean adapted families which are uniformly adapted to $I$, $J$ respectively. Further, $\text{supp}(\phi_R^{3,\tilde{k}}) \subseteq 2^{\tilde{k}}R$ for $\tilde{k}$ small enough, while $\phi_R^{3,\tilde{k}}$ is identically 0 otherwise. Now

$$\langle T^{1,2}(f, g), \chi_{E'} \rangle = \sum_{\tilde{k} \in \mathbb{N}^2} 2^{-10|\tilde{k}|} \sum_R \frac{1}{|R|^{1/2}} \langle \phi_R^{1,\tilde{k}}, f \rangle \langle \phi_R^{2,\tilde{k}}, g \rangle \langle \phi_R^{3,\tilde{k}}, \chi_{E'} \rangle.$$  

Hence, it suffices to show $|\sum |R|^{-1/2} \langle \phi_R^{1,\tilde{k}}, f \rangle \langle \phi_R^{2,\tilde{k}}, g \rangle \langle \phi_R^{3,\tilde{k}}, \chi_{E'} \rangle| \lesssim 2^{|\tilde{k}|}$, so long as the underlying constants are independent of $\tilde{k}$.

Let $SS^{\tilde{k}}$ be the double square operator with $\phi_R^{3,\tilde{k}}$. For each $\tilde{k} \in \mathbb{N}^2$, define

$$\Omega_{-3|\tilde{k}|} = \{MSf > C2^{3|\tilde{k}|}\} \cup \{SMg > C2^{3|\tilde{k}|}\},$$

$$\tilde{\Omega}_{\tilde{k}} = \{MS(\chi_{\Omega_{-3|\tilde{k}|}}) > 1/100\},$$

$$\bar{\Omega}_{\tilde{k}} = \{MS(\chi_{\tilde{\Omega}_{\tilde{k}}}) > 2^{-|\tilde{k}|-1}\}.$$

and

$$\Omega = \bigcup_{\tilde{k} \in \mathbb{N}^2} \bar{\Omega}_{\tilde{k}}.$$  

Observe, $C$ can be chosen independent of $f$ and $g$ so that $|\Omega| < |E|/2$. Set $E' = E - \Omega = E \cap \Omega^c$. Then, $E' \subseteq E$ and $|E'| > |E|/2$.

Fix $\tilde{k} \in \mathbb{N}^2$, and set $Z_{\tilde{k}} = \{MSf = 0\} \cup \{SMg = 0\} \cup \{SS^{\tilde{k}}(\chi_{E'}) = 0\}$. Let $\mathcal{D}$ be any finite collection of dyadic rectangles. Consider three subcollections. Set $\mathcal{D}_1 = \{R \in \mathcal{D} : R \cap Z_{\tilde{k}} \neq \emptyset\}$. For the remaining rectangles, let $\mathcal{D}_2 = \{R \in \mathcal{D} - \mathcal{D}_1 : R \subseteq \bar{\Omega}_{\tilde{k}}\}$ and $\mathcal{D}_3 = \{R \in \mathcal{D} - \mathcal{D}_1 : R \cap \tilde{\Omega}_{\tilde{k}} \neq \emptyset\}$.

If $R \in \mathcal{D}_1$, then there is some $(x, y) \in R \cap Z_{\tilde{k}}$. Namely, $MSf(x, y) = 0, SMg(x, y) = 0$, or $SS^{\tilde{k}}(\chi_{E'})(x, y) = 0$. If it is the first, $\langle \phi_R^{1,\tilde{k}}, f \rangle = 0$. If it is the second, then $\langle \phi_R^{2,\tilde{k}}, g \rangle = 0$, and if it is the third, $\langle \phi_R^{3,\tilde{k}}, \chi_{E'} \rangle = 0$. As this holds for all $R \in \mathcal{D}_1$, we have

$$\sum_{R \in \mathcal{D}_1} \frac{1}{|R|^{1/2}} |\langle \phi_R^{1,\tilde{k}}, f \rangle| |\langle \phi_R^{2,\tilde{k}}, g \rangle| |\langle \phi_R^{3,\tilde{k}}, \chi_{E'} \rangle| = 0.$$
Now suppose $R \in \mathcal{D}_2$, namely $R \subseteq \Omega_{\vec{k}}$. For some $\vec{k}$, $\phi_{\vec{k}}^{3,\vec{k}}$ is identically 0 and $\langle \phi_{\vec{k}}^{3,\vec{k}}, \chi_{E'} \rangle = 0$. For all others, $\phi_{\vec{k}}^{3,\vec{k}}$ is supported in $2^k R$. Let $(x, y) \in 2^k R$, and observe

$$MS(\chi_{\Omega_{\vec{k}}})(x, y) \geq \frac{1}{|2^k R|} \int_{2^k R} \chi_{\Omega_{\vec{k}}} dm \geq \frac{1}{2^{|k|}} \frac{1}{|R|} \int_R \chi_{\Omega_{\vec{k}}} dm = 2^{-|k|} > 2^{-|\vec{k}|-1}.$$ 

That is, $2^k R \subseteq \Omega_{\vec{k}} \subseteq \Omega$, a set disjoint from $E'$. Thus, $\langle \phi_{\vec{k}}^{3,\vec{k}}, \chi_{E'} \rangle = 0$. As this holds for all $R \in \mathcal{D}_2$, we have

$$\sum_{R \in \mathcal{D}_2} \frac{1}{|R|^{1/2}} |\langle \phi_R^1, f \rangle| |\langle \phi_R^2, g \rangle| |\langle \phi_R^3, \chi_{E'} \rangle| = 0.$$

Finally, we concentrate on $\mathcal{D}_3$. Define $\Omega_{-3[\vec{k}]+1}$ and $\Pi_{-3[\vec{k}]+1}$ by

$$\Omega_{-3[\vec{k}]+1} = \{ MSf > C2^{3|\vec{k}|+1-1} \},$$
$$\Pi_{-3[\vec{k}]+1} = \{ I \in \mathcal{D}_3 : |I \cap \Omega_{-3[\vec{k}]+1}| > |R|/100 \}.$$ 

Inductively, define for all $n > -3|\vec{k}| + 1$,

$$\Omega_n = \{ MSf > C2^{-n} \},$$
$$\Pi_n = \{ R \in \mathcal{D}_3 - \bigcup_{j=-3|\vec{k}|-1}^{n-1} \Pi_j : |R \cap \Omega_n| > |R|/100 \}.$$ 

As every $R \in \mathcal{D}_3$ is not in $\mathcal{D}_1$, that is $MSf > 0$ on $R$, it is clear that each $R \in \mathcal{D}_3$ will be in one of these collections.

Set $\Omega'_{-3[\vec{k}]} = \Omega_{-3[\vec{k}]}$ for symmetry. Define $\Omega'_{-3[\vec{k}]+1}$ and $\Pi'_{-3[\vec{k}]+1}$ by

$$\Omega'_{-3[\vec{k}]+1} = \{ SMg > C2^{3|\vec{k}|+1-1} \},$$
$$\Pi'_{-3[\vec{k}]+1} = \{ R \in \mathcal{D}_3 : |R \cap \Omega'_{-3[\vec{k}]+1}| > |R|/100 \}.$$ 

Inductively, define for all $n > -3|\vec{k}| + 1$,

$$\Omega'_n = \{ SMg > C2^{-n} \},$$
$$\Pi'_n = \{ R \in \mathcal{D}_3 - \bigcup_{j=-3|\vec{k}|-1}^{n-1} \Pi'_j : |R \cap \Omega'_n| > |R|/100 \}.$$ 

Again, all $R \in \mathcal{D}_3$ must be in one of these collections.
Choose an integer \( N \) big enough so that \( \Omega''_N = \{ SS^k(\chi_{E'}) > 2^N \} \) has very small measure. In particular, we take \( N \) big enough so that \(|R \cap \Omega''_N| < |R|/100 \) for all \( R \in \mathcal{D}_3 \), which is possible since \( \mathcal{D}_3 \) is a finite collection. Define

\[
\Omega''_{N+1} = \{ SS^k(\chi_{E'}) > 2^{N-1} \},
\]

\[
\Pi''_{N+1} = \{ R \in \mathcal{D}_3 : |R \cap \Omega''_{N+1}| > |R|/100 \},
\]

and

\[
\Omega''_n = \{ SS^k(\chi_{E'}) > 2^{-n} \},
\]

\[
\Pi''_n = \{ R \in \mathcal{D}_3 : \bigcup_{j=-N+1}^{n-1} \Pi''_j : |R \cap \Omega''_n| > |R|/100 \},
\]

Again, all \( R \in \mathcal{D}_3 \) must be in one of these collections.

Consider \( R \in \mathcal{D}_3 \), so that \( R \cap \tilde{\Omega}_k^c \neq \emptyset \). Then, there is some \((x, y) \in R \cap \tilde{\Omega}_k^c\) which implies \(|R \cap \Omega_{-3|k|}|/|R| \leq M_S(\chi_{\Omega_{-3|k|}})(x, y) \leq 1/100 \). Write \( \Pi_{n_1,n_2,n_3} = \Pi_{n_1} \cap \Pi_{n_2} \cap \Pi_{n_3}^c \). So,

\[
\sum_{R \in \mathcal{D}_3} \frac{1}{|R|^{1/2}} |\langle \phi_R^1, f \rangle||\langle \phi_R^2, g \rangle||\langle \phi_R^3, \chi_{E'} \rangle|
\]

\[
= \sum_{n_1,n_2>-3|k|, n_3>-N} \left[ \sum_{R \in \Pi_{n_1,n_2,n_3}} \frac{1}{|R|^{1/2}} |\langle \phi_R^1, f \rangle||\langle \phi_R^2, g \rangle||\langle \phi_R^3, \chi_{E'} \rangle| \right]
\]

\[
= \sum_{n_1,n_2>-3|k|, n_3>-N} \left[ \sum_{R \in \Pi_{n_1,n_2,n_3}} \frac{|\langle \phi_R^1, f \rangle||\langle \phi_R^2, g \rangle||\langle \phi_R^3, \chi_{E'} \rangle|}{|R|^{1/2} |R|^{1/2} |R|^{1/2}} |R| \right].
\]

Suppose \( R \in \Pi_{n_1,n_2,n_3} \). If \( n_1 > -3|k| + 1 \), then \( R \in \Pi_{n_1} \), which in particular says \( R \notin \Pi'_{n_1-1} \). So, \(|R \cap \Omega_{n_1-1}| \leq |R|/100 \). If \( n_1 = -3|k| + 1 \), then we still have \(|R \cap \Omega_{-3|k|}| \leq |R|/100 \), as \( R \in \mathcal{D}_3 \). Similarly, if \( n_2 > -3k + 1 \), then \( R \in \Pi_{n_2} \), which in particular says \( R \notin \Pi'_{n_2-1} \). So, \(|R \cap \Omega_{n_2-1}| \leq |R|/100 \). If \( n_2 = -3|k| + 1 \), then we still have \(|R \cap \Omega'_{-3|k|}| = |R \cap \Omega_{-3|k|}| \leq |R|/100 \), as \( R \in \mathcal{D}_3 \). Finally, if \( n_3 > -N + 1 \), then \( R \notin \Pi_{n_3-1} \) and \(|R \cap \Omega_{n_3-1}| \leq |R|/100 \). If \( n_3 = -N + 1 \), then \(|R \cap \Omega_{n_3-1}| \leq |R|/100 \) by the choice of \( N \). So, \(|R \cap \Omega_{n_1-1} \cap \Omega_{n_2-1} \cap \Omega_{n_3-1}^c| \geq \frac{97}{100} |R| \). Let \( \Omega_{n_1,n_2,n_3} = \bigcup \{ R : R \in \Pi_{n_1,n_2,n_3} \} \). Then,

\[
|R \cap \Omega_{n_1-1} \cap \Omega_{n_2-1} \cap \Omega_{n_3-1} \cap \Omega_{n_1,n_2,n_3}| \geq \frac{97}{100} |R| \]
for all \( R \in \Pi_{n_1,n_2,n_3} \). Further, 

\[
\sum_{R \in \Pi_{n_1,n_2,n_3}} \frac{|\langle \phi^1_R, f \rangle|}{|R|^{1/2}} \frac{|\langle \phi^2_R, g \rangle|}{|R|^{1/2}} \frac{|\langle \phi^{3,E}_R, \chi_E \rangle|}{|R|^{1/2}} \leq \sum_{R \in \Pi_{n_1,n_2,n_3}} \frac{|\langle \phi^1_R, f \rangle|}{|R|^{1/2}} \frac{|\langle \phi^2_R, g \rangle|}{|R|^{1/2}} \frac{|\langle \phi^{3,E}_R, \chi_E \rangle|}{|R|^{1/2}}
\]

\[
= \int_{\Omega_{n_1-1}^e \cap \Omega_{n_2-1}^e \cap \Omega_{n_3-1}^e \cap \Omega_{n_1,n_2,n_3}} \chi_R(x,y) dx dy 
\]

\[
\leq \int_{\Omega_{n_1-1}^e \cap \Omega_{n_2-1}^e \cap \Omega_{n_3-1}^e \cap \Omega_{n_1,n_2,n_3}} MSf(x,y)SMg(x,y)SS^E(\chi_E)(x,y) dx dy 
\]

\[
\leq C2^{-n_1-2^{-n_2-2^{-n_3}}}|\Omega_{n_1,n_2,n_3}|.
\]

Note,

\[
|\Omega_{n_1,n_2,n_3}| \leq |\bigcup \{R : R \in \Pi_{n_1}\}| \leq |\{MS(\chi_{\Omega_{n_1}}) > 1/100\}| 
\]

\[
\lesssim |\Omega_{n_1}| = |\{MSf > C2^{-n_1}\}| \lesssim C^{-p_1}2^{p_1n_1}.
\]

Repeating the argument,

\[
|\Omega_{n_1,n_2,n_3}| \lesssim |\Omega_{n_2}'| = |\{SMg > C2^{-n_2}\}| \lesssim C^{-p_2}2^{p_2n_2}, \text{ and}
\]

\[
|\Omega_{n_1,n_2,n_3}| \lesssim |\Omega_{n_3}'| = |\{SS^E(\chi_E) > 2^{-n_3}\}| \lesssim 2^{a_{n_3}}
\]

for any \( \alpha \geq 1 \). Thus, \(|\Omega_{n_1,n_2,n_3}| \lesssim C^{-p_1-p_2}2^{\theta_1p_1n_1}2^{\theta_2p_2n_2}2^{\theta_3n_3} \) for any \( \theta_1 + \theta_2 + \theta_3 = 1 \), 0 \( \leq \theta_1, \theta_2, \theta_3 \leq 1 \). Hence,

\[
\sum_{R \in D_1} \frac{1}{|R|^{1/2}} |\langle \phi^1_R, f \rangle| |\langle \phi^2_R, g \rangle| |\langle \phi^{3,E}_R, \chi_E \rangle| 
\]

\[
\lesssim \sum_{n_1,n_2 \geq -3|\bar{k}|, n_3 \geq 0} 2^{(\theta_1p_1-1)n_1}2^{(\theta_2p_2-1)n_2}2^{(\theta_1p_1-1)n_3} 
\]

\[
+ \sum_{n_1,n_2 \geq -3|\bar{k}|, -N < n_3 \leq 0} 2^{(\theta_1p_1-1)n_1}2^{(\theta_2p_2-1)n_2}2^{(\theta_3p_3-1)n_3}
\]

\[=: A + B.\]
For the first term, take $\theta_1 = 1/(2p_1)$, $\theta_2 = 1/(2p_2)$, $\theta_3 = 1 - 1/(2p)$, and $\alpha = 1$. For the second term, take $\theta_1 = 1/(3p_1)$, $\theta_2 = 1/(3p_2)$, $\theta_3 = 1 - 1/(3p) > 0$, and $\alpha = 2/\theta_3$ to see

$$A = \sum_{n_1,n_2>3|\vec{k}|, n_3>0} 2^{-n_1/2} 2^{-n_2/2} 2^{-n_3/2} p \lesssim 2^{|\vec{k}|} 2^{1/2} p \lesssim 2^{|\vec{k}|+1},$$

$$B = \sum_{n_1,n_2>3|\vec{k}|, -N<n_3\leq 0} 2^{-2n_1/3} 2^{-2n_2/3} 2^{n_3} \leq \sum_{n_1,n_2>3|\vec{k}|, n_3\leq 0} 2^{-2n_1/3} 2^{-2n_2/3} 2^{n_3} \lesssim 2^{|\vec{k}|}.$$ Note, there is no dependence on the number $N$, which depends on $D$, or $C$, which depends on $E$.

Combining the estimates for $D_1$, $D_2$, and $D_3$, we see

$$\sum_{R\in D} \frac{1}{|R|^{1/2}} |\langle \phi^1_R, f \rangle| |\langle \phi^2_R, g \rangle||\langle \phi^3_R, \chi_{E'} \rangle| \lesssim 2^{|\vec{k}|},$$

where the constant has no dependence on the collection $D$. Hence, as $D$ is arbitrary, we have

$$|\left| \sum_{R} \frac{1}{|R|^{1/2}} |\langle \phi^1_R, f \rangle| |\langle \phi^2_R, g \rangle||\langle \phi^3_R, \chi_{E'} \rangle| \right| \leq \sum_{R} \frac{1}{|R|^{1/2}} |\langle \phi^1_R, f \rangle| |\langle \phi^2_R, g \rangle||\langle \phi^3_R, \chi_{E'} \rangle| \lesssim 2^{|\vec{k}|},$$

which completes the proof.

It should now be clear that proving the above for $(a,b) \neq (1,2)$ follows by permuting the roles of $MM$, $MS$, $SM$, and $SS$. For instance, if $(a,b) = (1,1)$, then we consider $MMf$, $SSg$, and $SS^k \chi_{E'}$.

### 7. Multi-parameter multipliers

Finally, we would like to consider multipliers, and their corresponding operators, which are multi-parameter. That is, $m$ acts as if the product of $s$ standard multipliers.

For a vector $\vec{t} \in \mathbb{R}^d$ and $1 \leq j \leq s$, let $\rho_j(\vec{t}) = (t_j, t_{j+s}, \ldots, t_{j+s(d-1)}) \in \mathbb{R}^d$. Conversely, for $1 \leq j \leq d$, let $\vec{t}_j = (t_{s(j-1)+1}, \ldots, t_{js}) \in \mathbb{R}^s$ so that $\vec{t} = (\vec{t}_1, \ldots, \vec{t}_d)$.

Let $m : \mathbb{R}^d \to \mathbb{C}$ be smooth away from the origin and uniformly bounded. We say $m$ is an $s$-parameter multiplier if

$$|\partial^\alpha m(\vec{t})| \lesssim \prod_{j=1}^s ||\rho_j(\vec{t})||^{-|\alpha_j|}$$

for all indices $|\alpha| \leq sd(d + 3)$, where $|| \cdot ||$ is the Euclidean norm on $\mathbb{R}^d$. 

Given such a multiplier $m$ on $\mathbb{R}^{sd}$ and $L^1$ functions $f_1, \ldots, f_d : \mathbb{T}^s \to \mathbb{C}$, we define the associated multiplier operator $\Lambda_m^{(s)}(f_1, \ldots, f_d) : \mathbb{T}^s \to \mathbb{C}$ as

$$
\Lambda_m^{(s)}(f_1, \ldots, f_d)(\vec{x}) = \sum_{\vec{t} \in \mathbb{Z}^{sd}} m(\vec{t}) \hat{f}_1(\vec{t}_1) \cdots \hat{f}_d(\vec{t}_d) e^{2\pi i \vec{x} \cdot (\vec{t}_1 + \cdots + \vec{t}_d)}.
$$

The familiar $L^p$ estimates of still hold with minor modifications.

**Theorem 7.1.** For any $s$-parameter multiplier $m$ on $\mathbb{R}^{sd}$, it follows that $\Lambda_m^{(s)} : L^{p_1} \times \cdots \times L^{p_d} \to L^p$ for $1 < p_j < \infty$ and $1/p_1 + \cdots + 1/p_d = 1/p$. If any or all of the $p_j$ are equal to 1, this still holds with $L^p$ replaced by $L^p,\infty$ and $L^{p_j}$ replaced by $L(\log L)^{-1}$. In particular, $\Lambda_m^{(s)} : L(\log L)^{s-1} \times \cdots \times L(\log L)^{s-1} \to L^{1/d, \infty}$.

In view of these results, we now have a good perception of the heuristics. Away from $p_j = 1$, each of these operators act the same. However, it is these endpoint cases which are the most interesting. Each time we go up a parameter, we “gain a log” at the endpoint.

Just as in the bi-parameter case, we can reduce to paraproducts. We say $Q \subset \mathbb{T}^s$ is a dyadic rectangle if $Q = I_1 \times \cdots \times I_s$ for dyadic intervals $I_j$. Let $\varphi_Q : \mathbb{T}^s \to \mathbb{C}$ be the $s$-fold tensor product of adapted families. The appropriate (bi-linear) paraproducts in this setting are

$$
T_{\epsilon_1, \cdots, \epsilon_s}^{a_1, \cdots, a_s}(f, g)(\vec{x}) = \sum_Q \frac{1}{|Q|^{1/2}} \langle \phi_Q, f \rangle \langle \phi_Q, g \rangle \phi_Q^{a}(\vec{x})
$$

where the sum is over all dyadic rectangles $Q$. Each $a_j$ ranges over 1, 2, 3. If $\phi_Q^{a} = \phi_{I_1}^{a_1} \otimes \cdots \otimes \phi_{I_s}^{a_s}$, then $\int_{\mathbb{T}^s} \phi_Q^{a} d\vec{x} = 0$ whenever $i \neq a_j$.

To complete the proof on $s$-parameter multiplier operators, it suffices to show the associated paraproducts satisfy the same bounds. The same stopping time argument works equally well in all dimensions, given the correct $s$-fold hybrid operators. Therefore, we will understand the paraproducts if we can show each $s$-fold hybrid operator maps $L^{p} \rightarrow L^{p}$ for $1 < p < \infty$ and $L(\log L)^{s-1} \rightarrow L^{1/d, \infty}$.

For illustrative purposes, we show this for one specific operator when $s = 3$. For $f : \mathbb{T}^3 \to \mathbb{C}$ define

$$
SSM f(x, y, z) = \left( \sum_{I_1} \sum_{I_2} \left( \sup_{I_3} \frac{1}{|I_1|^{1/2}} |\langle \phi_{I_3}, f \rangle| \chi_{I_3}(z) \right)^2 \chi_{I_1}(x) \chi_{I_2}(y) \right)^{1/2}.
$$

Using the same notational conveniences as before,

$$
SSM f = \left( \sum_{I_1} \sum_{I_2} M_3'(\frac{\langle f, \phi_{I_1} \otimes \phi_{I_2} \rangle}{|I_1|^{1/2}|I_2|^{1/2}} \chi_{I_1} \chi_{I_2})^2 \right)^{1/2}.
$$
So,

\[ \|SSMf\|_p = \left\| \left( \sum_{I_1} \sum_{I_2} M_3' \left( \frac{\langle f, \phi_{I_1} \otimes \phi_{I_2} \rangle}{|I_1|^{1/2}|I_2|^{1/2}} \chi_{I_1} \chi_{I_2} \right)^2 \right)^{1/2} \right\|_p \]

\[ \lesssim \left\| \left( \sum_{I_1} \sum_{I_2} \frac{|\langle f, \phi_{I_1} \otimes \phi_{I_2} \rangle|^2}{|I_1||I_2|} \chi_{I_1} \chi_{I_2} \right)^{1/2} \right\|_p \]

\[ = \left\| \left( \sum_{I_1} S_2 \left( \frac{\langle f, \phi_{I_1} \rangle}{|I_1|^{1/2}} \chi_{I_1} \right)^2 \right)^{1/2} \right\|_p \lesssim \left\| \left( \sum_{I_1} \frac{|\langle f, \phi_{I_1} \rangle|^2}{|I_1|} \chi_{I_1} \right)^{1/2} \right\|_p \]

\[ = \|S_1f\|_p \lesssim \|f\|_p, \]

and

\[ \|SSMf\|_{1,\infty} = \left\| \left( \sum_{I_1} \sum_{I_2} M_3' \left( \frac{\langle f, \phi_{I_1} \otimes \phi_{I_2} \rangle}{|I_1|^{1/2}|I_2|^{1/2}} \chi_{I_1} \chi_{I_2} \right)^2 \right)^{1/2} \right\|_{1,\infty} \]

\[ \lesssim \left\| \left( \sum_{I_1} S_2 \left( \frac{\langle f, \phi_{I_1} \rangle}{|I_1|^{1/2}} \chi_{I_1} \right)^2 \right)^{1/2} \right\|_1 \lesssim \|S_1f\|_{L \log L} \lesssim \|f\|_{L(\log L)^2}. \]

The recipe for arbitrary s-fold hybrid operators should now be clear. Each such operator is pointwise smaller than one of the form SS...SMM...M. In this case, the M...MM part is bounded by \(M_j \circ M_{j+1} \circ \cdots \circ M_s\). Repeated iterations of Fefferman-Stein eliminate these \(M_j\), while the remaining SS...S part can be dealt with as usual.

**Acknowledgements.** The author would like to extend his sincere gratitude to Camil Muscalu. The author received support from both NSF and Department of Defense Graduate Fellowships.

**References**


Recibido: 11 de septiembre de 2008
Revisado: 22 de noviembre de 2008

John T. Workman
5811 Sable Dr.
Alexandria, VA 22303
United States
johntylerworkman@gmail.com