Integral Closure of Monomial Ideals on Regular Sequences

Karlheinz Kiyek and Jürgen Stückrad

Abstract

It is well known that the integral closure of a monomial ideal in a polynomial ring in a finite number of indeterminates over a field is a monomial ideal, again. Let \( R \) be a noetherian ring, and let \((x_1, \ldots, x_d)\) be a regular sequence in \( R \) which is contained in the Jacobson radical of \( R \). An ideal \( \mathfrak{a} \) of \( R \) is called a monomial ideal with respect to \((x_1, \ldots, x_d)\) if it can be generated by monomials \( x_1^{i_1} \cdots x_d^{i_d} \). If \( x_1 R + \cdots + x_d R \) is a radical ideal of \( R \), then we show that the integral closure of a monomial ideal of \( R \) is monomial, again. This result holds, in particular, for a regular local ring if \((x_1, \ldots, x_d)\) is a regular system of parameters of \( R \).

1. Introduction

Let \( A \) be a polynomial ring over a field in a finite number of indeterminates. It is well known that the integral closure \( \overline{\mathfrak{a}} \) of a monomial ideal \( \mathfrak{a} \) of \( A \) is a monomial ideal, again: \( \overline{\mathfrak{a}} \) is generated by all monomials \( m \) with \( m' \in \mathfrak{a}^l \) for some \( l \in \mathbb{N} \) [cf. [12], section 6.6, Example 6.6.1]. While studying a particular class of ideals in two-dimensional regular local rings [cf. the example at the end of this paper], the following question arose naturally: Let \( R \) be a noetherian ring, and let \((x_1, \ldots, x_d)\) be a regular sequence in \( R \) such that \( q := x_1 R + \cdots + x_d R \) is contained in the Jacobson radical of \( R \). Let \( \mathfrak{a} \) be an ideal of \( R \) that is generated by monomials in \( x_1, \ldots, x_d \); such ideals shall be called monomial ideals. Is the integral closure \( \overline{\mathfrak{a}} \) of \( \mathfrak{a} \) a monomial ideal, again?

In this paper the question is answered in the positive under the assumption that \( R/q \) is a reduced ring.

2000 Mathematics Subject Classification: Primary 13B22; Secondary 13B25.

Keywords: Regular sequences, monomial ideals, integral closure of monomial ideals.
In section 2 we collect some useful results on monomial ideals; in particular, we show that the usual ideal-theoretic operations, applied to monomial ideals, lead again to monomial ideals. It is also shown that for a monomial ideal \( \mathfrak{a} \) the ideal \( \text{gr}(\mathfrak{a}) \) in the associated graded ring \( \text{gr}_q(R) \) which is a polynomial ring over \( R/q \) is a monomial ideal.

In section 3 we introduce the notion of a monomial representation of an element of \( R \) and we show that, if \( R \) is complete, every element of \( R \) admits a monomial representation. In section 4 we associate with a monomial ideal \( \mathfrak{a} \) the ideal \( \overline{\mathfrak{a}} \) which is generated by all monomials \( m \) in \( R \) with \( m^l \in \mathfrak{a}^l \) for some \( l \in \mathbb{N} \). In section 5 we study monomial ideals in a polynomial ring over a reduced ring, and we show that for a monomial ideal \( \mathfrak{a} \) we have \( \overline{\mathfrak{a}} = \overline{\mathfrak{a}} \) where \( \overline{\mathfrak{a}} \) denotes the integral closure of \( \mathfrak{a} \). Let \( \mathfrak{a} \) be a monomial ideal in \( R \). Using the results of section 5 we show in section 6 that \( \overline{\mathfrak{a}} = \overline{\mathfrak{a}} \) if \( R \) is complete and \( q \) is a prime ideal. As a last step we show that this equality holds also if \( R \) is not necessarily complete, and if \( R/q \) is a reduced ring.

2. Monomial Ideals

2.1. Basic Definitions

**Notation 1** Let \( R \) be a ring. A sequence \( x := (x_1, \ldots, x_d) \) in \( R \) is called a weak regular sequence in \( R \) if

(a) \( x_i \) is regular for \( R/(x_1, \ldots, x_{i-1}) \) [i.e., the image of \( x_i \) in \( R/(x_1, \ldots, x_{i-1}) \) is a non-zero divisor] for every \( i \in \{1, \ldots, d\} \),

and it is called a regular sequence in \( R \) if, in addition,

(b) \( R \neq xR \).

In the sequel, we consider regular sequences \( x \) in \( R \) with the following additional property:

(c) every permutation \( (x_{\pi(1)}, \ldots, x_{\pi(d)}) \) of \( x \) is a regular sequence in \( R \).

Then every subsequence of \( x \) satisfies (a)-(c).

If \( R \) is noetherian, and if a regular sequence \( x \) in \( R \) is contained in the Jacobson radical [i.e., in the intersection of all maximal ideals] of \( R \), then (a) implies (c) [cf. [2], Ch. X, § 9, no. 7, Th. 1 and Cor. 1], and for the ideal \( q \) generated by \( x_1, \ldots, x_d \) we have \( \bigcap q^p = (0) \) [cf. [3], Ch. III, § 3, no. 3, Prop. 6].

If \( \varphi: R \to S \) is a flat homomorphism of rings, and if \( \varphi(x)S \neq S \), then the sequence \( \varphi(x) \) in \( S \) satisfies (a)-(c) [cf. [4], Ch. I, Prop. 1.1.1].
(1) For every $d$-tuple $i := (i_1, \ldots, i_d) \in \mathbb{N}_0^d$ we define $\deg(i) := i_1 + \cdots + i_d$, the degree of $i$, and we write

$$x^i := x_1^{i_1} \cdots x_d^{i_d}.$$ 

Since $x$ is a regular sequence, we have, for $i, j \in \mathbb{N}_0^d$, $x^i = x^j$ iff $i = j$.

(2) An element $m \in R$ is called a monomial with respect to $x$ if there exists $i \in \mathbb{N}_0^d$ with $m = x^i$; $i$ is determined uniquely by $m$. We call $\deg(m) := \deg(i)$ the degree of $m$.

(3) Let $x^i = x_1^{i_1} \cdots x_d^{i_d}$ be a monomial with respect to $x$. The set

$$\text{Supp}(x^i) := \{j \mid j \in \{1, \ldots, d\}, i_j \neq 0\}$$

is called the support of $x^i$.

(4) Let $M(x)$ be the set of all monomials of $R$ with respect to $x$. Clearly $M(x)$ is a commutative monoid with cancellation law, and $\deg : M(x) \rightarrow \mathbb{N}_0$ is a surjective homomorphism of monoids.

(5) An ideal $a$ of $R$ is called monomial with respect to $x$ if it is generated by elements in $M(x)$. In particular, the zero ideal and $R$ itself are monomial ideals.

**Remark 1** Let $i = (i_1, \ldots, i_d), j = (j_1, \ldots, j_d) \in \mathbb{N}_0^d$.

(1) If $x^i \in x^j R$, then we have $i_1 \geq j_1, \ldots, i_d \geq j_d$ and $x^i = x^j x^{i-j}$. In this case we say that $x^j$ divides $x^i$, and we write $x^j \mid x^i$.

(2) We define

$$k_\tau := \min \{i_\tau, j_\tau\}, \quad l_\tau := \max \{i_\tau, j_\tau\} \quad \text{for} \quad \tau \in \{1, \ldots, d\}$$

and

$$k := (k_1, \ldots, k_d), \quad l := (l_1, \ldots, l_d);$$

then

$$\gcd(x^i, x^j) := x^k, \quad \text{lcm}(x^i, x^j) := x^l$$

is the greatest common divisor resp. the least common multiple of $x^i$ and $x^j$. In particular, for monomials $m, n$ we have $mR : nR = (\text{lcm}(m, n)/n)R = (m/\gcd(m, n))R$. 


Notation 2 For the rest of this paper let \( R \) be a noetherian ring, and let \( x = (x_1, \ldots, x_d) \) be a fixed sequence in \( R \) which satisfies (a)-(c) above; all monomials of \( R \) are monomials with respect to \( x \), and all monomial ideals of \( R \) are monomial ideals with respect to \( x \). The set of all monomials of \( R \) shall be denoted by \( M \).

Definition 1 Let \( U \) be a subset of \( \{1, \ldots, d\} \); we define
\[
q_U := \sum_{i \in U} x_i R, \quad \mathcal{P}_U := \text{Ass}(R/q_U).
\]
If \( U = \{1, \ldots, d\} \), then we write
\[
q := q_U = \sum_{i=1}^{d} x_i R, \quad \mathcal{P} := \text{Ass}(R/q).
\]

Remark 2 (1) Note that \( \text{Ass}(R) = \mathcal{P}_\emptyset \).

(2) Let \( U \subset \{1, \ldots, d\}, \ i \in \{1, \ldots, d\} \setminus U \). Then \( x_i \) is regular for \( R/q_U \), hence, in particular, \( x_i / \in p \) for every \( p \in \mathcal{P}_U \).

Lemma 1 Let \( a \) be a monomial ideal of \( R \), and let \( \{m_1, \ldots, m_r\} \) be a system of generators of \( a \) consisting of monomials. Then we have
\[
\text{Ass}(R/a) \subset \bigcup_{U \subset \text{Supp}(m_1) \cup \cdots \cup \text{Supp}(m_r)} \mathcal{P}_U.
\]

Proof: There is nothing to prove if \( a = (0) \). We consider the case that \( a \neq (0) \). We define \( V := \text{Supp}(m_1) \cup \cdots \cup \text{Supp}(m_r) \). We prove the assertion by induction on \( s := \text{deg}(m_1) + \cdots + \text{deg}(m_r) - r \). If \( s = 0 \), then we have \( a = q_V \); in this case the assertion holds. Let \( s > 0 \), and assume that the assertion holds for all monomial ideals of \( R \) which admit a system of monomial generators \( m'_1, \ldots, m'_r \) with \( \text{deg}(m'_1) + \cdots + \text{deg}(m'_r) - r' < s \). Now let \( a \) be a monomial ideal of \( R \) having a system of monomial generators \( m_1, \ldots, m_r \) with \( \text{deg}(m_1) + \cdots + \text{deg}(m_r) - r = s \). Then there exists \( j \in \{1, \ldots, r\} \) with \( \text{deg}(m_j) \geq 2 \); by relabelling, we may assume that \( j = 1 \).

Let \( i \in \text{Supp}(m_1) \); let us label the monomials \( m_1, \ldots, m_r \) in such a way that \( i \in \text{Supp}(m_j) \) for \( j \in \{1, \ldots, t\} \) and \( i \notin \text{Supp}(m_j) \) for \( j \in \{t+1, \ldots, r\} \); here we have \( t \in \{1, \ldots, r\} \). For \( j \in \{1, \ldots, t\} \) we have \( m_j = x_i m'_j \) where \( m'_1, \ldots, m'_t \) are monomials. We put
\[
a_1 := m'_1 R + \cdots + m'_t R, \quad a_2 = m_{t+1} R + \cdots + m_r R, \quad b := a_1 + a_2,
\]
\[
V_1 := \bigcup_{j=1}^{t} \text{Supp}(m'_j), \quad V_2 := \bigcup_{j=t+1}^{r} \text{Supp}(m_j). \]
If $a_2 = (0)$, then we have $a : x_i = b$. This is also true if $a_2 \neq (0)$. In fact, by our induction assumption we get $\text{Ass}(R/a_2) \subset \bigcup_{U \in V_2} \mathcal{P}_U$. Using $i \notin V_2$, we see that $V_2 \subset \{1, \ldots, d\} \setminus \{i\}$. From Remark 2 we get the following: If $U \subset V_2$, then $x_i \notin p$ for every prime ideal $p \in \mathcal{P}_U$, hence $x_i \notin p$ for every $p \in \text{Ass}(R/a_2)$, hence $x_i$ is regular for $R/a_2$. This implies that $a : x_i = a_1 + a_2 = b$ since $a = x_i a_1 + a_2$.

Therefore the sequence

$$0 \longrightarrow \frac{R}{b} \xrightarrow{x_i} \frac{R}{a} \longrightarrow \frac{R}{(a + x_i R)} \longrightarrow 0$$

is exact; note that

$$\text{Ass}(R/a) \subset \text{Ass}(R/b) \cup \text{Ass}(R/(a + x_i R)). \quad (\ast)$$

We have $a + x_i R = x_i R + m_{i+1} R + \cdots + m_r R$. Applying our induction assumption to $b$ and to $a + x_i R$ we obtain

$$\text{Ass}(R/b) \subset \bigcup_{U \subset V_1 \cup V_2} \mathcal{P}_U \subset \bigcup_{U \subset V} \mathcal{P}_U,$$

$$\text{Ass}(R/(a + x_i R)) \subset \bigcup_{U \subset \{i\} \cup V_2} \mathcal{P}_U \subset \bigcup_{U \subset V} \mathcal{P}_U.$$

Therefore we get, using $(\ast)$, that $\text{Ass}(R/a) \subset \bigcup_{U \subset V} \mathcal{P}_U$. \hfill \blacksquare

**Corollary 1** If $i \notin \bigcup_{j=1}^t \text{Supp}(m_j)$, then we have $a : x_i = a$.

**Proof:** The element $x_i$ is not contained in any of the prime ideals in $\text{Ass}(R/a)$ [cf. Lemma 1]. \hfill \blacksquare

### 2.2. Operations on Monomial Ideals

**Lemma 2** Let $a = m_1 R + \cdots + m_r R$ with $m_1, \ldots, m_r \in M$ be a monomial ideal in $R$. For every $m \in M$ the ideal $a : m$ is monomial, again. More precisely, we have

$$a : m = \sum_{j=1}^r \frac{\text{lcm}(m_j, m)}{m} R.$$

**Proof:** We may assume that $a \neq (0)$. We prove the assertion by induction on $\text{deg}(m)$. The case $\text{deg}(m) = 0$, i.e., $m = 1$, is clear. Let $\text{deg}(m) > 0$; then there exists $i \in \{1, \ldots, d\}$ with $x_i \mid m$, and we write $m = x_i m'$ with $m' \in M$. As in the proof of Lemma 1 we label the monomials $m_1, \ldots, m_r$ in such a way that $x_i \mid m_j$ for $j \in \{1, \ldots, t\}$, $x_i \nmid m_j$ for $j \in \{t + 1, \ldots, r\}$
with \( t \in \{0, \ldots, r\} \), and we write, for \( j \in \{1, \ldots, t\} \), \( m_j = x_i m_j' \) with monomials \( m_1', \ldots, m_j' \). Then we have, as above,

\[
a : m = (a : x_i) : m' = \left( \sum_{j=1}^{t} m_j' R + \sum_{j=t+1}^{r} m_j R \right) : m' = \sum_{j=1}^{t} \frac{lcm(m_j', m')}{m'} R + \sum_{j=t+1}^{r} \frac{lcm(m_j, m')}{m'} R = \sum_{j=1}^{r} \frac{lcm(m_j, m)}{m} R.
\]

\[\Box\]

**Corollary 2** Let \( a = m_1 R + \cdots + m_r R \) with \( m_1, \ldots, m_r \in M \) be a monomial ideal in \( R \). Let \( m \in M \); then we have

\[
a \cap mR = \sum_{j=1}^{r} \text{lcm}(m_j, m) R.
\]

**Proof:** We have \( a \cap mR = (a : m)m \).

**Lemma 3** Let \( a = m_1 R + \cdots + m_r R \), \( b = n_1 R + \cdots + n_s R \) with \( m_1, \ldots, m_r, n_1, \ldots, n_s \in M \) be monomial ideals in \( R \). Then \( a \cap b \) is a monomial ideal; more precisely, we have

\[
a \cap b = \sum_{i=1}^{r} \sum_{j=1}^{s} \text{lcm}(m_i, n_j) R.
\]

\((*)\)

**Proof:** It is clear that the right-hand side of \((*)\) is contained in the left-hand side. We prove that the left-hand side of \((*)\) is contained in the right hand side by induction on \( s \). For \( s = 1 \) the assertion is clear, and for \( s = 1 \) the assertion follows from Cor. 2. Now we assume that \( s \geq 2 \), and we define \( b' = n_1 R + \cdots + n_{s-1} R \). Let \( z \in a \cap b \). We write \( z = a_1 m_1 + \cdots + a_r m_r = b_1 n_1 + \cdots + b_s n_s \) with \( a_1, \ldots, b_s \in R \). Since \( b_s n_s = a_1 m_1 + \cdots + a_r m_r - (b_1 n_1 + \cdots + b_{s-1} n_{s-1}) \), we have \( b_s n_s \in (a + b') \cap n_s R \), hence we can write [cf. Cor. 2]

\[
b_s n_s = \sum_{i=1}^{r} c_i \text{lcm}(m_i, n_s) + \sum_{j=1}^{s-1} d_j \text{lcm}(n_j, n_s) \quad \text{with} \quad c_1, \ldots, d_{s-1} \in R.
\]

We define

\[
w := \sum_{j=1}^{s-1} (b_j n_j + d_j \text{lcm}(n_j, n_s)).
\]
Then we have \( w \in b' \), and since \( w = z - \left( c_1 \text{lcm}(m_1, n_s) + \cdots + c_r \text{lcm}(m_r, n_s) \right) \in a \), we have

\[
  w \in a \cap b' = \sum_{i=1}^{r} \sum_{j=1}^{s} \text{lcm}(m_i, n_j) R
\]

by our induction assumption. Then we get

\[
  z = w + \sum_{i=1}^{r} c_i \text{lcm}(m_i, n_s) \in \sum_{i=1}^{r} \sum_{j=1}^{s} \text{lcm}(m_i, n_j) R,
\]

and therefore the left-hand side of \((*)\) lies in the right hand side. \(\blacksquare\)

Collecting our results, we have

**Proposition 1** Let \( a, b \) be monomial ideals in \( R \). Then \( a \cap b, a \cdot b, a : b \) are monomial ideals, again. More precisely, if \( a = m_1 R + \cdots + m_r R \) and \( b = n_1 R + \cdots + n_s R \) with monomials \( m_1, \ldots, n_s \in M \), then we have

\[
  a \cap b = \sum_{i=1}^{r} \sum_{j=1}^{s} \text{lcm}(m_i, n_j) R, \tag{2.1}
\]

\[
  a : b = \bigcap_{j=1}^{s} \sum_{i=1}^{r} \frac{\text{lcm}(m_i, n_j)}{n_j} R. \tag{2.2}
\]

If \( c \) is another monomial ideal, then we have

\[
  (a + b) \cap c = (a \cap c) + (b \cap c). \tag{2.3}
\]

**Proof:** (2.3) follows from (2.1), and (2.2) is a consequence of Lemma 2 since

\[
  a : b = \bigcap_{j=1}^{s} (a : n_j). \tag*{\blacksquare}
\]

**Corollary 3** Let \( a = m_1 R + \cdots + m_r R \) with \( m_1, \ldots, m_r \in M \) be a monomial ideal in \( R \), and let \( m \in M \). Then we have \( m \in a \) iff \( m_i \mid m \) for some \( i \in \{1, \ldots, r\} \).

**Proof:** We have \( m \in a \) iff

\[
  1 \in a : m = (\text{lcm}(m_1, m)/m) R + \cdots + (\text{lcm}(m_r, m)/m) R,
\]

hence iff \( \text{lcm}(m_i, m)/m = 1 \) for some \( i \in \{1, \ldots, r\} \), and this is the case iff \( m_i \mid m \) for some \( i \in \{1, \ldots, r\} \). \(\blacksquare\)
Corollary 4  Let \( \mathfrak{a} \) be a monomial ideal in \( R \), and let \( m_1, \ldots, m_r, n_1, \ldots, n_s \) be monomials with
\[
\mathfrak{a} = \sum_{i=1}^{r} m_i R = \sum_{j=1}^{s} n_j R.
\]

(1) We assume that \( m_i \nmid m_k \) for all \( i, k \in \{1, \ldots, r\} \) with \( i \neq k \). Then we have \( \{m_1, \ldots, m_r\} \subset \{n_1, \ldots, n_s\} \).
(2) We assume, furthermore, that \( n_j \nmid n_l \) for all \( j, l \in \{1, \ldots, s\} \) with \( j \neq l \). Then we have \( r = s \) and \( \{m_1, \ldots, m_r\} = \{n_1, \ldots, n_s\} \).

Proof: (1) Note that \( \#\{m_1, \ldots, m_r\} = r \). Let \( i \in \{1, \ldots, r\} \). Then, by Cor. 3, there exist \( j \in \{1, \ldots, s\} \) and \( k \in \{1, \ldots, r\} \) with \( m_i \mid n_j \) and \( n_j \mid m_k \), hence we have \( m_i \mid m_k \). Therefore we have \( i = k \) and \( m_i = n_j \in \{n_1, \ldots, n_s\} \). This implies that \( \{m_1, \ldots, m_r\} \subset \{n_1, \ldots, n_s\} \).
(2) This follows immediately from (1).

Remark 3  The result of Cor. 4 implies the following: Every monomial ideal of \( R \) admits a uniquely determined minimal set of monomial generators where “minimal” can be understood as “minimal with respect to number” or as “irredundant”. We denote this number by \( \nu(\mathfrak{a}) \). But we can even say more:

Corollary 5  Let \( \mathfrak{a} \) be a monomial ideal in \( R \), let \( r := \nu(\mathfrak{a}) \), and let \( \{m_1, \ldots, m_r\} \subset M \) be a minimal set of monomial generators of \( \mathfrak{a} \). Then we have
\[
\mu_{R_p}(\mathfrak{a}R_p) = r \quad \text{for all } p \in V((x_1, \ldots, x_r)).
\]
Moreover, every set of generators which generates \( \mathfrak{a} \) contains at least \( r \) elements.

(In a local ring \( A \) we denote by \( \mu_A(M) \) the minimal number of generators of a finitely generated \( A \)-module \( M \).)

Proof: The second statement follows from the first one, and the first statement is obtained from Cor. 4 by replacing \( R \) by \( R_p \).

2.3. The Associated Graded Ring

Remark 4  The associated graded ring
\[
\text{gr}(R) := \text{gr}_q(R) = \bigoplus_{p \geq 0} q^p/q^{p+1} = R/q[\overline{x}_1, \ldots, \overline{x}_d]
\]
is a polynomial ring over \( R/q \) in \( \overline{x}_1 := x_1 \mod q^2, \ldots, \overline{x}_d := x_d \mod q^2 \) [cf. [2], Ch. X, § 9, no. 7, Th. 1]. Notice that the sequence \( (\overline{x}_1, \ldots, \overline{x}_d) \) is a sequence in \( \text{gr}(R) \) which satisfies (a)-(c) above.
(1) Let $\overline{M} = \{ \bar{x}^i : \bar{x}^i = \bar{x}_1^i \cdots \bar{x}_d^i | i \in \mathbb{N}_0^d \}$ be the set of monomials of the polynomial ring $R/\mathfrak{q}[\bar{x}_1, \ldots, \bar{x}_d]$; the map $\bar{x}^i \mapsto \bar{x}^i : M \to \overline{M}$ is an isomorphism of monoids. An ideal $\mathfrak{A}$ of $\text{gr}(R)$ is called a monomial ideal if it can be generated by elements in $\overline{M}$; such an ideal is a homogeneous ideal of the graded ring $\text{gr}(R)$. Every non-zero element $z \in \text{gr}(R)$ has a unique representation $z = \bar{e}_1 \bar{m}_1 + \cdots + \bar{e}_r \bar{m}_r$ with pairwise distinct monomials $\bar{m}_1, \ldots, \bar{m}_r \in \overline{M}$ and non-zero elements $\bar{e}_1, \ldots, \bar{e}_r \in R/\mathfrak{q}$; we call this the monomial representation of $z$.

(2) For every $z \in R$ with $z \notin \bigcap \mathfrak{q}^p$ we define the order $\text{ord}(z)$ to be the largest integer $p$ with $z \in \mathfrak{q}^p$. Let $p := \text{ord}(z)$; then we define the initial form of $z$ as $\text{In}(z) := z \mod \mathfrak{q}^{p+1} \in \text{gr}(R)_p$, note that $\text{In}(z)$ is a homogeneous non-zero polynomial of degree $p$. In particular, for a monomial $m \in M$ $\text{ord}(m)$ is defined, and we have $\text{ord}(m) = \deg(m)$ and $\text{In}(m) = \bar{m}$.

(3) For every ideal $\mathfrak{a}$ of $R$ we define

$$\text{gr}(\mathfrak{a}) := \bigoplus_{p \geq 0} (\mathfrak{a} \cap \mathfrak{q}^p + \mathfrak{q}^{p+1})/\mathfrak{q}^{p+1} \subset \text{gr}(R);$$

$\text{gr}(\mathfrak{a})$ is a homogeneous ideal in $\text{gr}(R)$. If $\mathfrak{b}$ is another ideal in $R$, then we have $\text{gr}(\mathfrak{a})\text{gr}(\mathfrak{b}) \subset \text{gr}(\mathfrak{ab})$.

(4) Let $\mathfrak{a} = m_1 R + \cdots + m_r R$ with $m_1, \ldots, m_r \in M$ be a monomial ideal in $R$. Then we have $\text{gr}(\mathfrak{a}) = \bar{m}_1 \text{gr}(R) + \cdots + \bar{m}_r \text{gr}(R)$, hence, in particular, $\text{gr}(\mathfrak{a})$ is a monomial ideal in $\text{gr}(R)$ [note that, for $p \in \mathbb{N}_0$, $\mathfrak{a} \cap \mathfrak{q}^p$ is generated by the elements $m_{ij} := \text{lcm}(m_i, n_j)$ where $n_j \in M$ is of degree $p$ by Lemma 3, and that $m_{ij} \in \mathfrak{q}^{p+1}$ if $\deg(m_{ij}) > p$]. In particular, for monomial ideals $\mathfrak{a}, \mathfrak{b}$ in $R$ we have $\text{gr}(\mathfrak{ab}) = \text{gr}(\mathfrak{a})\text{gr}(\mathfrak{b})$ and $\text{gr}(\mathfrak{a}^i) = (\text{gr}(\mathfrak{a}))^i$ for every $i \in \mathbb{N}$.

Remark 5 Now we assume that $\mathfrak{q}$ is a prime ideal of $R$ which is contained in the Jacobson radical of $R$ and we equip $R$ with the $\mathfrak{q}$-adic topology. Then $\bigcap \mathfrak{q}^p = (0)$ [cf. [3], Ch. III, § 3, no. 3, Prop. 6], $\text{gr}(R)$ is a domain, hence $R$ is a domain, also, and the order function is a valuation of the quotient field of $R$ [cf. [13], vol. II, Ch. VIII, § 1, Th. 1]. Moreover, all the ideals $\mathfrak{q}_U$ for every $U \subset \{1, \ldots, d\}$ are prime ideals as is easily seen by considering the sequence $(x_i \mod \mathfrak{q}_U)_{i \in \{1, \ldots, d\}\setminus U}$ in $R/\mathfrak{q}_U$. Therefore all the associated ideals of a monomial ideal $\mathfrak{a}$ of $R$ are of the form $\mathfrak{q}_U$ for some $U \subset \{1, \ldots, d\}$ [cf. Lemma 1], and therefore, by considering a primary representation of $\mathfrak{a}$, we get: if $em \in \mathfrak{a}$ with $e \in R \setminus \mathfrak{q}$ and $m \in M$, then we have $m \in \mathfrak{a}$.

Let $\hat{R}$ be the $\mathfrak{q}$-adic completion of $R$. Then $\bar{x}$ is a sequence in $\hat{R}$ which satisfies (a)-(c), $\hat{\mathfrak{q}} = \mathfrak{q} \hat{R}$ is a prime ideal in $\hat{R}$, and $\hat{R}$ is a faithfully flat $R$-module [cf. [3], Ch. III, § 3, no. 3, Prop. 6].
3. Monomial Representations

Assumption 1 In this section we assume that $q$ is a prime ideal of $R$ which is contained in the Jacobson radical of $R$.

Notation 3 Let $w \in R$ be different from 0. Then $\mathrm{In}(w) \in \mathrm{gr}(R)$ is a homogeneous polynomial of degree $\mathrm{ord}(w)$; therefore there exist uniquely determined and pairwise distinct monomials $m_1, \ldots, m_r \in M$ having degree $\mathrm{ord}(w)$ and elements $e_1, \ldots, e_r \in R \setminus q$ such that $\mathrm{In}(w) = \mathrm{In}(e_1m_1 + \cdots + e_rm_r)$; we define the set of terms of $w$ by
\[ T_m(w) := \{m_1, \ldots, m_r\}. \]
For $w = 0$ we put $\mathrm{In}(w) = 0$ and $T_m(w) = \emptyset$.

Definition 2 We say that $w \in R$, $w \neq 0$, admits a monomial representation (with respect to $x$), if there exist monomials $m_1, \ldots, m_r \in M$ and elements $e_1, \ldots, e_r \in R \setminus q$ such that
\[ w = e_1m_1 + \cdots + e_rm_r \quad \text{and} \quad \nu(m_1R + \cdots + m_rR) = r. \quad (*) \]
In (*) we have $m_i \nmid m_j$ for all $i, j \in \{1, \ldots, r\}$ with $i \neq j$; in particular, the monomials $m_1, \ldots, m_r$ are pairwise distinct. For every nonempty subset $U \subset \{1, \ldots, r\}$ clearly $\sum_{i \in U} e_i m_i =: z$ is a monomial representation of $z$.

Lemma 4 Let $w \in R \setminus \{0\}$. If $w$ admits a monomial representation $w = e_1m_1 + \cdots + e_rm_r$, then we have
\[ \mathrm{In}(w) = \sum_{\deg(m_i) = \mathrm{ord}(w)}^{r} \mathrm{In}(e_i)\mathrm{In}(m_i), \]
\[ \mathrm{ord}(w) = \min\{\deg(m_i) \mid i \in \{1, \ldots, r\}\}, \]
\[ T_m(w) = \{m_i \mid i \in \{1, \ldots, r\}, \deg(m_i) = \mathrm{ord}(w)\}. \]

Proof: Let $s := \min\{\deg(m_i) \mid i \in \{1, \ldots, r\}\}$. Then
\[ \mathrm{In}\left( \sum_{\deg(m_i) = s}^{r} e_im_i \right) = \sum_{\deg(m_i) = s}^{r} \mathrm{In}(e_i)\mathrm{In}(m_i), \]
and since $\mathrm{In}(e_i) \neq 0$ for $i \in \{1, \ldots, r\}$, we obtain
\[ \mathrm{ord}\left( \sum_{\deg(m_i) = s}^{r} e_im_i \right) = s, \]
hence ord(w) = s. Clearly we have
\[ \text{In}\left(\sum_{i=1}^{r} e_i m_i\right) = \text{In}\left(\sum_{i=1}^{r} e_i m_i\text{ }_{\text{deg}(m_i)=s}\right) = \text{In}(w). \]

\[\blacksquare\]

**Proposition 2** Let \( R \) be complete with respect to the \( q \)-adic topology. Every \( w \in R, w \neq 0 \), admits a monomial representation.

**Proof:** (1) Let \( w \in R, w \neq 0 \). Let \( \text{Tm}(w) = \{m_1, \ldots, m_r\} \). There exist elements \( e_1, \ldots, e_r \in R \setminus q \) such that
\[ \text{In}(w) = \text{In}(e_1 m_1 + \cdots + e_r m_r); \]

let us put \( \iota(w) := e_1 m_1 + \cdots + e_r m_r \). Then we have \( \text{ord}(w) = \text{ord}(\iota(w)) \) and \( \text{ord}(w - \iota(w)) > \text{ord}(w) \). If \( w = 0 \), then we put \( \iota(w) = 0 \).

(2) Let \( w \in R, w \neq 0 \). We define a sequence \( (w_p)_{p \in \mathbb{N}_0} \) in \( R \): Let \( w_0 := w \); if \( p \in \mathbb{N}_0 \), and if \( w_p \) is defined, then we define \( w_{p+1} := w_p - \iota(w_p) \).

Note the following: If \( w_p = 0 \) for one \( p \in \mathbb{N}_0 \), then \( w_q = 0 \) for every \( q \in \mathbb{N}_0 \) with \( q \geq p \), and if \( w_p \neq 0 \) for one \( p \in \mathbb{N}_0 \), then the elements \( w_0, \ldots, w_{p-1} \) are different from 0, and we have
\[ \text{ord}(w) = \text{ord}(w_0) < \text{ord}(w_1) < \cdots < \text{ord}(w_p); \]
in particular, we have \( \text{ord}(w_p) \geq p \).

For every \( p \in \mathbb{N}_0 \) let \( a_p \) be that monomial ideal of \( R \) which is generated by the monomials in \( \text{Tm}(w_0), \ldots, \text{Tm}(w_p) \). Then \( (a_p)_{p \in \mathbb{N}_0} \) is an increasing sequence of ideals in \( R \), and therefore it becomes stationary, i.e., there exists \( q \in \mathbb{N}_0 \) with \( a_q = a_{q+1} = \cdots = a \). We can write \( a = m_1 R + \cdots + m_r R \) where \( m_1, \ldots, m_r \in M \) and \( r := \nu(a) \).

(3) We have
\[ w = w_{p+1} + \sum_{j=0}^{p} \iota(w_j) \text{ for every } p \in \mathbb{N}_0; \]

note that \( w_{p+1} = 0 \) or \( \text{ord}(w_{p+1}) \geq p + 1 \), hence \( w_{p+1} \in q^{p+1} \).

Let \( j \in \mathbb{N}_0 \) with \( w_j \neq 0 \). Then we can write \( \iota(w_j) \) as a sum
\[ \iota(w_j) = \sum_{i=1}^{r} a_{ji} m_i \]
where the elements \( a_{ji} \in R \) for \( i \in \{1, \ldots, r\} \) satisfy the following condition: If \( \text{ord}(w_{ji}) < \text{deg}(m_i) \), then \( a_{ji} = 0 \), and if \( \text{ord}(w_{ji}) \geq \text{deg}(m_i) \) and \( a_{ji} \neq 0 \), then \( a_{ji} \) is a linear combination of monomials of degree \( \text{ord}(w_{ji}) - \text{deg}(m_i) \) with coefficients which lie in \( R \setminus q \) [note that the monomials in \( T_m(w_{ji}) \) lie in \( a \)]. For \( p \in \mathbb{N}_0 \) we have

\[
\sum_{j=0}^{p} t(w_j) = \sum_{i=1}^{r} e_{pi} m_i
\]

with

\[
e_{pi} := \sum_{j=0}^{p} a_{ji} \quad \text{for every } i \in \{1, \ldots, r\}.
\]

Let \( i \in \{1, \ldots, r\} \). There exists a unique \( j_i \in \{0, \ldots, q\} \) with \( \text{ord}(w_{j_i}) = \text{deg}(m_i) \) [cf. (2) and note that \( \{m_1, \ldots, m_r\} \) is a minimal system of generators of \( a \)].

We consider any integer \( p \geq q \). Then we have \( a_{ji} = 0 \) for \( j \in \{0, \ldots, j_i - 1\} \), \( a_{j_i} \in R \setminus q \), and \( a_{ji} \in q^{j-\text{deg}(m_i)} \) for \( j \in \{j_i + 1, \ldots, p\} \). In particular, \( e_{pi} \in R \setminus q \). Furthermore, we have

\[
e_{p+1,i} - e_{pi} = a_{p+1,i} \in q^{p+1-\text{deg}(m_i)};
\]

therefore, the sequence \( (e_{pi})_{p \geq 0} \) is a Cauchy sequence in \( R \setminus q \). Since \( q \) is an open ideal in the \( q \)-adic topology, we have

\[
e_i := \lim_{p \to \infty} e_{pi} \in R \setminus q.
\]

From

\[
\sum_{i=1}^{r} e_i m_i = \sum_{i=1}^{r} \left( \lim_{p \to \infty} e_{pi} \right) m_i = \lim_{p \to \infty} \left( \sum_{i=1}^{r} e_{pi} m_i \right)
\]

\[
= \lim_{p \to \infty} \left( \sum_{j=0}^{p} t(w_j) \right) = \lim_{p \to \infty} (w - w_{p+1})
\]

and \( w_{p+1} \in q^{p+1} \) for every \( p \in \mathbb{N}_0 \) we obtain

\[
w = \sum_{i=1}^{r} e_i m_i.
\]
Proposition 3 Let $a \neq (0)$ be an ideal in $R$. The following statements are equivalent:

1. $a$ is a monomial ideal.
2. For every $w \in a$, $w \neq 0$, we have $Tm(w) \subseteq a$.

Now we assume, in addition, that $R$ is complete in the $q$-adic topology. Then the following statements are equivalent with (1) and (2):

3. Every $w \in a$, $w \neq 0$, admits a monomial representation $w = e_1m_1 + \cdots + e_rm_r$ with $m_1, \ldots, m_r \in a$.
4. Let $w \in a$, $w \neq 0$, and let $w = e_1m_1 + \cdots + e_rm_r$ be a monomial representation of $w$, then $m_1, \ldots, m_r \in a$.

Proof: (1) $\Rightarrow$ (2): Let $w \in a$, $w \neq 0$, and let $Tm(w) = \{m_1, \ldots, m_r\}$; let $s := \text{ord}(w)$, hence we have $\deg(m_1) = \cdots = \deg(m_r) = s$ [cf. Lemma 4].

There exist elements $e_1, \ldots, e_r \in R \setminus q$ with $\text{ord}(w - (e_1m_1 + \cdots + e_rm_r)) > s$. Let $i \in \{1, \ldots, r\}$, and define

$$b_i := a + m_1R + \cdots + m_{i-1}R + m_{i+1}R + \cdots + m_R + q^{s+1};$$

$b_i$ is a monomial ideal of $R$. Note that $e_im_i \in b_i$, and therefore we have $m_i \in b_i$ [cf. Remark 5]. For no monomial $m \in q^{s+1}$ we have $m | m_i$ [since $\deg(m_i) = s < \deg(m)$], and we have $m_j \nmid m_i$ for $j \in \{1, \ldots, r\}$, $j \neq i$.

Therefore, by Cor. 3, there exists a monomial $m \in a$ with $m | m_i$, hence we have $m_i \in a$, and therefore we have shown that $Tm(w) \subseteq a$.

(2) $\Rightarrow$ (1): Suppose that $a$ is not a monomial ideal. This means, in particular, that $a \neq R$. Let $a'$ be the monomial ideal which is generated by all the monomials which lie in $a$; then we have $a' \nsubseteq a$. By assumption we have $Tm(w) \subset a'$ for every $w \in a$, $w \neq 0$. The prime ideals in $\text{Ass}(R/a')$ are of the form $q_U$ for $U \subset \{1, \ldots, d\}$, hence are contained in $q$ [cf. Remark 5].

By Krull’s intersection theorem [cf. [13], Vol. I, Ch. 4, § 7, Th. 12] we have $\bigcap_{n \geq 0}(a' + q^n) = a'$. Therefore there exists $n \in \mathbb{N}_0$ with $a \subset a' + q^n$, $a \not\subset a' + q^{n+1}$. We choose $w \in a$, $w \notin a' + q^{n+1}$; we can write $w = w_1 + z$ with $w_1 \in a'$, $z \in q^n$ and $z \notin q^{n+1}$. This implies that $z = w - w_1 \in a$, $z \neq 0$, and, by assumption, we have $Tm(z) \subset a$, hence $Tm(z) \subset a'$. Let $Tm(z) = \{m_1, \ldots, m_r\}$. Then there exist elements $e_1, \ldots, e_r \in R \setminus q$ such that, putting $z_1 := e_1m_1 + \cdots + e_rm_r$, we have $z_1 \in a'$ and $z \neq z_1 \in q^{n+1}$. This implies that $w = w_1 + z = w_1 + z_1 + (z - z_1) \in a' + q^{n+1}$, in contradiction with the choice of $w$.

Now we assume that $R$ is complete; then every $w \in R$, $w \neq 0$, admits a monomial representation [cf. Prop. 2].
Let $w \in a, w \neq 0$, and let $w = e_1m_1 + \cdots + e_rm_r$ be a monomial representation of $w$. We show by induction on $r$ that $\{m_1, \ldots, m_r\} \subset a$. Let $r = 1$, hence $T_m(w) = \{m_1\} \subset a$. Now let $r > 1$. It is clear that $T_m(w) \subset \{m_1, \ldots, m_r\}$. We label the elements $m_1, \ldots, m_r$ in such a way that $T_m(w) = \{m_1, \ldots, m_q\}$ with $q \leq r$. We put $w_1 := e_1m_1 + \cdots + e_qm_q$. Now we have $w_1 \in a$ by assumption. If $q = r$, then the elements $m_1, \ldots, m_q$ lie in $a$. If $q < r$, then we have $w - w_1 = e_{q+1}m_{q+1} + \cdots + e_rm_r$, and since $w - w_1 \in a$, we get by our induction assumption that $m_{q+1}, \ldots, m_r \in a$.

$(4) \Rightarrow (3)$ and $(3) \Rightarrow (1)$ are trivial.  

\section{Integral Elements}

\textbf{Remark 6} Let $S$ be a ring, and let $a$ be an ideal in $S$. The integral closure of the Rees ring

$$R(a, S) = \bigoplus_{p \geq 0} a^pT^p \subset S[T]$$

in the polynomial ring $S[T]$ is the graded ring $\bigoplus_{p \geq 0} \overline{a^p}T^p$ where, for every $p \in \mathbb{N}$, $\overline{a^p}$ is the integral closure of $a^p$ in $S$ [cf. \cite{10}, Ch. II, § 5]. In particular, an element $z \in S$ is integral over $a$ iff $zT \in S[T]$ is integral over $\bigoplus_{p \geq 0} a^pT^p$.

\textbf{Notation 4} Let $a, b$ be monomial ideals in $R$.

1. We define

$$\tilde{a} := (\{m \in M \mid \text{there exists } l \in \mathbb{N} \text{ with } m^l \in a^l\});$$

$\tilde{a}$ is a monomial ideal of $R$. Since the monomials which generate $\tilde{a}$ are integral over $a$, $\tilde{a}$ is an ideal which is integral over $a$, and therefore $\tilde{a}$ is contained in the integral closure $\overline{a}$ of $a$ in $R$, and we have

$$a \subset \tilde{a} \subset \overline{a}.$$  

It is clear that $\tilde{a} \subset a \subset \tilde{b}$, and if $a \subset b$, then we have $\tilde{a} \subset \tilde{b}$.

2. We show that

$$\tilde{\tilde{a}} = \tilde{a}.$$  

In fact, let $\tilde{a} = m_1R + \cdots + m_R R$. For every $i \in \{1, \ldots, r\}$ there exists $l_i \in \mathbb{N}$ with $m_i^l \in a^l$. Let $m$ be a monomial in $a$. Then there exists $l \in \mathbb{N}$ with $m^l \in a^l$. This implies that there exist $(i_1, \ldots, i_r) \in \mathbb{N}^r_0$ with $i_1 + \cdots + i_r = l$ and such that $m_{i_1}^l \cdot \cdots \cdot m_{i_r}^l$ divides $m^l$ [cf. Cor. 3]. Since $(m_{i_1}^l \cdot \cdots \cdot m_{i_r}^l)^{l_{i_1} \cdot \cdots \cdot l_{i_r}}$ lies in $a^{l_{i_1} \cdot \cdots \cdot l_{i_r}}$, we see that $m^{l_{i_1} \cdot \cdots \cdot l_{i_r}}$ lies in $a^{l_{i_1} \cdot \cdots \cdot l_{i_r}}$, also, and this means that $m \in \tilde{a}$.  

(3) By (1) we get \( \tilde{a}^p \tilde{a}^q \subset \tilde{a}^{p+q} \) for all \( p, q \in \mathbb{N}_0 \). Therefore
\[
\tilde{R}(a, R) := \bigoplus_{p \geq 0} \tilde{a}^p T^p \subset R[T]
\]
is a graded \( R \)-algebra and a graded \( R \)-subalgebra of \( R[T] \), and it contains the Rees ring \( \tilde{R}(a, R) := \bigoplus_{p \geq 0} a^p T^p \) of \( a \) as a graded \( R \)-subalgebra.

(4) Since \( \tilde{a}^p \subset \tilde{a}^p \) for every \( p \in \mathbb{N} \), the integral closure of \( \tilde{R}(a, R) \) in \( R[T] \) is the ring \( \bigoplus_{p \geq 0} \tilde{a}^p T^p \) [cf. Remark 6].

(5) Just as in [8], Prop. 4.6, one may prove, using (4): For \( z \in R \) we have \( z \in \tilde{a} \) iff there exist \( p \in \mathbb{N} \) and elements \( a_i \in \tilde{a}^i, i \in \{1, \ldots, p\} \), such that
\[
z^p + a_1 z^{p-1} + \cdots + a_p = 0.
\]

Assumption 2 For the rest of this section we again assume that \( q \) is a prime ideal of \( R \) which is contained in the Jacobson radical of \( R \). The \( q \)-adic completion of \( R \) shall be denoted by \( \hat{R} \).

Proposition 4 Let \( a \) be a monomial ideal of \( R \), and let \( m = x_1^{j_1} \cdots x_d^{j_d} \in M \). The following statements are equivalent:

1. \( m \) is integral over \( a \).
2. \( m \) is integral over \( a\hat{R} \).
3. There exists \( l \in \mathbb{N} \) with \( ml \in a^l \).
4. \((j_1, \ldots, j_d)\) lies in the convex hull of \( \Gamma + \mathbb{R}_{\geq 0}^d \) where \( \Gamma \subset \mathbb{N}_0^d \) is the set of exponents of monomials appearing in \( a \).

In particular, every monomial in \( \tilde{a} \) lies in \( \tilde{a} \).

Proof: (1) \( \Rightarrow \) (2) and (3) \( \Rightarrow \) (1) hold trivially.

(2) \( \Rightarrow \) (3): Let \( T^p + a_1 T^{p-1} + \cdots + a_p \in \hat{R}[T] \) with \( a_i \in (a\hat{R})^i = a^i \hat{R} \) for \( i \in \{1, \ldots, p\} \) be an equation of integral dependence for \( m \) over \( a\hat{R} \). Let \( i \in \{1, \ldots, p\} \). Since \( a^i \) is a monomial ideal of \( \hat{R} \), the ideal \( a^i \hat{R} \) is a monomial ideal of \( \hat{R} \), and, by Prop. 2, there exist elements \( e_{i1}, \ldots, e_{ir_i} \in \hat{R} \setminus q\hat{R} \) and monomials \( m_{i1}, \ldots, m_{ir_i} \in M \) with
\[
a_i = \sum_{j=1}^{r_i} e_{ij} m_{ij},
\]

From Prop. 3 we obtain \( m_{ij} \in a^i \hat{R} \cap R = a^i \) for \( i \in \{1, \ldots, p\}, j \in \{1, \ldots, r_i\} \) [note that \( \hat{R} \) is a faithfully flat extension of \( R \)]. Therefore the monomial \( m^p \)
lies in the $\check{R}$-ideal which is generated by the set $\{m_{ij} m^{p-i} \mid i \in \{1, \ldots, p\}, j \in \{1, \ldots, r_i\}\}$. Using Cor. 3 we find $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, r_i\}$ with $m_{ij} m^{p-i} \mid m^p$, hence $m_{ij} \mid m^l$. Thus, we have shown that $m^l \in m_{ij} R \subset a_i$.

(3) $\iff$ (4) This is an easy consequence of Cor. 3 and Carathéodory’s theorem [for Carathéodory’s theorem cf. [11], Th. 17.1].

**Corollary 6** Let $a$ be a monomial ideal of $R$.

1. We have $\check{a} \check{R} = \check{\check{a}} \check{R}$ and $\check{a} \check{R} \subset \check{R}$.

2. We have $\overline{\text{gr}(a)} = \text{gr}(\check{a})$.

**Proof:** (1) The first assertion is an easy consequence of Prop. 4, and the second assertion is clear.

(2) Let $\check{a}$ be generated by the monomials $m_1, \ldots, m_r$. Then $\text{gr}(\check{a})$ is generated by the monomials $\overline{m_1}, \ldots, \overline{m_r}$ [cf. (4) in Remark 4]. For every $i \in \{1, \ldots, r\}$ there exists $l_i \in \mathbb{N}$ with $\overline{m_i} \in a_i$, hence $\overline{m_i} \in \text{gr}(a)^{l_i} = \text{gr}(a_i)$, and therefore we have $\overline{m_i} \in \check{\check{a}}$. Conversely, let $m \in M$ be a monomial with $\overline{m} \in \text{gr}(a)$. Then there exists $l \in \mathbb{N}$ with $\overline{m} \in (\text{gr}(a))^{l} = \text{gr}(a^l)$, hence $m^l \in a^l$, and therefore $m \in \check{a}$, hence $\overline{m} \in \text{gr}(\check{a})$.

**5. Monomial Ideals in Polynomial Rings**

The following result in Prop. 5 should be known, but we could not find a source for it.

**Notation 5** Let $(\Gamma, \prec)$ be a totally ordered commutative monoid with neutral element 0 satisfying the following condition:

Every non-empty subset of $\Gamma$ has a smallest element.

This condition is satisfied if $\prec$ is a well-ordering; in particular, a monomial ordering on $\mathbb{N}_0^d$ satisfies this condition.

Let $R = \bigoplus_{\gamma \in \Gamma} R_{\gamma}$ be a $\Gamma$-graded ring. For $z \in R$ let $z_{\gamma} \in R_{\gamma}$ be the homogeneous component of $z$ of degree $\gamma$, and if $z \neq 0$, then define

$$\text{Supp}(z):= \{\gamma \in \Gamma \mid z_{\gamma} \neq 0\}, \quad \text{deg}(z):= \max_{\prec}\{\gamma \mid \gamma \in \text{Supp}(z)\}, \quad z^*:= z_{\text{deg}(z)}.$$ 

Let $z, w \in R \setminus \{0\}$; then we have $\text{deg}(zw) \leq \text{deg}(z) + \text{deg}(w)$ if $zw \neq 0$ and $\text{deg}(z + w) \leq \max_{\prec}\{\text{deg}(z), \text{deg}(w)\}$ if $z + w \neq 0$. Notice that, if $z$ is not homogeneous, then we have $\text{deg}(z - z^*) \prec \text{deg}(z)$.
Proposition 5 Let $S$ be a $\Gamma$-graded ring, and let $R$ be a $\Gamma$-graded subring of $S$. Then the integral closure $\overline{R}$ of $R$ in $S$ is a $\Gamma$-graded subring of $S$.

Proof: (1) Firstly, we consider the case that every homogeneous element of $S$ which is integral over $R$ already lies in $R$. Then we have to show that $\overline{R} = R$. Suppose that $R \subsetneq \overline{R}$, and choose $z \in \overline{R} \setminus R$ in such a way that $(\#(\text{Supp}(z))) \leq (\#(\text{Supp}(w)))$ for every $w \in \overline{R} \setminus R$. Now $z$ is not homogeneous by our assumption on $R$. If $z^* \in \overline{R}$, then we would have $z^* \in R$ since $z^*$ is homogeneous, hence $z - z^* \in \overline{R}$, and therefore $z - z^* \in R$ by the choice of $z$ [note that $(\#(\text{Supp}(z - z^*))) < (\#(\text{Supp}(z)))$. Therefore we have $z^* \notin \overline{R}$. In particular, we have $(z^*)^i \neq 0$ for every $i \in \mathbb{N}$, hence $(z^*)^i = (z^*)^i$ and $\deg(z^i) = i \deg(z)$ for every $i \in \mathbb{N}$.

Let

$$\mathcal{V} := \{ \mathbf{a} = (a_1, \ldots, a_p) \mid a_1, \ldots, a_p \in R, z^p + a_1 z^{p-1} + \cdots + a_p = 0 \}.$$

Obviously $\mathcal{V}$ is not empty. For every $\mathbf{a} = (a_1, \ldots, a_p) \in \mathcal{V}$ we define

$$\gamma(\mathbf{a}) := \max_{a_i \neq 0} \{ \deg(a_i) - i \deg(z) \mid a_i \neq 0, i \in \{0, 1, \ldots, p\} \} \in \Gamma,$$

$$s(\mathbf{a}) := \min \{ i \in \{0, 1, \ldots, p\} \mid a_i \neq 0, \deg(a_i) - i \deg(z) = \gamma(\mathbf{a}) \} \in \{0, 1, \ldots, p\}$$

[we define $a_0 := 1$]. Then we have $\gamma(\mathbf{a}) \geq 0$ [since $a_0 = 1 \in R_0$]. Suppose that there exists $\mathbf{a} = (a_1, \ldots, a_p) \in \mathcal{V}$ with $\gamma(\mathbf{a}) = 0$. Then we have for every $i \in \{1, \ldots, p\}$ with $a_i z^{p-i} \neq 0$

$$\deg(a_i z^{p-i}) \leq \deg(a_i) + \deg(z^{p-i}) = \deg(a_i) + (p - i) \deg(z)$$

$$\leq p \deg(z) + \gamma(\mathbf{a}) = p \deg(z).$$

In $z^p + a_1 z^{p-1} + \cdots + a_p = 0$ we consider the homogeneous component of degree $p \deg(z) = \deg(z^p)$. Then we get $(z^*)^p + a'_1 (z^*)^{p-1} + \cdots + a'_p = 0$ with

$$a'_i := \begin{cases} a_i^* & \text{if } a_i z^{p-i} \neq 0 \text{ and } \deg(a_i z^{p-i}) = p \deg(z), \\ 0 & \text{else} \end{cases} \text{ for } i \in \{1, \ldots, p\}.$$

But this would imply that $z^* \in \overline{R}$, in contradiction with our observation above.

Therefore we have $\gamma(\mathbf{a}) > 0$ for every $\mathbf{a} \in \mathcal{V}$. This implies that $s(\mathbf{a}) > 0$; moreover, we have $s(\mathbf{a}) \leq p - 1$ since otherwise $a_p^* = 0$.

Let

$$\gamma_0 := \min_{\mathbf{a} \in \mathcal{V}} \{ \gamma(\mathbf{a}) \mid \mathbf{a} \in \mathcal{V} \}, \quad \mathcal{V}_0 := \{ \mathbf{a} \in \mathcal{V} \mid \gamma(\mathbf{a}) = \gamma_0 \}.$$
Then we have $\gamma_0 > 0$. We choose $a = (a_1, \ldots, a_p) \in V_0$ with $s(b) \leq s(a)$ for every $b \in V_0$. We define

$$a'_j := \begin{cases} a_j & \text{if } a_j \neq 0, \deg(a_j) - j \deg(z) = \gamma_0, \\ 0 & \text{else} \end{cases} \quad \text{for } j \in \{1, \ldots, p\}.$$ 

By the choice of $s$ we have $a'_1 = \cdots = a'_{s-1} = 0$, $a'_s = a_s^* \neq 0$, and

$$a'_s(z^*)^{p-s} + a'_{s+1}(z^*)^{p-s-1} + \cdots + a'_p = 0 \quad (*)$$

[consider in $z^p + a_1 z^{p-1} + \cdots + a_p = 0$ the homogeneous component of degree $\gamma_0 + p \deg(z)$]. We multiply $(*)$ by $a'_s^{p-s-1}$ and obtain

$$(a'_s z^*)^{p-s} + a'_{s+1}(a'_s z^*)^{p-s-1} + \cdots + a'_p a_s^* = 0.$$ 

Therefore the homogeneous element $a'_s z^*$ is integral over $R$, hence lies in $R$. Since $a'_s z - a'_s z^*$ is integral over $R$, and since either $a'_s z = a'_s z^*$ or $\#(\text{Supp}(a'_s z - a'_s z^*)) < \#(\text{Supp}(a'_s z))$, we have $a'_s z - a'_s z^* \in R$ by the choice of $z$, hence $a'_s z \in R$. We define

$$\overline{a}_i := \begin{cases} a_i & \text{if } i \neq s, s+1, \\ a_s - a'_s & \text{if } i = s, \\ a_{s+1} + a'_s z & \text{if } i = s+1 \end{cases} \quad \text{for } i \in \{1, \ldots, p\}.$$ 

Then we have $\overline{a} = (\overline{a}_1, \ldots, \overline{a}_p) \in R^p$, and since $z^p + \overline{a}_1 z^{p-1} + \cdots + \overline{a}_p = 0$, we have $\overline{a} \in V$. We show that we even have $\overline{a} \in V_0$. We have $\overline{a}_s = 0$ or $\deg(a_s - a'_s) - s \deg(z) < \deg(a_s) - s \deg(z) \leq \gamma_0$, and we have $\overline{a}_{s+1} = 0$ or $\deg(a_{s+1} + a'_s z) - (s+1) \deg(z) \leq \gamma_0$, and therefore we have $\gamma(\overline{a}) = \gamma_0$. Obviously we have $s(\overline{a}) \geq s + 1$, in contradiction with the choice of $\overline{a}$. Therefore we have $\overline{R} = R$.

(2) Now we consider the general case. Let $R' := R[\Sigma]$, where $\Sigma$ is the set of homogeneous elements of $S$ which are integral over $R$; then $R'$ is a $\Gamma$-graded subring of $S$. We have $R \subseteq R' \subseteq \overline{R}$, hence $\overline{R} = \overline{R'}$. Since $\overline{R'} = R'$ by (1), we have $\overline{R} = R'$.

**Corollary 7** Let $R$ be a $\Gamma$-graded ring, and let $a$ be a $\Gamma$-homogeneous ideal of $R$. Then the integral closure of $a$ in $R$ is a $\Gamma$-homogeneous ideal of $R$, again.

**Proof:** We equip the polynomial ring $R[T]$ in a natural way with a $\Gamma \times \mathbb{N}_0$-grading; then we can consider the Rees ring $\mathcal{R}(a, R)$ as a $\Gamma \times \mathbb{N}_0$-graded...
subring of $R[T]$. The integral closure of $\mathcal{R}(a, R)$ in $R[T]$ is a $\Gamma \times \mathbb{N}_0$-graded subring by Prop. 5, and $w \in R$ is integral over $a$ iff $wT \in R[T]$ lies in

$$\overline{\mathcal{R}(a, R)} = \bigoplus_{p \geq 0} a^p T^p$$

[cf. Remark 6].

**Notation 6** For the rest of this section let $k$ be a ring, and let $A = k[x_1, \ldots, x_d]$ be the polynomial ring over $k$ in $d$ variables $x_1, \ldots, x_d$. Then $(x_1, \ldots, x_d)$ is a regular sequence in $A$ which satisfies (a)-(c) above; let $M$ be the set of monomials $x^i = x_1^{i_1} \cdots x_d^{i_d}$, $i \in \mathbb{N}_0^d$. Every non-zero $z \in A$ has a unique representation $z = c_1m_1 + \cdots + c_rm_r$ with non-zero elements $c_1, \ldots, c_r \in k$ and pairwise distinct monomials $m_1, \ldots, m_r \in M$; we call this the monomial representation of $z$.

An ideal $\mathfrak{a}$ of $A$ is called a monomial ideal if it is generated by a set of monomials. Let $\mathfrak{a}$ be a monomial ideal in $A$; then $\mathfrak{a}$ is generated by a finite set of monomials [Dickson’s Lemma, cf. [1], Ch. 4, Cor. 4.48 and Th. 5.2 or [5], Ch. II, § 4, in particular Exercise 7] and a monomial $m \in M$ belongs to $\mathfrak{a}$ iff it is a multiple of a monomial in $\mathfrak{a}$. Moreover, if $cm \in \mathfrak{a}$ with $c \in k \setminus \{0\}$ and $m \in M$, then $m \in \mathfrak{a}$.

**Corollary 8** Let $\mathfrak{a}$ be a monomial ideal in $A$. Then we have

$$\overline{\mathfrak{a}} = \text{rad}_k(0)A + \mathfrak{a}.$$  

**Proof:** Clearly we have $\text{rad}_k(0) \subset \overline{\mathfrak{a}}$ and $\mathfrak{a} \subset \overline{\mathfrak{a}}$. Let $z \in \overline{\mathfrak{a}}$, $z \neq 0$; since $\mathfrak{a}$ is an $\mathbb{N}_0^d$-homogeneous ideal of $A$ [cf. Cor. 7], there exist $s \in \mathbb{N}$, non-zero elements $c_1, \ldots, c_s \in k$ and monomials $n_1, \ldots, n_s \in M$ with $z = c_1n_1 + \cdots + c_sn_s$ and such that $c_i n_i$ is integral over $\mathfrak{a}$ for $i \in \{1, \ldots, s\}$. Let $i \in \{1, \ldots, s\}$. There then exist $p \in \mathbb{N}$, elements $d_1, \ldots, d_p \in k$ and monomials $m_1 \in \mathfrak{a}, \ldots, m_p \in \mathfrak{a}$ such that

$$(c_dn_d)^p + d_1m_1(c_dn_d)^{p-1} + \cdots + d_pm_p = 0.$$  

If $d_1 = \cdots = d_p = 0$, then we have $c_i^n = 0$, hence $c_i \in \text{rad}_k(0)$. Otherwise, there exists $l \in \{1, \ldots, p\}$ with $n_l^p = m_l n_l^{p-1}$, hence $n_l = m_l \in \mathfrak{a}$, hence $n_l \in \overline{\mathfrak{a}}$. Therefore we have $z \in \text{rad}_k(0)A + \overline{\mathfrak{a}}$.

**Corollary 9** The following statements are equivalent:

1. $k$ is a reduced ring.
2. There exists a monomial ideal $\mathfrak{a}$ in $A$ such that $\mathfrak{a} = \overline{\mathfrak{a}}$.
3. For every monomial ideal $\mathfrak{a}$ of $A$ we have $\mathfrak{a} = \overline{\mathfrak{a}}$. 
6. The Main Theorem

We keep the notations and assumptions introduced in section 2.

**Notation 7** (1) A monomial ordering \( \prec \) of \( \mathbb{N}_0^d \) is said to be degree-compatible if it satisfies the following condition: for any \( i, j \in \mathbb{N}_0^d \) with \( \deg(i) < \deg(j) \) we have \( i \prec j \).

(2) Let \( \prec \) be a degree-compatible ordering on \( \mathbb{N}_0^d \). Then every subset of \( \mathbb{N}_0^d \) which is bounded above is finite.

(3) Let \( \prec \) be a monomial ordering on \( \mathbb{N}_0^d \). Let \( i \neq j \) be in \( \mathbb{N}_0^d \). We define \( i \prec_g j \) if \( \deg(i) < \deg(j) \) or if \( \deg(i) = \deg(j) \) and \( i \prec j \). Then \( \prec_g \) is a degree-compatible monomial ordering on \( \mathbb{N}_0^d \).

(4) If \( \prec \) is the lexicographical ordering \( \text{lex} \) on \( \mathbb{N}_0^d \), then \( \prec_g \) is the degree-lexicographical ordering \( \text{deglex} \) on \( \mathbb{N}_0^d \).

(5) Every monomial ordering \( \prec \) on \( \mathbb{N}_0^d \) induces an ordering on \( M \) which will be denoted by \( \prec \), again.

**Proposition 6** We assume that \( R/\mathfrak{q} \) is a reduced ring. Let \( a \) be a monomial ideal of \( R \); then \( \text{gr}(\tilde{a}) \) is the integral closure of the monomial ideal \( \text{gr}(a) \) in \( \text{gr}(R) \).

**Proof:** Since \( \tilde{a} \) is integral over \( a \), obviously \( \text{gr}(\tilde{a}) = \text{gr}(\tilde{a}) \) [cf. Cor. 9(2)] is integral over \( \text{gr}(a) \). Let \( m \in M \) be a monomial, and assume that \( \text{In}(m) = m \) is integral over \( \text{gr}(a) \). Then there exists \( h \in \mathbb{N} \) with \( \text{In}(m)^h \in (\text{gr}(a))^h = \text{gr}(a^h) \) [cf. Cor. 9], hence we see that \( m^h \in a^h \cap q^{h \deg(m)} \subseteq a^h \), hence \( m \in \tilde{a} \), and therefore we obtain that \( \text{In}(m) \in \text{gr}(\tilde{a}) \).

**Remark 7** We assume that \( R \) is complete, and that \( \mathfrak{q} \) is a prime ideal which is contained in the Jacobson radical of \( R \). Let \( \prec \) be a degree-compatible monomial ordering on \( M \), and let \( z \in R \setminus \{0\} \); we define

\[
\text{lm}(z) := \min_\prec \{Tm(z)\}.
\]

Let

\[
z = e_1m_1 + \cdots + e_rm_r
\]

be a monomial representation of \( z \), then we have \( \text{lm}(z) \leq m_j \) for every \( j \in \{1, \ldots, r\} \) [cf. Lemma 4 and note that \( \prec \) is a degree-compatible ordering], hence we even have

\[
\text{lm}(z) = \min_\prec \{m_i \mid i \in \{1, \ldots, r\}\}.
\]

For \( z, w \in R \setminus \{0\} \) we obviously have

\[
\text{lm}(zw) = \text{lm}(z)\text{lm}(w).
\]
Proposition 7 We assume that $R$ is complete, and that $q$ is a prime ideal which is contained in the Jacobson radical of $R$. For every monomial ideal $a$ of $R$ we have $\bar{a} = \tilde{a}$.

Proof: (1) We have $\bar{a} \subseteq \tilde{a}$ for every monomial ideal $a$ of $R$ [cf. (1) in Notation 4]. Suppose that the proposition does not hold. Then the family

$$\mathcal{I} := \{ a | a \text{ monomial ideal of } R, \bar{a} \subseteq \tilde{a} \}$$

is not empty. For every $a \in \mathcal{I}$ we define $r(a) \in \mathbb{N}$ in the following way: If $y \in \bar{a} \setminus \tilde{a}$, and if $y = e_1m_1 + \cdots + e_rm_r$ is a monomial representation of $y$ [cf. Prop. 2], then we have $r \geq r(a)$. We choose $a \in \mathcal{I}$ in such a way that $r(a) \leq r(b)$ for every $b \in \mathcal{I}$. We define $r := r(a)$, and we choose $y \in \bar{a} \setminus \tilde{a}$ such that $y$ admits a monomial representation $y = e_1m_1 + \cdots + e_rm_r$ having $r$ terms. By Prop. 4 we have $r \geq 2$. By (5) in Notation 4 there exist $p \in \mathbb{N}$ and $a_i \in \tilde{a}^i$ for $i \in \{1, \ldots, p\}$ with

$$y^p + a_1y^{p-1} + \cdots + a_p = 0.$$

(2) Let $\prec$ be a degree-compatible monomial ordering on $M$. Without loss of generality we may assume that in the monomial representation of $y$ we have $m_1 \prec m_2 \prec \cdots \prec m_r$, hence that $\text{lm}(y) = m_1$, and that $\text{deg}(m_1) \leq \text{deg}(m_2) \leq \cdots \leq \text{deg}(m_r)$. We choose $t \in \{1, \ldots, r\}$ with $\text{deg}(m_1) = \text{deg}(m_2) = \cdots = \text{deg}(m_t) < \text{deg}(m_{t+1})$, and we define $y_t := e_1m_1 + \cdots + e_tm_t$; then we have $\text{In}(y) = \text{In}(y_t)$.

(3) Let

$$\mathcal{S} := \{ b = (b_1, \ldots, b_p) | b_i \in \tilde{a}^i \text{ for } i \in \{1, \ldots, p\}, y^p + b_1y^{p-1} + \cdots + b_p = 0 \}.$$ 

The set $\mathcal{S}$ is not empty [cf. (1)]; we define for $b \in \mathcal{S}$

$$\rho(b) := \min \{ \text{lm}(b_iy^{p-i}) \mid i \in \{1, \ldots, p\}, b_i \neq 0 \} \in M,$$

$$s(b) := \min \{ i \in \{1, \ldots, p\} \mid b_i \neq 0, \text{lm}(b_iy^{p-i}) = \rho(b) \} \in \{1, \ldots, p\}.$$ 

(4) There exists $b \in \mathcal{S}$ with

$$\rho(b) \geq \text{lm}(y^p).$$

Proof: Let us suppose, on the contrary, that

$$\rho(b) < \text{lm}(y^p) \quad \text{for every } b \in \mathcal{S}.$$
This implies that \(s(b) \leq p - 1\) for every \(b \in S\). The set \(\{\rho(b) \mid b \in S\}\) is bounded above, hence finite; we define
\[
\rho := \max_{\rho} \{\rho(b) \mid b \in S\} \in M.
\]
Furthermore, we define
\[
S' := \{b \in S \mid \rho(b) = \rho\}.
\]
We choose \(b' = (b'_1, \ldots, b'_p) \in S'\) in such a way that \(s(b) \leq s(b')\) for every \(b \in S'\), and we define \(s := s(b')\); note that \(1 \leq s \leq p - 1\).

Let \(i \in \{1, \ldots, p\}\) with \(b'_i \neq 0\). We consider a monomial representation
\[
b'_i = e_{i_1}m_{i_1} + \cdots + e_{i_r}m_{i_r}.
\]
Since \(\tilde{a}\) is a monomial ideal, we have \(m_{i_1}, \ldots, m_{i_r} \in \tilde{a}\) [cf. Prop. 3]. Without loss of generality we may assume that \(m_{i_1} < m_{i_2} < \cdots < m_{i_r}\). We choose \(t_i \in \{1, \ldots, r_i\}\) with \(\deg(m_{i_1}) = \cdots = \deg(m_{i_{t_i}}) < \deg(m_{i_{t_i+1}})\), and we define \(b''_i := e_{i_1}m_{i_1} + \cdots + e_{i_{t_i}}m_{i_{t_i}}\); then we have \(\text{In}(b'_i) = \text{In}(b''_i)\) in \(\text{gr}(R)\).

For \(i \in \{1, \ldots, p\}\) we define
\[
d_i := \begin{cases} 
0 & \text{if } b'_i = 0 \text{ or if } b'_i \neq 0 \text{ and } \text{lm}(b'_iy^{p-i}) \succ \rho, \\
\rho & \text{if } b'_i \neq 0 \text{ and } \text{lm}(b'_iy^{p-i}) = \rho.
\end{cases}
\]
Then we have \(d_i \in \tilde{a}\) for every \(i \in \{1, \ldots, p\}\).

We consider the equation
\[
y^p + b'_1y^{p-1} + \cdots + b'_p = 0. \quad \text{(*)}
\]
For \(i \in \{1, \ldots, p\}\) we replace \(b'_i\) by \(d_i\), and we replace \(y\) by \(y_1\); using the inequality \(\rho \ll \text{lm}(y^p)\), we obtain the following equation in \(\text{gr}(R)\)
\[
\text{In}(d_s)\text{In}(y_1^{p-s}) + \text{In}(d_{s+1})\text{In}(y_1^{p-s-1}) + \cdots + \text{In}(d_p) = 0. \quad \text{(**)}
\]
We multiply (** with \(\text{In}(d_s^{p-s-1})\), and we obtain
\[
(\text{In}(d_s)\text{In}(y_1))^{p-s} + \text{In}(d_{s+1})(\text{In}(d_s)\text{In}(y_1))^{p-s-1} + \text{In}(d_{s+2}d_s)(\text{In}(d_s)\text{In}(y_1))^{p-s-2} + \cdots + \text{In}(d_pd_s^{p-s-1}) = 0.
\]
We have
\[
d_{s+l}d_s^{l-1} \in \tilde{a}^{s+l}(\tilde{a})^{l-1} \subset \tilde{a}^{(s+1)l} \quad \text{for } l \in \{1, \ldots, p - s\}.
\]
Therefore we have $\text{In}(d_s d_s^{d_s-1}) \in \text{gr}(\overline{a^{(s+1)l}}) = (\text{gr}(\overline{a^{s+1}})^l)$ [cf. Cor. 6(2) and (4) in Remark 4] for $l \in \{1, \ldots, p-s\}$, hence $\text{In}(d_s y_1)$ is integral over $(\text{gr}(\overline{a}))^{s+1}$ [cf. (5) in Notation 4]. $\text{In}(m_s m_1)$ is integral over $(\text{gr}(\overline{a}))^{s+1}$, also [cf. Cor. 9], and therefore $e_s e_1 m_s m_1$ is an element of $\overline{a^{s+1}}$. We multiply (*) with $(e_s m_1)^p$ and we obtain

$$(e_s m_s y)^p + b'_1 e_s m_s (e_s m_s y)^{p-1} + \cdots + b'_p (e_s m_s)^p = 0.$$ 

Note that

$$b'_l (e_s m_s)^l \in \overline{a^l (\overline{a^{s+1}})^l} \text{ for } l \in \{1, \ldots, p\},$$

and therefore $e_s m_s y$ is integral over $\overline{a^{s+1}}$ [cf. (5) in Notation 4]. Let $y' := y - e_1 m_1$; then $e_s m_s y'$ is integral over $\overline{a^{s+1}}$, and $e_s m_s y' = \sum_{i=2} e_i e_s m_s m_i$ admits a monomial representation having only $r - 1$ terms.

We have $e_s m_s y' \in \overline{a^{s+1}}$ [this is clear if $\overline{a^{s+1}} = \overline{a^{s+1}}$, and if $\overline{a^{s+1}} \supset \overline{a^{s+1}}$, then $\overline{a^{s+1}}$ lies in $\mathcal{I}$, and by the choice of $r$ [cf. (1)] we get $e_s m_s y' \in \overline{a^{s+1}}$ in this case, also]. Since $e_s m_s y'$ and $e_1 e_s m_1 m_s$ lie in $\overline{a^{s+1}}$, the element $e_s m_s y$ lies in $\overline{a^{s+1}}$, also.

We define [note that $s \leq p - 1$]

$$\overline{b}_i := \begin{cases} b'_i & \text{if } i \neq s, s + 1, \\ b'_s - e_s m_s & \text{if } i = s, \\ b'_{s+1} + e_s m_s y & \text{if } i = s + 1. \end{cases}$$

We have $\overline{b}' \in \mathcal{S}$, $e_s m_s \in \overline{a^s}$ and $e_s m_s y \in \overline{a^{s+1}}$, hence we have $\overline{b}_i \in \overline{a^i}$ for $i \in \{1, \ldots, p\}$. Clearly we have

$$y^p + \overline{b}_1 y^{p-1} + \cdots + \overline{b}_p = 0,$$

and therefore $\overline{b} := (\overline{b}_1, \ldots, \overline{b}_p)$ lies in $\mathcal{S}$, and this implies that $\rho(\overline{b}) \lessdot \rho$ by the choice of $\rho$. We show that $\overline{b}$ even lies in $\mathcal{S}'$.

We have $\overline{b}_s = 0$ or $\overline{b}_s = e_s m_s + \cdots + e_{s,r} m_s r_s$ and $\text{In}(\overline{b}_s) = m_s \geq m_s = \text{In}(b'_s) = \rho$. We have $\text{In}(e_s m_s y^{p-s}) = \rho$, and if $b'_{s+1} \neq 0$, then we have $\text{In}(b'_{s+1} y^{p-s-1}) \supset \rho$. Therefore we have $\text{In}(\overline{b}_{s+1} y^{p-s-1}) \supset \rho$, and since $\rho(\overline{b}') = \rho$, we obtain $\rho(\overline{b}) \supset \rho$. This implies that $\rho(\overline{b}) = \rho$, hence we get, in fact, that $\overline{b} \in \mathcal{S}'$.

Now we have $\overline{b}_s = 0$ or $\text{In}(\overline{b}_s) \supset \rho$ and $\overline{b}_i = b'_i$ for $i \in \{1, \ldots, s - 1\}$, and this implies $s(\overline{b}) > s(\overline{b}') = s$, in contradiction with the choice of $\overline{b}'$. 

Integral Closure of Monomial Ideals on Regular Sequences 505
(5) By (4) there exists \( b \in S \) with \( \text{lm}(b_iy^{p_i-i}) \geq \text{lm}(y^p) \) for every \( i \in \{1, \ldots, p\} \) with \( b_i \neq 0 \).

Let \( i \in \{1, \ldots, p\} \) with \( b_i \neq 0 \), and let \( b_i = e_{i1}m_{i1} + \cdots + e_{i,r_i}m_{i,r_i} \in \tilde{a}^i \) be a monomial representation of \( b_i \); without loss of generality we may assume that \( m_{i1} < m_{i2} < \cdots < m_{i,r_i} \), which implies that \( m_{i1} = \text{lm}(b_i) \). We choose \( t_i \in \{1, \ldots, r_i\} \) with \( \deg(m_{i1}) = \cdots = \deg(m_{i,t_i}) < \deg(m_{i,t_i+1}) \), and we define

\[
b'_i := e_{i1}m_{i1} + \cdots + e_{i,t_i}m_{i,t_i};
\]

note that \( \text{In}(b_i) = \text{In}(b'_i) \). We have \( m_{ij} \in \tilde{a}^i \) for \( j \in \{1, \ldots, r_i\} \) [cf. Prop. 3], hence, in particular, \( b'_i \in \tilde{a}^i \).

Now let \( i \in \{1, \ldots, p\} \); we define

\[
c_i := \begin{cases} 
0 & \text{if } b_i = 0 \text{ or if } b_i \neq 0 \text{ and } \text{lm}(b_iy^{p_i-i}) \geq \text{lm}(y^p), \\
b'_i & \text{if } b_i \neq 0 \text{ and } \text{lm}(b_iy^{p_i-i}) = \text{lm}(y^p).
\end{cases}
\]

Clearly we have \( c_i \in \tilde{a}^i \). From \( y^p + b_1y^{p_1-1} + \cdots + b_p = 0 \) we obtain the following equation in \( \text{gr}(R) \)

\[
\text{In}(y_1)^p + \text{In}(c_1)\text{In}(y_1)^{p_1-1} + \cdots + \text{In}(c_p) = 0.
\]

Now we have \( \text{In}(c_i) \in \text{gr}(\tilde{a}^i) \) for every \( i \in \{1, \ldots, p\} \). Just as in (4) we see that \( \text{In}(y_1) \) is integral over \( \text{gr}(a) \) and that therefore \( \text{In}(m_1) \) is integral over \( \text{gr}(a) \), hence we have \( m_1 \in \tilde{a} \), hence \( e_1m_1 \in \tilde{a} \). Now \( y' := y - e_1m_1 \) lies in \( \tilde{a} \), and therefore \( y' \) lies in \( \tilde{a} \) by the choice of \( r \). From this we get that \( y = y' + e_1m_1 \) lies in \( \tilde{a} \), in contradiction with the choice of \( y \).

\[\blacksquare\]

**Theorem 1** Let \( R \) be a noetherian ring, let \( x = (x_1, \ldots, x_d) \) be a regular sequence in \( R \), and assume that \( q := xR \) is contained in the Jacobson radical of \( R \) and that \( R/q \) is a reduced ring. For every monomial ideal \( a \) of \( R \) we have \( \tilde{a} = \tilde{a}; \) in particular, \( \tilde{a} \) is a monomial ideal, also.

**Proof:** (1) Firstly, let \( q \) be a prime ideal. Let \( y \in \tilde{a} \). We have \( \tilde{a}R \subset (aR) \) and \( \tilde{a}R = \tilde{a}R \) [cf. Cor. 6], hence \( y \in \tilde{a}R = \tilde{a}R = \tilde{a}R \) [cf. Prop. 7], and since \( \tilde{a}R \cap R = \tilde{a} \) we obtain \( y \in \tilde{a} \). Thus, we have shown that \( \tilde{a} = \tilde{a} \).

(2) Now we consider the case that \( R/q \) is reduced.

(a) Let \( p \in \text{Ass}(R/q) \). Then \( qR_p \) is the maximal ideal of \( R_p \), hence we have \( \tilde{a}R_p = \tilde{a}R_p \) by (1). Obviously we have \( \tilde{a}R_p = \tilde{a}R_p \) and \( \tilde{a}R_p \subset \tilde{a}R_p \). Therefore we have \( \tilde{a}R_p \subset \tilde{a}R_p \).

(b) For every \( p \in \text{Ass}(R/q) \) there exists, by (a), an element \( s_p \in R \setminus p \) with \( \tilde{a} \subset \tilde{a} : s_p \). Let \( b \) be the ideal generated by the elements \( s_p \); then we
have $\overline{a} \subset \overline{a} : b$. Let $p' \in \text{Ass}(R/\overline{a})$. Since $\overline{a}$ is a monomial ideal, there exists $U \subset \{1, \ldots, d\}$ with $p' \in \text{Ass}(R/q_U)$ [cf. Lemma 1]. Repeated application of Lemma 1 in [13], vol. II, Appendix 6, shows that there exists a prime ideal $p \in \text{Ass}(R/q)$ with $p' \subset p$. Therefore $b$ is not contained in any prime ideal in $\text{Ass}(R/\overline{a})$, hence $\overline{a} : b = \overline{a}$, hence $\overline{a} \subset \overline{a}$. The inclusion $\overline{a} \subset \overline{a}$ was noticed in (1) of Notation 4, and therefore we have $\overline{a} = \overline{a}$.

**Example 1** Let $R$ be a regular local two-dimensional ring, and let $\{x, y\}$ be a regular system of parameters of $R$. Let $m > n > 1$ be coprime integers, and write $m = s_1 n + n_1$ with $1 \leq n_1 < n$. Let $a$ be the ideal of $R$ generated by $x^m$ and $y^n$. Then $a$ is a monomial ideal. It can be shown [cf. [7]] that the integral closure $\mathfrak{P}$ of $a$ has a minimal system of generators $\{x^{m-\sigma_{m,n}(i)}y^i \mid i \in \{0, \ldots, n\}\}$ where $\sigma_{m,n} : \{0, \ldots, n\} \to \{0, \ldots, m\}$ is a strictly increasing function; in particular, one has

$$\sigma_{m,n}(0) = 0, \sigma_{m,n}(1) = s_1, \sigma_{m,n}(n-1) = m-(s_1+1), \sigma_{m,n}(n) = m,$$

and

$$\sigma_{m,n}(i+j) \geq \sigma_{m,n}(i) + \sigma_{m,n}(j) \text{ for } i, j \in \{0, \ldots, n\} \text{ with } i+j \leq n.$$

Moreover, the polar ideal $\mathfrak{P}_\varphi$ of $\varphi$ has

$$\{x^{m-\sigma_{m,n}(i+1)}y^i \mid i \in \{0, \ldots, n-1\}\}$$

as minimal set of generators.

**References**


Recibido: 20 de febrero de 2002
Revisado: 23 de octubre de 2002