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Cauchy problem for semilinear parabolic equations with initial data in $H_p^s(\mathbb{R}^n)$ spaces

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Abstract. We study local and global Cauchy problems for the Semilinear Parabolic Equations $\partial_t U - \Delta U = P(D)F(U)$ with initial data in fractional Sobolev spaces $H_p^s(\mathbb{R}^n)$. In most of the studies on this subject, the initial data $U_0(x)$ belongs to Lebesgue spaces $L^p(\mathbb{R}^n)$ or to supercritical fractional Sobolev spaces $H_p^s(\mathbb{R}^n)$ $(s > n/p)$. Our purpose is to study the intermediate cases (namely for $0 < s < n/p$). We give some mapping properties for functions with polynomial growth on subcritical $H_p^s(\mathbb{R}^n)$ spaces and we show how to use them to solve the local Cauchy problem for data with low regularity. We also give some results about the global Cauchy problem for small initial data.

1. Introduction and results.

1.1. The evolution equation.

We study the Cauchy problem for the Semilinear Parabolic Equation

$$(1) \qquad \begin{cases} \partial_t U - \Delta U = P(D)F(U)\,, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n\,, \\ U(0,x) = U_0(x)\,, \end{cases}$$

where $P(D)$ is a pseudodifferential operator of order $d \in [0,2[$ and where F is a nonlinear function which behaves like $|x|^\alpha$ or $x\,|x|^{\alpha-1}$ ($\alpha > 1$). The most classical examples of such evolution equations are the semilinear heat equations

$$\partial_t u - \Delta u = a\,u\,|u|^{\alpha-1}\,,$$

the Burgers viscous equations

$$\partial_t u - \Delta u = a\,\partial_x(|u|^\alpha)$$

and the Navier-Stokes equation

$$\partial_t u - \Delta u = \mathcal{P}\,\nabla(u \otimes u)\,,$$

where $\mathcal{P}$ denotes the projector on the divergence free vector field (see [Ca] for instance).

We look for mild solutions of (1), *i.e.* for solutions of the integral equation

$$U(t,x) = e^{t\Delta}\,U_0 + \int_0^t e^{(t-\tau)\Delta} P(D)F(U(\tau))\,d\tau\,, \tag{2}$$

where $e^{t\Delta}$ is the heat kernel. As usual the fractional Sobolev spaces and their homogeneous versions are defined by

$$H_p^s(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n) :\ \Lambda_s f \in L^p\}$$

and

$$\dot{H}_p^s = \{f \in \mathcal{S}'(\mathbb{R}^n) :\ \dot{\Lambda}_s f \in L^p\}\,,$$

where Λ_s and $\dot{\Lambda}_s$ are the operators with symbols $\Lambda_s(\xi) = (1+|\xi|^2)^{s/2}$ and $\dot{\Lambda}_s(\xi) = |\xi|^s$ (these spaces are sometimes also denoted $L^{p,s}(\mathbb{R}^n)$, see [Me]). In the sequel we will say that $H_p^s(\mathbb{R}^n)$ is supercritical if $s > n/p$, *i.e.* if the embedding $H_p^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ is verified and, on the contrary, we will say that $H_p^s(\mathbb{R}^n)$ is subcritical.

In the proofs of existence and uniqueness for (2), there always exists a tight connection between the regularity of the Cauchy's data U_0 and the properties of the nonlinear term $P(D)F(U)$. Thus, for $F(x) \approx |x|^\alpha$, Giga [Gi] proved existence and uniqueness for Equation (2) as long as U_0 belongs to an $L^p(\mathbb{R}^n)$ for p large enough. When U_0 belongs to supercritical $H_p^s(\mathbb{R}^n)$ spaces, Taylor [Ta] proved existence and uniqueness for (2) under the assumptions $F(0) = 0$ and $F \in C^{[s]+1}(\mathbb{R})$. One of our purpose is to study all the intermediate range of regularity, namely, to solve (2) for initial data in $H^s_p(\mathbb{R}^n)$ with s in $]0, n/p[$. About this problem, partial results have been found by Henry [He] who proved that, if $s < 2 - d$ and if F maps bounded sets from $H^s_p(\mathbb{R}^n)$ into bounded sets in $L^p(\mathbb{R}^n)$, then (2) is well posed. Let us remark that, in the examples considered by these authors, the action of F on the functional space of the initial data is well understood. This allows to obtain crucial estimates on the nonlinear terms to solve (2): in the first two cases $F : L^p \longrightarrow L^{p/\alpha}$ and $F : H^s_p(\mathbb{R}^n) \longrightarrow H^s_p(\mathbb{R}^n)$ is bounded and in the third one, the hypothesis on F implies some similar properties.

In this paper our goal is to improve Henry's results for the local Cauchy problem and Giga's results for the global Cauchy problem (for small initial data). We give the minimal regularity of U_0 (see Remark 3 after Theorem 1.3 about this), measured on the scale of $H^s_p(\mathbb{R}^n)$ spaces, which ensures both existence and uniqueness for (2). So, for a fixed p in $]1, +\infty[$, we are looking for the smallest exponent of regularity such that, for all U_0 in $H^s_p(\mathbb{R}^n)$ with s greater than this smallest exponent, existence and uniqueness occur. In such a framework one of the most important difficulty arises from the fact that the action of the nonlinear function F on subcritical $H^s_p(\mathbb{R}^n)$ spaces is badly understood. So, to solve (2) in subcritical $H^s_p(\mathbb{R}^n)$ spaces, we will need to prove some mapping properties on those spaces for functions with polynomial growth: this will be realized using harmonic analysis and paradifferential calculus techniques in Section 4.

As an example, let us consider the nonlinear heat equations

$$\partial_t U - \Delta U = a\, U\, |U|^{\alpha-1}\,, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n\,. \tag{3}$$

When $U(t,x)$ is a solution of (3) then, for each $\lambda > 0$, the functions U_λ defined by $U_\lambda(t,x) = \lambda^{(2-d)/(\alpha-1)} U(\lambda^2 t, \lambda x)$ are also solutions (here $d = 0$) and, one can check that U and U_λ have the same norm in $L^\infty(\mathbb{R}^+, \dot{H}^s_p)$ if and only if

$$s = s_c = \frac{n}{p} - \frac{2-d}{\alpha - 1}\,. \tag{4}$$

Without further assumptions on the nonlinear term, this scaling argument suggests that, for all data in $H^s_p(\mathbb{R}^n)$, there exists a unique solution of (3) as long as $s > s_c$. This also suggests that the "right spaces" for the study of global existence are the spaces $H^{s_c}_p(\mathbb{R}^n)$. For instance, we show (see Theorem 1.3) that, for all $U_0(x) \in H^1(\mathbb{R}^3)$, one can find a unique local solution of (3) as long as $\alpha \in]1, 5]$ and, furthermore (see Theorem 1.5), this solution is global as long as $\|U_0\|_{H^1}$ is sufficiently small. This result improves Henry's results because, using his criterion, one can only prove existence and uniqueness in $H^1(\mathbb{R}^3)$ for $\alpha \in]1, 3]$.

In fact, we will show that this scaling argument is true for Equation (2) even if $P(D)$ and F do not possess the exact homogeneity of Equation (3). For these reasons we will say that $H^s_p(\mathbb{R}^n)$ is supercritical (respectively critical) for (2) if $s > s_c$ (respectively if $s = s_c$).

To avoid technical problems we will always assume that

$$s \geq \frac{n}{p} - \frac{n}{\alpha} \tag{5}$$

and that

$$s \geq 0\,. \tag{6}$$

Indeed, according to the Sobolev embedding theorem, if $u \in C([0, T[, H^s_p)$ with s as in (5) and as in (6) then $u \in C([0, T[, L^{\tilde{p}})$ with $\tilde{p} \geq \alpha$. Hence, the term $F(u)$ in (2) is well defined in $\mathcal{D}'(]0, T[\times\mathbb{R}^n)$. On the contrary, if (5) or (6) is not satisfied, solutions in $C([0, T[, H^s_p)$ cannot be defined in a simple way: for instance, if $u \in C([0, T[, H^s_p)$ with $s < 0$, then $F(u)$ has no sense *a priori*. For the study of such cases, when (5) or (6) are not fulfilled, we refer to [Ri] where we show that (2) can sometimes be solved using some smoothing properties of the heat kernel.

1.2. Hypotheses on the nonlinear terms.

About the nonlinear terms $P(D)$ and $F(u)$ we will make the following assumptions.

H1) $P(D)$ is a pseudodifferential operator of degree $d \in [0, 2[$ with constant coefficients (and so $P(D)$ is bounded from $H^{s+d}_p(\mathbb{R}^n)$ to $H^s_p(\mathbb{R}^n)$ for all $s \in \mathbb{R}$ and for all $p \in]1, +\infty[$).

About F we will assume that

H2) there exists $\alpha > 1$ such that,

i) $s \leq n\,(\alpha - 1)/(p\,\alpha)$,

ii) $F : \mathbb{R} \longrightarrow \mathbb{R}$ verifies $|F(x) - F(y)| \leq C\,|x-y|(|x|^{\alpha-1} + |y|^{\alpha-1})$ or,

H3) there exists $\alpha > 1$ such that,

i) $n\,(\alpha-1)/(p\,\alpha) < s < \min\{(n/p+1)\,(\alpha-1)/\alpha, n/p\}$,

ii) $F : \mathbb{R} \longrightarrow \mathbb{R}$ is $[\alpha]$ time differentiable, $D^j F(0) = 0$ for $j = 0, \dots, [\alpha]-1$, $D^{[\alpha]}F(0) = 0$ if $\alpha \notin \mathbb{N}$, and $|D^{[\alpha]}F(x) - D^{[\alpha]}F(y)| \leq C\,|x-y|^{\alpha-[\alpha]}$

or,

H4)

i) $n/p < s$,

ii) $F : \mathbb{R} \longrightarrow \mathbb{R}$ verifies $F(0) = 0$ and $F \in C^{[s]+1}(\mathbb{R})$.

Note that those assumptions on the nonlinear term F depend in a crucial way of the smoothness of the initial data $U_0(x)$. Indeed, when $U_0(x)$ belongs to a supercritical $H_p^s(\mathbb{R}^n)$ space then, since we look for a solution in $C([0,T], H_p^s)$, we look for a bounded solution of (2). Hence, in H4), we do not need any assumptions on the asymptotic behavior of F; we just need smoothness assumptions on F. On the contrary, when $U_0(x)$ belongs to a subcritical $H_p^s(\mathbb{R}^n)$ space, then $U_0(x)$ is possibily unbounded in a neighbourhood of some point x_0 and then we need assumptions on the behavior of F at infinity to “control” $F(U_0(x))$ near x_0.

Note also that, from the assumptions on F, we can easily deduce from H3.ii) the following properties for the intermediate derivatives of F.

Lemma 1.1. *If* H3.ii) *holds then there exists a constant C such that,*

$$|D^j F(x) - D^j F(y)| \leq C\,|x-y|\,(|x|^{\alpha-j-1} + |y|^{\alpha-j-1})\,, \tag{7}$$

for all $j = 0, \dots, [\alpha]-1$,

$$|D^j F(x)| \leq C\,|x|^{\alpha-j}\,, \tag{8}$$

for all $j = 0, \dots, [\alpha]$.

1.3. Statement of main results.

To solve (2) the main idea is to counterbalance the loss of smoothness coming from the nonlinear terms by the smoothing effects of the heat kernel. In the framework of $L^p(\mathbb{R}^n)$ spaces, according to H2) and Hölder's inequality, $F : L^p \longrightarrow L^{p/\alpha}$ is continuous. If H4) holds there is no loss of smoothness on the $H^s_p(\mathbb{R}^n)$ scale thanks to the following Theorem (see [Me] or [Ta]).

Theorem 1.1. *Let* $p \in]1,+\infty[$*. If* H4) *is fulfilled then, for all* $u \in H^s_p(\mathbb{R}^n)$*,* $F(u)$ *belongs to* $H^s_p(\mathbb{R}^n)$ *and furthermore*

$$\|F(u)\|_{H^s_p} \le C\left(\|u\|_{L^\infty}\right)\|u\|_{H^s_p} \,.$$

On the other hand, in the case of subcritical $H^s_p(\mathbb{R}^n)$ spaces, there is no stability by composition with nonlinear functions. For instance, the $H^s_p(\mathbb{R}^n)$ spaces are algebras if and only if $s > n/p$. For $s \in [1+1/p, n/p[$ and $p \in]1,+\infty[$ one can also prove that the functional calculus is trivial in $H^s_p(\mathbb{R}^n)$ (see G. Bourdaud [Bo] for instance): if F maps $H^s_p(\mathbb{R}^n)$ into itself for s in this range then $f(x) = a\,x$.

To measure the loss of smoothness on the $H^s_p(\mathbb{R}^n)$ scale coming from the composition by F, we will prove the following Theorem in Section 4.

Theorem 1.2. *Let* $p \in]1,+\infty[$ *and* s *such that*

$$\max\left\{0, \frac{n}{p} - \frac{n}{\alpha}\right\} < s < \frac{n}{p} \,.$$

Let s_α *defined by*

$$s_\alpha = s - (\alpha-1)\left(\frac{n}{p} - s\right). \tag{9}$$

If H2) *or* H3) *is fulfilled then, for all* $u \in H^s_p(\mathbb{R}^n)$*,* $F(u)$ *belongs to* $H^{s_\alpha}_p(\mathbb{R}^n)$ *and furthermore, there exists a constant* C *independent of* u *such that*

$$\|F(u)\|_{H^{s_\alpha}_p} \le C\,\|u\|^\alpha_{H^s_p} \,.$$

REMARKS.

1) Note that the condition $s \le n\,(\alpha-1)/(p\,\alpha)$ in H2.i) is equivalent to

$$s_\alpha \le 0\,. \tag{10}$$

In the same way, the conditions $n\,(\alpha-1)/(p\,\alpha) < s < (n/p+1)(\alpha-1)/\alpha$ in H3.i) are equivalent to

$$0 < s_\alpha < \alpha - 1\,. \tag{11}$$

2) The hypothesis $s > \max\{0, n/p - n/\alpha\}$ ensures that $F(u)$ is well defined as an element of $\mathcal{D}'$.

3) The restriction $s < (1+n/p)(\alpha-1)/\alpha$ in H3.i) (*i.e.* $s_\alpha < \alpha-1$) comes from the lack of smoothness of F at $x=0$. However, if F is C^∞ ($F(x) = x^m$ for instance), then in H3.i) we must only assume that

$$\frac{n\,(\alpha-1)}{p\,\alpha} < s < \frac{n}{p}$$

to obtain Theorem 1.3.

4) The value of s_α given by Theorem 1.2 is optimal. To see this we have just to consider the example of $u(x) = \psi(x)\,x^{-\beta}$ and $F(x) = |x|^\alpha$ where ψ is a cut of function near 0.

5) In order to solve nonlinear Schrödinger equations, T. Colin [Co] established a related result to Theorem 1.2 for the spaces $H^s_p(\mathbb{R}^n) \cap L^z(\mathbb{R}^n)$. Recently another proof of Theorem 1.2 has been found by T. Runst and W. Sickel in [RS]. First, using paraproduct techniques, they prove Theorem 1.2 in the special case of polynomial functions. Then, using a Taylor expantion of F and Poisson approximations of u, they prove Theorem 1.2 in the general seeting of H3). Our proof is in fact very different. First, we use different techniques (we only use paradifferential calculus) and, second, we do not need to distingue between the polynomial case and the general case.

Using the nonlinear estimates given by Theorem 1.2 and the fixed point Theorem, in Section 2 we prove the following result about the local Cauchy problem.

Theorem 1.3. *Let* $p \in]1, +\infty[$. *Assume that* (5) *and* (6) *holds, and that* H2) *or* H3) *or* H4) *is fulfilled.*

a) *For all initial data* U_0 *in* $H^s_p(\mathbb{R}^n)$ *with* $s > s_c$ *there exists a unique maximal solution* $U(t,x)$ *of* (2) *in* $C([0,T_m[\,,H^s_p)$ *with*

$$T_m \geq C\,\|U_0\|_{H^s_p}^{-\nu^{-1}}\,, \qquad \text{where } \nu = \frac{s-s_c}{2}\,,$$

and, if $T_m < +\infty$, *then*

$$\lim_{t\to T_m} \|U(t,\cdot)\|_{H^s_p} = +\infty\,.$$

b) *Furthermore the following smoothing effects occur*:

• $U(t,x) - e^{t\Delta}U_0 \in C([0,T_m[\,,H^{s+\theta}_p)$ *for all* $\theta < (\alpha-1)\,\nu$ *if* $s < n/p$ *and for all* $\theta < 2-d$ *if* $s > n/p$.

• *If* F *is* $C^\infty(\mathbb{R})$ *then,*

$$U(t,x) \in C^\infty([\delta, T_m[\times\mathbb{R}^n)\,,$$

for all $\delta > 0$.

c) *Let us assume that* $s < 2-d$. *Let* $U \in C([0,T_1[\,,H^s_p)$ *and* $V \in C([0,T_2[\,,H^s_p)$ *be the maximal solutions for the respective initial data* U_0 *and* V_0. *Then,*

$$\|U-V\|_{C([0,T[,H^s_p)} \leq C(T)\,\|U_0-V_0\|_{H^s_p}^{1/2}\,.$$

for all $T < \min\{T_1,T_2\}$.

REMARKS.

1) Let us consider Equation (3) with $U_0 \in H^s_p(\mathbb{R}^n)$. If $s < 2-d$ and if $\alpha < 1/(1-sp/n)$ then Henry's results [He] give existence and uniqueness of a solution in $C([0,T[\,,H^s_p)$. Theorem 1.3 improves this because one can consider larger values of α (see the example in Section 1.1) and because the condition $s < 2-d$ is not needed.

2) Because of (4) we see that $L^p(\mathbb{R}^n)$ is supercritical for (2) if and only if $p > p_c$ where p_c is defined as

$$p_c = \frac{n\,(\alpha-1)}{2-d}\,. \tag{12}$$

So, for $U_0 \in L^p(\mathbb{R}^n)$ with $p > p_c$ and $p \geq \alpha$ (to make sure that (6) and that (5) are fulfilled with $s = 0$), there exists a unique solution of (2) in $C([0,T[, L^p)$: this had ever been proved in [Gi]. However, when $U_0 \in H_p^s(\mathbb{R}^n)$ with $H_p^s(\mathbb{R}^n) \hookrightarrow L^{\tilde{p}}$ for supercritical $L^{\tilde{p}}(\mathbb{R}^n)$ space, Giga's results give existence and uniqueness only in $C([0,T[\,, L^{\tilde{p}})$ but nothing is said about the $H_p^s(\mathbb{R}^n)$ regularity of the solution. Theorem 1.3 answers precisely to this question.

3) If $U_0 \in L^p(\mathbb{R}^n)$ with $p < p_c$, phenomena of non-existence and non-uniqueness may occur (see [We1] and [HW]). Note also that non uniqueness could also occur in the space $H^s(\mathbb{R}^n)$ for subcritical value of s: see Tayachi [T] for the nonlinear heat equations and Dix [Di] for the nonlinear Burgers equations. Theorem 1.3 shows that this could occur only for subcritical $H_p^s(\mathbb{R}^n)$ spaces since it is sufficient to assume that U_0 belongs to $H_p^s(\mathbb{R}^n)$ with $s > s_c$ to ensure both existence and uniqueness. Thus, with no further assumptions than H2) or H3) on the nonlinear terms, our results are optimal. However, for some more specific nonlinear terms, one can sometimes prove that (2) is well posed in some subcritical spaces: for instance for the nonlinear heat equations with the "good" sign and for the Burgers viscous equation with nonlinear term in divergence form (see [EZ]).

4) We mentioned earlier that the restrictions (5) and (6) are only technical. Indeed, when $U_0(x) \in H_p^s(\mathbb{R}^n)$ with $0 \leq s_c < s \leq n/p - n/\alpha$, using $L^q([0,T[\,, L^z)$ estimates for the heat kernel we can always solve (2). Also, when $U_0(x) \in H_p^s(\mathbb{R}^n)$ with $s_c < s < 0$ we can sometimes solve (2): this allows us to solve (2) with measures or distributions as initial data: see [Ri].

In the critical case, we obtain existence of a solution but uniqueness occurs (*a priori*) only in a subspace of $C([0,T[\,, H_p^{s_c})$. However, in this case, we prove global existence for small initial data. We also prove some time decay estimates for those solutions in various $L^q(\mathbb{R}^n)$ norms.

For the study of the global Cauchy problem, we will assume that

H1') $P(D)$ is a pseudodifferential of order $d < 2$ with homogeneous symbol $P(\xi)$,

and

H5) $F(0) = 0$ and there exists $\alpha > 1$ such that,

$$|F(x) - F(y)| \leq C\,|x - y|\,(|x|^{\alpha-1} + |y|^{\alpha-1})\,.$$

First, let us recall a useful result about the Cauchy problem for small initial data in $L^{p_c}(\mathbb{R}^n)$ which has been proved by F. Weissler [We2] for the nonlinear heat equations, by T. Kato [Ka] for the Navier-Stokes equations and by Y. Giga [Gi] for the general problem (2).

Theorem 1.4. *Assume that* H1') *and that* H5) *are fulfilled. Assume furthermore that* $p_c > 1$. *Let* $\gamma(q)$ *defined as*

$$\gamma(q) = \frac{n}{2}\Big(\frac{1}{p_c} - \frac{1}{q}\Big)\,. \tag{13}$$

Then, there exists an absolute constant A *such that, for all* $U_0 \in L^{p_c}(\mathbb{R}^n)$ *with* $\|U_0\|_{L^{p_c}} \le A$, *there is a unique global solution* $U(t,x)$ *of* (2) *such that*

$$t \longrightarrow t^{\gamma(q)}\|U(t,\cdot)\|_{L^q} \in BC([0,+\infty[)\,, \tag{14}$$

for all q *and* $\gamma(q)$ *such that*

$$p_c \le q < +\infty \qquad \textit{and} \qquad 0 \le \gamma(q) < \alpha^{-1}\,, \tag{15}$$

and such that

$$\lim_{t\to 0^+} t^{\gamma(q)}\,\|U(t,\cdot)\|_{L^q} = 0\,, \tag{16}$$

for all q *and* $\gamma(q)$ *such that*

$$p_c < q < +\infty\,,\ \alpha < q \qquad \textit{and} \qquad 0 < \gamma(q) < \alpha^{-1}\,. \tag{17}$$

REMARKS.

1) Generally, the assumption $p_c > 1$ is sharp. For the nonlinear heat equations (3) with $a > 0$ the blow-up for non-negative $C_0^2(\mathbb{R}^n)$ initial data has been proved when $p_c \le 1$ (see [Fu] and [We2]).

2) Note that uniqueness in $BC(\mathbb{R}^+, L^{p_c})$ occurs only on the subspace defined by (14)-(15) and (16)-(17): if $V(t,x)$ is a solution of (2) in $BC(\mathbb{R}^+, L^{p_c})$, we do not know if V satisfies (14)-(15) and (16)-(17) or not.

3) Note that, from Theorem 1.4, the asymptotic decay of $U(t,x)$ in $L^q(\mathbb{R}^n)$ norm is exactly the same as the asymptotic decay of $e^{t\Delta}\,U_0(x)$ as long as the decay rate $\gamma(q)$ satisfies $\gamma(q) < \alpha^{-1}$.

4) Note also that, since $p_c > 1$, there always exists a q_0 such that (17) holds: if $\alpha \le p_c$ this is obvious since $\gamma(p_c) = 0$ and if $p_c < \alpha$ one can check that for $q \in]\alpha, \alpha\, p_c[$ then $0 < \gamma(\alpha) < \gamma(q) < \gamma(\alpha\, p_c) = (2-d)/(2\,\alpha) \le \alpha$.

First, we will prove a slight improvement of the Giga's result,

Lemma 1.2. *Assume that* $\|U_0\|_{L^{p_c}} \le A$ *and let us consider* $U(t,x)$ *the Giga's solution of* (2). *Then,*

$$\|U(t,\cdot)\|_{L^q} \le C\, t^{-\gamma(q)}\, \|U_0\|_{L^{p_c}}\,, \qquad \text{for all } q \in [p_c, +\infty[\,. \tag{18}$$

REMARK. Note that, in the estimate (18), there is no any restrictions on the size of the decay rate $\gamma(q)$.

Then, using the Lemma 1.2, we will consider the case of initial data with arbitrarily high norm in subcritical $L^p(\mathbb{R}^n)$ spaces and small norm in the critical space $L^{p_c}(\mathbb{R}^n)$.

Proposition 1.1. *Let* $U_0 \in L^{p_c}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ *with* $p \le p_c$ *and assume that* $\|U_0\|_{L^{p_c}} \le A$. *Let us consider* $U(t,x)$ *the global solution of* (2) *given by Theorem* 1.4. *Then,*

$$U(t,x) \in BC(\mathbb{R}^+, L^p) \cap BC(\mathbb{R}^+, L^{p_c})\,, \tag{19}$$

$$\|U(t,\cdot)\|_{L^r} \le C\, t^{-n/2(1/p-1/r)}\, \|U_0\|_{L^p}\,, \tag{20}$$

for all $r \ge p$ *and* $t > 0$.

REMARK. One more time we see that $U(t,x)$ decay in $L^r(\mathbb{R}^n)$ with the same rate than $e^{t\Delta}\, U_0(x)$ this, without any restriction on the decay rate. For the Navier-Stokes equations, such a result has ever been proved in [Ka] but only when $n\,(1/p - 1/r)/2 < 1/2$ in (20).

Using Proposition 1.1, in Section 3.3 we will prove the following result on the global Cauchy problem for initial data in $H^s_p(\mathbb{R}^n)$ spaces.

Theorem 1.5. *Assume that* H1') *and* H5) *hold. Assume that* $p_c > 1$ *and that* $p \in]p_c\, \alpha^{-1}, p_c]$. *Then,*

a) *There exists an absolute constant* A' *such that, for all* $U_0 \in H_p^{s_c}(\mathbb{R}^n)$ *with* $\|U_0\|_{H_p^{s_c}} \le A'$, *there is a unique global solution* $U(t,x)$ *of* (2) *in* $C([0,+\infty[, H_p^{s_c})$ *which satisfies* (14)-(15) *and* (16)-(17). *Furthermore* $U(t,x)$ *satisfies the estimates* (19) *and* (20).

b) *Let* $U_0 \in H_p^s(\mathbb{R}^n)$ *with* $s > s_c$. *If* $\|U_0\|_{H_p^{s_c}} \le A'$, *then the local solution of* (2) *given by Theorem* 1.3 *belongs to* $BC(\mathbb{R}^+, H_p^s)$ *and satisfies the estimates* (19) *and* (20).

REMARKS.

1) For data with an arbitrarily norm in $H_p^{s_c}(\mathbb{R}^n)$ one can also prove local existence and uniqueness in a subspace of $C([0,T[, H_p^{s_c})$ defined by a local version of (14)-(15) and (16)-(17).

2) There is no restriction on the size of $\|U_0\|_{H_p^s}$ in Part b) of Theorem 1.5: we just assume that $\|U_0\|_{H_p^{s_c}(\mathbb{R}^n)}$ is small enough (the only norm invariant by scaling).

3) For the Navier-Stokes equations, using Besov spaces of nonpositive order, one can also prove global existence under a weaker assumption than the natural assumption $\|U_0\|_{H_p^{s_c}} \le A$ (for instance see [GM], [KM] or [Ca]).

In Section 2 we will study the local Cauchy problem under the assumptions of Theorem 1.3: we prove existence, uniqueness and continuous dependance with respect to the initial data; we also prove smoothing effects for the solution of (2). In Section 3 we study the global Cauchy problem for small initial data in the critical space $L^{p_c}(\mathbb{R}^n)$, for initial data in the space $L^p(\mathbb{R}^n) \cap L^{p_c}(\mathbb{R}^n)$ for subcritical $L^p(\mathbb{R}^n)$ spaces and then, for initial data in the Sobolev spaces $H_p^s(\mathbb{R}^n)$: we will prove Lemma 1.2, Proposition 1.1 and Theorem 1.5. Next, in Section 4, we will prove the nonlinear estimate of Theorem 1.2 which is the key estimate to prove the Theorem 1.3.

2. The local Cauchy problem.

We first prove existence of a solution (Section 2.1) and then uniqueness (Section 2.2). In Section 2.3 we study smoothing effects for (2) and, in Section 2.4, we study continuous dependence of the solutions with respect to the initial data.

2.1. Existence.

First we assume that Theorem 1.2 holds and that U_0 belongs to subcritical $H_p^s(\mathbb{R}^n)$ spaces. In the sequel C will denote a non-negative constant which may be changed from one line to another. We also forget the time dependance of C since in this section we are only dealing with a local problem. To simplify the notations we define

$$L(u)(t,x) = \int_0^t e^{(t-\tau)\Delta} P(D) F(u(\tau))\, d\tau\,.$$

We introduce the exponent $\tilde{p}$ given by

$$\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{s}{n}\,, \tag{21}$$

and by (5), (6) and since $s > s_c$,

$$\tilde{p} \geq \alpha \qquad \text{and} \qquad \tilde{p} > p_c\,. \tag{22}$$

We define the spaces

$$Y = C([0,T[\,, H_p^s) \tag{23}$$

and

$$X = C([0,T[\,, L^{\tilde{p}})\,. \tag{24}$$

Hence, by the Sobolev embedding Theorem, $Y \hookrightarrow X$. Now, let us consider the sequence of functions

$$u^0 = e^{t\Delta}\, U_0(x)\,, \qquad u^{j+1} = u^0 + L(u^j)\,. \tag{25}$$

First we are going to prove that $\{u^j\}$ converges strongly in X to a limit U which verifies (2) (this proof follows closely Giga's proof but we detail it for the reader's convenience) and second, using the new estimates given by Theorem 1.2, we will show that U belongs also to Y. Let us recall the $(L^p - L^q)$ and $(H_p^{s+\theta} - H_p^s)$ estimates for the semigroup $e^{\tau\Delta}$ (see [Tr]).

Lemma 2.1.

a) *For all* $q \geq p$ *and* $\tau > 0$, *there exists* C *such that*

$$\|e^{\tau\Delta} f\|_{L^q} \leq C\,\tau^{-n/2(1/p-1/q)}\,\|f\|_{L^p}\,.$$

b) *For all* $\theta \geq 0$ *and* $\tau \in]0,T]$, *there exists* $C(T)$ *such that*

$$\|e^{\tau\Delta} f\|_{H_p^{s+\theta}} \leq C(T)\,\tau^{-\theta/2}\,\|f\|_{H_p^s}\,.$$

c) *For all* $\theta \geq 0$ *and* $\tau > 0$, *there exists* C *such that*

$$\|e^{\tau\Delta} f\|_{\dot{H}_p^{s+\theta}} \leq C\,\tau^{-\theta/2}\,\|f\|_{\dot{H}_p^s}\,.$$

By Part a) of Lemma 2.1

$$\|u^0\|_X \leq \|U_0\|_{L^{\tilde{p}}} \leq C\,\|U_0\|_{H_p^s}\,. \tag{26}$$

Let u and v in X then,

$$\|L(u)(t) - L(v)(t)\|_{L^{\tilde{p}}} \leq \int_0^t \left\|e^{(t-\tau)\Delta} P(D)\,(F(u)(\tau) - F(v)(\tau))\right\|_{L^{\tilde{p}}} d\tau\,.$$

Since we are working in the whole Euclidian space $\mathbb{R}^n$, the operators $e^{\tau\Delta}$ and $P(D)$ are some Fourier multipliers and so,

$$e^{(t-\tau)\Delta} P(D) = P(D)\,e^{(t-\tau)\Delta} = e^{\Delta(t-\tau)/2}\,P(D)\,e^{\Delta(t-\tau)/2}\,.$$

Furthermore by H1), $P(D) : H_p^d(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$ is bounded and so, using Lemma 2.1,

$$\begin{aligned}
&\|e^{(t-\tau)\Delta}\,P(D)\,(F(u)(\tau) - F(v)(\tau))\|_{L^{\tilde{p}}} \\
&\qquad\qquad \leq C\,(t-\tau)^{-d/2}\,\|e^{\Delta(t-\tau)/2}\,(F(u)(\tau) - F(v)(\tau))\|_{L^{\tilde{p}}} \\
&\qquad\qquad \leq C\,(t-\tau)^{-\beta}\,\|F(u)(\tau) - F(v)(\tau)\|_{L^{\tilde{p}/\alpha}}\,,
\end{aligned}$$

(note that the first part of (22) is needed) where, by (4),

$$\beta = \frac{d}{2} + \frac{n}{2}\,\frac{\alpha-1}{\tilde{p}} = 1 - \frac{(\alpha-1)\,(s-s_c)}{2} < 1\,. \tag{27}$$

Using this last estimate and Hölder's inequality, we obtain

$$\|L(u)(t) - L(v)(t)\|_{L^{\tilde p}} \le C\int_0^t (t-\tau)^{-\beta}\,\|u(\tau)-v(\tau)\|_{L^{\tilde p}}\big(\|u(\tau)\|_{L^{\tilde p}}^{\alpha-1} + \|v(\tau)\|_{L^{\tilde p}}^{\alpha-1}\big)\,d\tau$$

and, since $\beta < 1$,

$$\|L(u) - L(v)\|_X \le C\,T^{1-\beta}\,\|u-v\|_X\big(\|u\|_X^{\alpha-1} + \|v\|_X^{\alpha-1}\big)\,. \tag{28}$$

Furthermore $L(0) = 0$ and from (26) and (28) we deduce that

$$\begin{cases} \|u^{j+1}\|_X \le \|U_0\|_{H^s_p} + C\,T^{1-\beta}\|u^j\|_X^{\alpha}\,, \\ \|u^{j+1} - u^j\|_X \le C\,T^{1-\beta}\|u^j - u^{j-1}\|_X \\ \qquad\qquad\qquad \cdot\big(\|u^j\|_X^{\alpha-1} + \|u^{j-1}\|_X^{\alpha-1}\big)\,. \end{cases} \tag{29}$$

Then, a standard fixed point argument shows that, for

$$T < \frac{C}{4}\,\|U_0\|_{H^s_p}^{-(\alpha-1)/(1-\beta)}\,, \tag{30}$$

the sequence $\{u^j\}$ converges strongly in X to a limit U which obviously solves (2) since $\tilde p \ge \alpha$ by (22).

Now, we must prove that this solution belongs also to Y. Let $u \in Y$, then,

$$\|L(u)(t)\|_{H^s_p} \le \int_0^t \|e^{(t-\tau)\Delta}P(D)F(u)(\tau)\|_{H^s_p}\,d\tau\,.$$

As previously

$$\begin{aligned}\|e^{(t-\tau)\Delta}P(D)F(u)(\tau)\|_{H^s_p} &\le C\,(t-\tau)^{-d/2}\,\|e^{\Delta(t-\tau)/2}F(u)(\tau)\|_{H^s_p} \\ &\le C\,(t-\tau)^{-(d+s-s_\alpha)/2}\,\|F(u)(\tau)\|_{H^{s_\alpha}_p}\,.\end{aligned}$$

But now, using Theorem 1.2, we can bound the term $\|F(u)(\tau)\|_{H^{s_\alpha}_p}$ by $C\,\|u(\tau)\|_{H^s_p}^{\alpha}$ and furthermore, thanks to (27) and to (4), we obtain

$$\|e^{(t-\tau)\Delta}P(D)F(u)(\tau)\|_{H^s_p} \le C\,(t-\tau)^{-\beta}\,\|u(\tau)\|_{H^s_p}^{\alpha}\,.$$

This last inequality leads then to

$$\|L(u)(t)\|_{H_p^s} \le C\int_0^t (t-\tau)^{-\beta}\,\|u(\tau)\|_{H_p^s}^{\alpha}\,d\tau \le C\,T^{1-\beta}\,\|u\|_Y^{\alpha}$$

and so by (26),

$$\|u^{j+1}\|_Y \le \|U_0\|_{H_p^s} + C\,T^{1-\beta}\,\|u^j\|_Y^{\alpha}\;. \tag{31}$$

As previously, if T satisfies (30), thanks to (31) we see that the $\|u^j\|_Y$ remain bounded and so, we can always extract a subsequence $\{u^{j_k}\}$ which converges weakly-$\star$ to a limit $\tilde{U}\in Y$. Now the u^{j_k} converge to U and converge to $\tilde{U}$ in $\mathcal{D}'(]0,T[\times\mathbb{R}^n)$ and so U agrees with $\tilde{U}$. Thus we have proved the existence of a solution in $C([0,T[,H_p^s)$.

The estimate for T_m comes from (30) which gives

$$T_m \ge \frac{C}{8}\,\|U_0\|_{H_p^s}^{-2/(s-s_c)}\;.$$

If $T_m<+\infty$, this explicit lower bound obviously allows us to show the blow-up in $H_p^s(\mathbb{R}^n)$ norm (one can also prove the blow up in $L^{\tilde{p}}(\mathbb{R}^n)$ when it holds in $H_p^s(\mathbb{R}^n)$).

If $s>n/p$, using Theorem 1.1 instead of Theorem 1.2, the same proof gives existence under the hypothesis H4).

2.2 Uniqueness.

Let $U(t,x)\in Y$ and $V(t,x)\in Y$ be two solutions for the same initial data U_0 and let $T<\max\{T_m(V),T_m(U)\}$. Then, since U and V solve (2),

$$\|U-V\|_X = \|L(U)-L(V)\|_X$$

and so, by (28),

$$\|U-V\|_X \le 2\,T^{1-\beta}\,C\,M^{\alpha-1}\,\|U-V\|_X\;,$$

where

$$M=\sup_{[0,T]}\{\|U(t)\|_{L^{\tilde{p}}},\|V(t)\|_{L^{\tilde{p}}}\}\,.$$

So, for T small enough,

$$\|U-V\|_X \le \frac{1}{2}\,\|U-V\|_X$$

and so $U = V$ on $[0, T]$. To conclude we just have to iterate this in order to prove that $T_m(U) = T_m(V)$ and that $U = V$ on $[0, T_m(U)[$.

2.3. Smoothing effects.

Let U be a solution of (2). Using Lemma 2.1 we easily see that

$$\|U(t,x) - e^{t\Delta}U_0\|_{H_p^{s+\theta}} \leq C \int_0^t (t-\tau)^{-\theta-\beta} \|U(\tau,\cdot)\|_{H_p^s}^\alpha \, d\tau$$

and so, for all $\theta < 1 - \beta = (\alpha - 1)\nu$,

$$\|U(t,x) - e^{t\Delta}U_0\|_{H_p^{s+\theta}} \leq C\, T^{1-\beta-\theta} \|U\|_Y^\alpha \,,$$

which gives the first part of Theorem 1.3.b). If $s > n/p$, the proof is the same using Theorem 1.1 instead of Theorem 1.2.

Now let us assume that $F \in C^\infty(\mathbb{R})$. For all $t > 0$, $e^{t\Delta}U_0$ is $C^\infty(\mathbb{R}^n)$ and so $U(\delta/2, \cdot) \in H_p^{s+\theta}(\mathbb{R}^n)$. Taking $\delta/2$ as initial time, we just have to repeat this argument to prove that $U(\delta/2 + \delta/4, \cdot) \in H_p^{s+2\theta}(\mathbb{R}^n) \cdots$ finally, $U \in C([\delta, T[\,, C^\infty)$ for each $\delta > 0$. Thus we have proved the second part of Theorem 1.3.b).

2.4. Continuous dependence with respect to the data.

First we deal with continuity in X norm. Let U and V be two solutions of (2) for the respective initial data U_0 and V_0. Let

$$T < \min\{T_m(U_0), T_m(V_0)\}$$

and

$$M = \sup_{t\in[0,T]} \{\|U(t)\|_{H_p^s}, \|V(t)\|_{H_p^s}\}\,.$$

By (28),

$$\|U - V\|_X \leq \|U_0 - V_0\|_{L^{\tilde{p}}} + C\, T^{1-\beta} \|U - V\|_X \, 2\, M^{\alpha-1}\,.$$

Taking $T' \leq T$ such that $4\, C\, {T'}^{1-\beta} M^{\alpha-1} \leq 1$ then,

$$\|U - V\|_{C([0,T'[\,,L^{\tilde{p}})} \leq 2\, \|U_0 - V_0\|_{L^{\tilde{p}}}$$

and, if one can take $T' = T$, this ends the proof. On the contrary, solving (2) for the initial data $U(T')$ and $V(T')$, the uniqueness and the uniform bound for U and V in X norm allow us to iterate this last argument N times until $N\,T' \geq T$ and thus

$$\|U - V\|_X \leq C(T)\,\|U_0 - V_0\|_{L^{\tilde{p}}} \leq C(T)\,\|U_0 - V_0\|_{H_p^s}\,. \tag{32}$$

Now let us assume that $s_\alpha \leq 0$, *i.e.* that $s \leq n\,(\alpha - 1)/(p\,\alpha)$. Then,

$$\begin{aligned}
&\|U(t) - V(t)\|_{H_p^s} \\
&\qquad \leq \|U_0 - V_0\|_{H_p^s} + C\int_0^t (t-\tau)^{-\beta}\,\|F(U(\tau)) - F(V(\tau))\|_{H_p^{s_\alpha}}\,d\tau\,.
\end{aligned}$$

Since $s_\alpha \leq 0$ and $\alpha/\tilde{p} = 1/p - s_\alpha/n$, we can use the Sobolev embedding

$$L^{\tilde{p}/\alpha} \hookrightarrow H_p^{s_\alpha}\,,$$

which leads to

$$\begin{aligned}
&\|U(t) - V(t)\|_{H_p^s} \\
&\qquad \leq \|U_0 - V_0\|_{H_p^s} + C\int_0^t (t-\tau)^{-\beta}\,\|F(U(\tau)) - F(V(\tau))\|_{L^{\tilde{p}/\alpha}}\,d\tau \\
&\qquad \leq \|U_0 - V_0\|_{H_p^s} + C\,T^{1-\beta}\,\|U - V\|_X\big(\|U\|_X^{\alpha-1} + \|V\|_X^{\alpha-1}\big)
\end{aligned}$$

and, according to (32) and to this last inequality, we obtain that

$$\|U - V\|_Y \leq C(T)\,\|U_0 - V_0\|_{H_p^s}\,. \tag{33}$$

To conclude we have to relax our assumption on s. Since U and V are solutions of (2),

$$\|U - V\|_Y \leq \|U_0 - V_0\|_{H_p^s} + \|L(U) - L(V)\|_Y\,.$$

First, let us recall the following interpolation inequality.

Lemma 2.2. *Let $p \in]1, +\infty[$, $\theta \in \mathbb{R}$ and $s \in \mathbb{R}$. Then, for all $f \in H_p^{s+\theta}(\mathbb{R}^n)$,*

$$\|f\|_{H_p^s}^2 \leq C\,\|f\|_{H_p^{s+\theta}}\,\|f\|_{H_p^{s-\theta}}\,.$$

For a proof see [Tr].

By Lemma 2.2 we see that

$$
\begin{aligned}
&\|L(U)(t) - L(V)(t)\|_{H^s_p} \\
&\le \big(\|L(U)(t))\|_{H^{s+\theta}_p} + \|L(V)(t)\|_{H^{s+\theta}_p}\big)^{1/2} \, \|L(U)(t) - L(V)(t)\|^{1/2}_{H^{s-\theta}_p}
\end{aligned}
$$

Now since $s < 2 - d$, one can choose $\theta < (\alpha - 1)\,(s - s_c)/2$ such that

$$
s_c < s - \theta < \frac{\alpha - 1}{\alpha}\,\frac{n}{p}\,. \tag{34}
$$

Using the smoothing effects, the first term of the left-hand side of the last inequality can be bounded by

$$
C(T)\,(\|U\|_Y + \|V\|_Y)^{1/2} \le C'(T)\,M\,,
$$

and, using (33), since $(s - \theta)$ satisfies (34), we bound the second term by

$$
\begin{aligned}
\|U_0 - V_0\|^{1/2}_{H^{s-\theta}_p} + \|U(t) - V(t)\|^{1/2}_{H^{s-\theta}_p} &\le C(T)\,\|U_0 - V_0\|^{1/2}_{H^{s-\theta}_p} \\
&\le C(T)\|U_0 - V_0\|^{1/2}_Y\,.
\end{aligned}
$$

Combining this two inequalities we obtain that

$$
\|U - V\|_Y \le C(T)\,\|U_0 - V_0\|^{1/2}_Y
$$

and the proof of Part c) is completed.

3. The global Cauchy problem.

In this section we study the global Cauchy problem for small initial data in $L^{p_c}(\mathbb{R}^n)$. First in Section 3.1 we study the case of initial data which belongs only to $L^{p_c}(\mathbb{R}^n)$ and we prove Lemma 1.2. In Section 3.2 we study the global Cauchy problem for initial data in $L^p(\mathbb{R}^n) \cap L^{p_c}(\mathbb{R}^n)$ when $L^p(\mathbb{R}^n)$ is subcritical for (2) and we prove the Proposition 1.1. Then, in Section 3.3, we consider initial data in $H^s_p(\mathbb{R}^n)$ space and we prove the Theorem 1.5.

3.1. Initial data in $L^{p_c}(\mathbb{R}^n)$.

Let us consider $U_0 \in L^{p_c}(\mathbb{R}^n)$. In [Gi] Giga proved that there exists a non-negative absolute constant A such that, if $\|U_0\|_{L^{p_c}} \leq A$, then there exists a unique global solution of (2) in $BC(\mathbb{R}^+, L^{p_c})$ which satisfies

$$t \longmapsto t^{\gamma(q)} \, \|U(t,\cdot)\|_{L^q} \in BC(\mathbb{R}^+)\,,$$

for all q and $\gamma(q)$ such that

$$p_c \leq q < +\infty \qquad \text{and} \qquad 0 \leq \gamma(q) < \alpha^{-1}\,,$$

and which satisfies

$$\lim_{t \to 0^+} t^{\gamma(q)} \, \|U(t,\cdot)\|_{L^q} = 0\,,$$

for all q and $\gamma(q)$ such that

$$p_c < q < +\infty\,,\ \alpha < q \qquad \text{and} \qquad 0 < \gamma(q) < \alpha^{-1}\,.$$

First we are going to prove that, for $p_c \leq q < +\infty$ and $0 \leq \gamma(q) < \alpha^{-1}$,

$$\|U(t,\cdot)\|_{L^q} \leq C\, t^{-\gamma(q)} \, \|U_0\|_{L^{p_c}}\,, \tag{35}$$

which is a little more precise than the estimate

$$\|U(t,\cdot)\|_{L^q} \leq C\, t^{-\gamma(q)}\,.$$

Second, we are going to relax the restriction $\gamma(q) < \alpha^{-1}$ in this estimate. Indeed, when $p_c \geq n\alpha/2$, the reader will check that the assumption $\gamma(q) < \alpha^{-1}$ is fulfilled for all $q \in [p_c, +\infty[$ and so, the asymptotic estimates (35) to. On the contrary, when $p_c < n\,\alpha/2$, on must assume that $q \in [p_c, (1/p_c - 2/(n\,\alpha))^{-1}[$ to be sure that $\gamma(q) < \alpha^{-1}$ holds. So, when $p_c < n\,\alpha/2$, the asymptotic estimates are proved only for q in the range $[p_c, (1/p_c - 2/(n\,\alpha))^{-1}[$ and we want to show that they holds for all exponent q in $[p_c, +\infty[$.

To prove Lemma 1.2 let us come back to the proof of Theorem 1.4 given in [Gi]. In the critical case (when $U_0 \in L^{p_c}(\mathbb{R}^n)$), to prove the existence of a solution for (2), one introduce, for $p_c < q < +\infty$, $\alpha < q$ and $0 < \gamma(q) < \alpha^{-1}$, the Banach spaces

$$X_q = \{f(t,x) :\ t \longmapsto t^{\gamma(q)} \, \|f(t,x)\|_{L^q} \in BC(\mathbb{R}^+)\}$$

and the space

$$Y = \{f(t,x) :\ t \longmapsto \|f(t,x)\|_{L^{p_c}} \in BC(\mathbb{R}^+)\}\,.$$

Then, if we consider $\{u^j\}$ the sequence of functions defined by (25), we have the estimate (see [Gi])

$$\|u^{j+1}\|_{X_q} \leq C_1\,\|u^0\|_{X_q} + C_2\,\|u^j\|^{\alpha}_{X_q}\,, \tag{36}$$

where,

$$\|f(t,x)\|_{X_q} = \sup_{t>0} t^{\gamma(q)}\,\|f(t,x)\|_{L^q}\,.$$

Then, when $\|U_0\|_{L^{p_c}} \leq A$, using (36) and (16)-(17), one can prove that the $\{u^j\}$ converge in X_q to $U(t,x)$ the unique solution of (2) such that (16)-(17) is fulfilled (see [Gi] for a proof). Furthermore, to prove that $U(t,x)$ belongs also to $BC([0,+\infty[\,, L^{p_c}(\mathbb{R}^n))$, one can easily check that the nonlinear map $L : X_q \longrightarrow Y$ defined at the beginning of the Section 2.1 satisfies

$$\|L(U)\|_Y \leq C\|U\|^{\alpha}_{X_q} \tag{37}$$

as soon as $p_c < q < +\infty$, $\alpha < q$ and as soon as $0 < \gamma(q) < \alpha^{-1}$.

Now let us come back to the proof of Lemma 1.2. By (36), it is obvious that the sequence $\{u^j\}$ stay in the ball $B(0, 2C_1\|u^0\|_{X_q})$ for the X_q topology as soon as

$$C_2\,(2\,C_1\,\|u^0\|_{X_q})^{\alpha} \leq C_1\,\|u^0\|_{X_q}\,,$$

which holds for

$$\|u_0\|_{X_q} \leq \Big(\frac{1}{2^{\alpha}\,C_1^{\alpha-1}\,C_2}\Big)^{1/(\alpha-1)}.$$

Now, by Lemma 2.1,

$$\|u^0\|_{X_q} \leq C\,\|U_0\|_{L^{p_c}} \tag{38}$$

and so, for

$$\|U_0\|_{L^{p_c}} \leq A = \frac{1}{C}\Big(\frac{1}{2^{\alpha}C_1^{\alpha-1}C_2}\Big)^{1/(\alpha-1)},$$

there exists a global solution $U(t,x)$ of (2) which belongs to the ball

$$B(0, 2\,C_1\,\|u^0\|_{X_q}) \subset B(0, 2\,C_1\,C\,\|U_0\|_{L^{p_c}})\,,$$

for the X_q topology. Thus the proof of Lemma 1.2 is completed for the exponents q such that $p_c < q < +\infty$, $\alpha < q$ and $0 < \gamma(q) < \alpha^{-1}$. To conclude in the special case of $L^{p_c}(\mathbb{R}^n)$, we have just to use this last result and the estimate (37). Thus, if $p_c \geq \alpha$ the proof is over. On the contrary, if $p_c < \alpha$, we have just to interpolate the estimates in $L^q(\mathbb{R}^n)$ norm and in $L^{p_c}(\mathbb{R}^n)$ norm to end the proof.

Now we are going to prove that the asymptotic estimates

$$\|U(t,\cdot)\|_{L^q} \leq C\,t^{-\gamma(q)}\,\|U_0\|_{L^{p_c}}$$

holds also when $\gamma(q) \geq \alpha^{-1}$. First, for U_0 such that $\|U_0\|_{L^{p_c}} \leq A$, let us consider $U(t,x)$ the Giga's solution of (2) and let us consider q_0 an exponent such that $q_0 > p_c$ and such that $\gamma(q_0) < \alpha^{-1}$ (such a q_0 always exists since $p_c > 1$: see the Remark 4 after Theorem 1.4). Next let us consider the sequence $\{q_i\}$ defined by

$$\frac{n}{2}\Big(\frac{1}{q_i} - \frac{1}{q_{i+1}}\Big) = \delta < \alpha^{-1} \tag{39}$$

and note that $\{q_i\}$ is increasing and that there exists q_l such that $n/(2\,q_l) < \alpha^{-1}$.

Let us define

$$I(q_i, q_{i+1}) = \int_0^1 (1-s)^{-d/2 - n(\alpha-1)/(2q_{i+1})}\,s^{-\delta\alpha}\,ds\,.$$

Then, by (39), for all $i \geq 0$,

$$I(q_i, q_{i+1}) < +\infty\,.$$

Now we pick $t_0 > 0$ and we consider V the solution of

$$\begin{cases} V(t,x) = e^{t\Delta}V_0 + L(V)(t,x)\,, \\ V(0,x) = V_0(x) = U(t_0,x)\,. \end{cases} \tag{40}$$

First, by the previous result and since $0 < \gamma(q_0) < \alpha^{-1}$, it follows that

(41) $V_0 \in L^{p_c}(\mathbb{R}^n) \cap L^{q_0}(\mathbb{R}^n)$ with $\|V_0\|_{L^{q_0}} \leq C\,t_0^{-\gamma_0}\,\|U_0\|_{L^{p_c}}$.

Lemma 3.1. *Let* $T = T(t_0)$ *such that*

$$2^\alpha C^\alpha T^{(2-d)/2(1-p_c/q_o)} I(q_0, q_1) \, \|V_0\|_{L^{q_0}}^{\alpha-1} < 1 \tag{42}$$

then,

$$\|V(t)\|_{L^{q_1}} \leq 2\, C\, t^{-\delta} \, \|V_0\|_{L^{q_0}} \ , \tag{43}$$

for all $t \in]0, T[$.

Indeed, since $V_0(x) \in L^{q_0}(\mathbb{R}^n)$ with $q_0 > p_c$, using the proof of Theorem 1.3 we see that the sequence

$$v^0 = e^{t\Delta} V_0 \ , \qquad v^{j+1}(t,x) = v^0 + L(v^j)(t,x) \, ,$$

converges strongly to $V(t,x)$ in $C([0,T], L^{q_0})$. By Lemma 2.1, v^0 obviously satisfies (43) for all $t > 0$. Now, if v^j satisfies (43), then

$$\begin{aligned} \|v^{j+1}(t)\|_{L^{q_1}} &\leq C\, \|V_0\|_{L^{q_0}}\, t^{-\delta} \\ &\qquad + C \int_0^t (t-\tau)^{-d/2 - n(\alpha-1)/(2q_1)} \, \|v^j(\tau)\|_{L^{q_1}}^\alpha \, d\tau \\ &\leq C\, \|V_0\|_{L^{q_0}}\, t^{-\delta} \\ &\qquad + C\, 2^\alpha\, C^\alpha\, \|V_0\|_{L^{q_0}}^\alpha \int_0^t (t-\tau)^{-d/2-n(\alpha-1)/(2q_1)}\, \tau^{-\delta\alpha}\, d\tau \end{aligned}$$

for all $t \in]0, T[$, and so,

$$\begin{aligned} &\|v^{j+1}(t)\|_{L^{q_1}} \\ &\leq \frac{2\, C\, \|V_0\|_{L^{q_0}}}{t^{n/2(1/q_0 - 1/q_1)}} \left(\frac{1}{2} + 2^{\alpha-1}\, C^\alpha\, \|V_0\|_{L^{q_0}}^{\alpha-1}\, I_{q_0,q_1}\, T^{(2-d)/2(1-p_c/q_0)} \right), \end{aligned}$$

for all $t \in]0, T[$. Hence, if T satisfies (42),

$$\|v^{j+1}(t,\cdot)\|_{L^{q_1}} \leq 2\, C\, t^{-\delta}\, \|V_0\|_{L^{q_0}} \ .$$

So, by induction, (43) holds for all j and thus the Lemma is proved.

Using the uniqueness result in the supercritical case and (40) we see that

$$V(t,x) = U(t + t_0, x)$$

and so, by Lemma 3.1,

$$\|U(t+t_0)\|_{L^{q_1}} \leq 2\,C\,t^{-\delta}\,\|U(t_0)\|_{L^{q_0}}\ ,$$

for each $t \in [0, T(t_0)[$ where $T(t_0)$ satisfies (42). Now, we claim that there exists an absolute constant A' such that, when $\|U_0\|_{L^{p_c}} \leq A'$, one can always take $T(t_0) = t_0/2$ in the previous inequality. Indeed by Lemma 3.1 we have only to make sure that

$$2^\alpha\,C^\alpha \Big(\frac{t_0}{2}\Big)^{(2-d)/2(1-p_c/q_0)}\,\|V_0\|_{L^{q_0}}^{\alpha-1}\,I(q_0,q_1) < 1\,,$$

which, combined with (41), leads to

$$2^\alpha\,C^\alpha \Big(\frac{t_0}{2}\Big)^{(2-d)/2(1-p_c/q_0)-n(\alpha-1)/2(1/p_c-1/q_0)}\,\|U_0\|_{L^{p_c}}^{\alpha-1}\,I(q_1,q_0) < 1$$

and, since

$$\frac{2-d}{2}\Big(1-\frac{p_c}{q_0}\Big) - \frac{n(\alpha-1)}{2}\Big(\frac{1}{p_c}-\frac{1}{q_0}\Big) = 0$$

it is sufficient to make sure that

$$2^\alpha\,C^\alpha\,\|U_0\|_{L^{p_c}}^{\alpha-1}\,I(q_1,q_0) < 1\,.$$

Thus, when U_0 is small enough in $L^{p_c}(\mathbb{R}^n)$, (42) holds for each $t_0 > 0$ and so

$$\Big\|U\Big(\frac{3\,t_0}{2}\Big)\Big\|_{L^{q_1}} \leq 2\,C\,t_0^{-\delta}\,t_0^{-n/2(1/p_c-1/q_0)}\,\|U_0\|_{L^{p_c}}$$

and, since t_0 is arbitrary,

$$\|U(t)\|_{L^{q_1}} \leq 2\,C\,t^{-n/2(1/p_c-1/q_1)}\,\|U_0\|_{L^{p_c}}\ ,$$

for all $t > 0$. Now, since $I(q_i, q_{i+1}) < +\infty$ and since we have prove the required estimate for q_1 defined by (39), we have just to iterate this proof to get the required estimate in L^{q_2} norm... Thus, for each q_i, the proof follows by induction. Now, if $q \in]q_i, q_{i+1}[$, we get the result by interpolation. Thus we have proved that $U(t,x)$, the global solution of (2), satisfies

$$\|U(t,x)\|_{L^q} \leq C\,t^{-\gamma(q)}\,\|U_0\|_{L^{p_c}}\ ,$$

for all $q \in [p_c, +\infty[$.

3.2. Initial data in $L^{p_c}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$.

Let $p < p_c$. We consider now an initial data U_0 which belongs to $L^p(\mathbb{R}^n) \cap L^{p_c}(\mathbb{R}^n)$, we assume that $\|U_0\|_{L^{p_c}} \leq A$ and we denote by $U(t,x)$ the Giga's solution of (2) which belongs to $BC(\mathbb{R}^+, L^{p_c})$ and satisfies the estimates (14)-(15) and (16)-(17). Using the slight improvement about the decay of the $L^q(\mathbb{R}^n)$ norms (estimates (18) of Lemma 1.2) that we previously proved, we are first going to show that the Giga's solution belongs to $L^p(\mathbb{R}^n)$ for all t (step one), then we will prove that $U(t,x)$ belongs to $BC(\mathbb{R}^+,\ L^p(\mathbb{R}^n))$ (step two) and next, that $U(t,x)$ satisfies the asymptotic estimates (20) (step three).

Step one. Here we consider $U_0(x) \in L^p(\mathbb{R}^n) \cap L^{p_c}(\mathbb{R}^n)$ and we want to prove that,

$$\|U(t)\|_{L^p} \leq C(T)\,, \qquad \text{for all } T > 0 \text{ and } t \in [0,T]\,. \tag{44}$$

First let us assume that

$$\max\left\{1, \frac{p_c}{\alpha}\right\} \leq p < p_c\,. \tag{45}$$

Then, since $U(t,x)$ is a solution for (2), for all $T > 0$ and $t \in [0,T]$

$$\begin{aligned}
\|U(t)\|_{L^p} &\leq \|U_0\|_{L^p} + \|L(U)(t)\|_{L^p} \\
&\leq \|U_0\|_{L^p} + \int_0^t \|e^{(t-\tau)\Delta} P(D) F(U)(\tau)\|_{L^p}\, d\tau \\
&\leq \|U_0\|_{L^p} + \int_0^t |t-\tau|^{-d/2}\, \|F(U)(\tau)\|_{L^p}\, d\tau \\
&\leq \|U_0\|_{L^p} + \int_0^t |t-\tau|^{-d/2}\, \|U(\tau)\|^{\alpha}_{L^{p\alpha}}\, d\tau
\end{aligned}$$

Now, by (45), $p\,\alpha \geq p_c$ and so, using the estimates (18) of the Lemma 1.2, we obtain that

$$\begin{aligned}
\|U(t)\|_{L^p} &\leq \|U_0\|_{L^p} + \int_0^t |t-\tau|^{-d/2}\, \tau^{-\alpha\gamma(p\alpha)}\, \|U_0\|^{\alpha}_{L^{p_c}}\, d\tau \\
&\leq \|U_0\|_{L^p} + C(T)\, \|U_0\|^{\alpha}_{L^{p_c}}
\end{aligned}$$

since $d < 2$ and

$$0 < \gamma(p\,\alpha) < \gamma(p_c\,\alpha) = \frac{2-d}{2\,\alpha} \leq \frac{1}{\alpha}\,,$$

for all $p_c/\alpha < p < p_c$.

Thus, the estimate (44) is proved for all p which verify (45) and, if $p_c < \alpha$, the proof is over.

Assume now that

$$\max\left\{1, \frac{p_c}{\alpha^2}\right\} \leq p < \frac{p_c}{\alpha}\,. \tag{46}$$

First, if $U_0(x) \in L^p \cap L^{p_c}$, then $U_0(x)$ belongs to $L^q(\mathbb{R}^n)$ for all $q \in [p_c\,\alpha^{-1}, p_c]$ and then, by the previous result, $\|U(t)\|_{L^q} \leq C(T, U_0)$ for all q in the range $[p_c\,\alpha^{-1}, p_c]$. Second, since $U(t,x)$ is a solution of (2)

$$\begin{aligned}
\|U(t)\|_{L^p} &\leq \|U_0\|_{L^p} + \|L(U)(t)\|_{L^p} \\
&\leq \|U_0\|_{L^p} + \int_0^t \|e^{(t-\tau)\Delta} P(D) F(U)(\tau)\|_{L^p}\, d\tau \\
&\leq \|U_0\|_{L^p} + \int_0^t |t-\tau|^{-d/2} \|F(U)(\tau)\|_{L^p}\, d\tau \\
&\leq \|U_0\|_{L^p} + \int_0^t |t-\tau|^{-d/2}\, \|U(\tau)\|_{L^{p\alpha}}^{\alpha}\, d\tau\,.
\end{aligned}$$

Next, we remark that $d < 2$ and that, by (46), $p\,\alpha$ belongs to the range $[p_c\,\alpha^{-1}, p_c[$. Hence, by the previous result, we can use the bound

$$\|U(t)\|_{L^{p\alpha}} \leq C(T, U_0)\,,$$

which leads to

$$\|U(t)\|_{L^p} \leq \|U_0\|_{L^p} + C(T, U_0)$$

and so, the estimate (44) holds for all p in the range $[\max\{1, p_c/\alpha^2\}, p_c]$. Next, for $p \in [p_c\,\alpha^{-n-1}, p_c\,\alpha^{-n}[$, the proof of (44) follows easily by induction.

Step two. In step one, we have proved that $U(t,x)$ the Giga's solution of (2) belongs to $L^p(\mathbb{R}^n)$ for all $t \geq 0$ when U_0 belongs to $L^p(\mathbb{R}^n) \cap L^{p_c}(\mathbb{R}^n)$ and when U_0 is small enough in $L^{p_c}(\mathbb{R}^n)$. Now, we are going to prove that $U(t,x)$ belongs to $BC(\mathbb{R}^+, L^p)$. Let us consider $T>0$ and t in $[0,T]$. First, since $U(t,x)$ is a mild solution of (1),

$$U(t,x) = e^{t\Delta}U_0(x) + L(U)(t,x)\,.$$

So, by Lemma 2.1,

$$\begin{aligned}\|U(t)\|_{L^p} &\le \|U_0\|_{L^p} + \|L(U)(t)\|_{L^p}\\ &\le \|U_0\|_{L^p} + \int_0^t \|e^{(t-\tau)\Delta}P(D)F(U)(\tau)\|_{L^p}\,d\tau\\ &\le \|U_0\|_{L^p} + C\int_0^t (t-\tau)^{-\xi(q)}\,\|F(U(\tau))\|_{L^q}\,d\tau\,,\end{aligned}$$

where q is any exponent in $[1,p[$ which will be fixed latter and where $\xi(q)$ is defined by

$$\xi(q) = \frac{d}{2} + \frac{n}{2}\Big(\frac{1}{q} - \frac{1}{p}\Big)\,, \tag{47}$$

Using Hölder's inequality we get

$$\|U(t)\|_{L^p} \le \|U_0\|_{L^p} + C\int_0^t (t-\tau)^{-\xi(q)}\,\|U(\tau)\|_{L^{qq_1}}\,\|U(\tau)\|^{\alpha-1}_{L^{qq_2(\alpha-1)}}\,d\tau\,,$$

where $1/q_1 + 1/q_2 = 1$ and furthermore, we choose q_1 such that $q\,q_1 = p$ to obtain

$$\|U(t)\|_{L^p} \le \|U_0\|_{L^p} + C\,\|U\|_{L^\infty([0,T],L^p)}\int_0^t (t-\tau)^{-\xi(q)}\|U(\tau)\|^{\alpha-1}_{L^{qq_2(\alpha-1)}}\,d\tau\,.$$

Now if we choose q such that $q\approx p$ with $q<p$ then, since $q\,q_1 = p$, $q_1\approx 1$. Hence it follows that $z = q\,q_2\,(\alpha-1)\ge p_c$. Next, for $z = q\,q_2\,(\alpha-1)\ge p_c$, by Lemma 1.2, we can bound $U(t,x)$ in $L^{qq_2(\alpha-1)}(\mathbb{R}^n)$ norm by

$$\|U(t,x)\|_{L^{qq_2(\alpha-1)}} \le C\,t^{-\gamma(qq_2(\alpha-1))}\,\|U_0\|_{L^{p_c}}$$

and so,

$$\|U(t)\|_{L^p} \le C\,\|U_0\|_{L^p} + \|U\|_{L^\infty([0,T],L^p)}\,\|U_0\|^{\alpha-1}_{L^{p_c}}\int_0^t (t-\tau)^{-\xi(q)}\tau^{-\theta(q)}\,d\tau\,,$$

where

$$\theta(q) = (\alpha - 1)\,\gamma(q\,q_2\,(\alpha-1)) = \frac{1}{2}\Big(2 - d - \frac{n}{q\,q_2}\Big)\,. \tag{48}$$

One can easily check that choosing $q \approx p$ then q_2 is large enough to makes sure that $0 < \xi(q) < 1$ and that $0 < \theta(q) < 1$ (since $d < 2$). Furthermore $\xi(q) + \theta(q) = 1$ and so,

$$\|U\|_{L^\infty([0,T],L^p)} \le \|U_0\|_{L^p} + C\,\|U\|_{L^\infty([0,T],L^p)}\,\|U_0\|^{\alpha-1}_{L^{p_c}}\,. \tag{49}$$

Now, if $\|U_0\|_{L^{p_c}}$ is small enough then

$$1 - C\,\|U_0\|^{\alpha-1}_{L^{p_c}} \ge \frac{1}{2}$$

and then, by (49), and since $\|U\|_{L^\infty([0,T[,L^p)} < +\infty$ for all $T > 0$,

$$\|U\|_{L^\infty([0,T],L^p)} \le \frac{\|U_0\|_{L^p}}{1 - C\,\|U_0\|^{\alpha-1}_{L^{p_c}}} \le 2\,\|U_0\|_{L^p}\,.$$

To conclude, we have just to remark that the right side of this estimate do not depend of T. Thus, we have proved that $U(t,x)$ the mild solution of (1) belongs to $BC(\mathbb{R}^+, L^p(\mathbb{R}^n))$.

Step three. Now we have to prove the $L^r(\mathbb{R}^n)$ estimates (20) of Theorem 1.4. They hold obviously for the term $e^{t\Delta}U_0$ by Lemma 2.1, hence, we just deal with the nonlinear term $L(U)$. First let us suppose that

$$\delta(r) = \frac{n}{2}\Big(\frac{1}{p} - \frac{1}{r}\Big) < \frac{2-d}{2}\,. \tag{50}$$

Then,

$$\begin{aligned}
\|L(U)(t)\|_{L^r} &\le \int_0^t \|e^{(t-\tau)\Delta}P(D)F(U(\tau))\|_{L^r}\,d\tau \\
&\le C\int_0^t (t-\tau)^{-\delta(r)}\|e^{(t-\tau)\Delta}P(D)F(U(\tau))\|_{L^p}\,d\tau \\
&\le C\int_0^t (t-\tau)^{-\delta(r)-\xi(q)}\,\|U(\tau)\|_{L^{qq_1}}\,\|U(\tau)\|^{\alpha-1}_{L^{qq_2(\alpha-1)}}\,d\tau\,,
\end{aligned}$$

where $q \in [1, p[$, $1/q_1 + 1/q_2 = 1$ and $\xi(q)$ is given by (47). Now, taking $q\, q_1 = p$ with $q \approx p$ then $q_1 \approx 1$ and $q\, q_2\,(\alpha - 1) \geq p_c$ and so, we using the estimates of Lemma 1.2 we obtain

$$\|L(U)(t)\|_{L^r} \leq C\big(\sup_{t\in\mathbb{R}^+} \|U(t)\|_{L^p}\big)\|U_0\|_{L^{p_c}}^{\alpha-1} \int_0^t (t-\tau)^{-\delta(r)-\xi(q)}\,\tau^{-\theta(q)}\,d\tau\,,$$

where $\theta(q)$ is given by (48). If (50) holds then one can choose q, q_1 and q_2 such that $\theta(q) < 1$, $\delta(r) + \xi(q) < 1$ and $\xi(q) + \theta(q) = 1$, and so,

$$\|L(U)(t)\|_{L^r} \leq C\, t^{-\delta(r)}\,\big(\sup_{t\in\mathbb{R}^+} \|U(t)\|_{L^p}\big)\|U_0\|_{L^{p_c}}^{\alpha-1}\,.$$

Then, since $\|U(t)\|_{L^p} \leq C\,\|U_0\|_{L^p}$ (by step two),

$$\|L(U)(t)\|_{L^r} \leq C\, t^{-\delta(r)}\,\|U_0\|_{L^{p_c}}^{\alpha-1}\,\|U_0\|_{L^p} \leq C\, t^{-\delta(r)}\,\|U_0\|_{L^p}\,,$$

which completes the proof.

Now, if (50) is not fulfilled, we build a sequence $\{r_i\}$ defined by

$$r_0 = p\,, \qquad \frac{n}{2}\Big(\frac{1}{r_i} - \frac{1}{r_{i+1}}\Big) = \delta < \max\Big\{\frac{(2-d)}{2}, \alpha^{-1}\Big\}\,.$$

And, if $p < r_1 < r_2 < p_c$, since $U(t,\cdot)$ is bounded in $L^p \cap L^{p_c}$, then $U(t,\cdot)$ is also bounded in L^r for all r in $[p, p_c]$ and for each $t \geq 0$.

Now let $t_0 > 0$ and let W be the solution of

$$\text{(51)} \qquad \begin{cases} W(t,x) = e^{t\Delta} V_0 + L(W)(t,x)\,,\\ W(0,x) = W_0(x) = U(t_0,x)\,. \end{cases}$$

We have already proved that

$$\text{(52)} \quad W_0 \in L^{p_c}(\mathbb{R}^n) \cap L^{r_1}(\mathbb{R}^n) \quad \text{with} \quad \|W_0\|_{L^{r_1}} \leq C\, t_0^{-\delta(r_1)}\,\|U_0\|_{L^{p_c}}$$

and furthermore $W(t,\cdot)$ is bounded in $L^{r_1}(\mathbb{R}^n) \cap L^{p_c}(\mathbb{R}^n)$. So we just have to iterate the previous proof to estimate $W(t_0,x) = U(2\,t_0,x)$ in $L^{r_2}(\mathbb{R}^n)$ norm with respect to $W_0(x) = U(t_0,x)$ in $L^{r_1}(\mathbb{R}^n)$ norm to obtain the required estimate and we can do this until $r_i \leq p_c$.

Now let us denote by I the first index such that $r_I > p_c$. We have proved that

$$\text{(53)} \qquad \begin{cases} \|U(t)\|_{L^{p_c}} \leq C\,(1+t)^{-\delta(p_c)}\,,\\ \|U(t)\|_{L^{r_I}} \leq C\, t^{-\delta(r_I)}\,. \end{cases}$$

and, to conclude, we just have to use the same proof as in Section 3.1 with the estimate (53) instead of the estimates (41). This end the proof of the Proposition.

3.3. Initial data in $H^s_p(\mathbb{R}^n)$.

We give now the proof of Theorem 1.5. Let us consider an initial data U_0 such that $\|U_0\|_{H^{s_c}_p} \leq A'$. Then, by the Sobolev embedding theorem, U_0 belongs to $L^{p_c}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ and, if A' is small enough, then $\|U_0\|_{L^{p_c}} \leq A$. So, according to Theorem 1.4, there exists a unique global solution $U(t,x)$ of (2) and this solution satisfies (19) and (20). Hence, to prove that U belongs to $BC(\mathbb{R}^+, H^{s_c}_p)$, we have only to check that U remains bounded in the homogeneous space $\dot{H}^{s_c}_p(\mathbb{R}^n)$ thanks to the following well know inequality

$$\|f\|_{H^s_p} \leq C\,(\|f\|_{L^p} + \|f\|_{\dot{H}^s_p})\,, \qquad \text{for all } s \geq 0\,.$$

Now since U is a solution of (2),

$$\begin{aligned}
\|U(t)\|_{\dot{H}^{s_c}_p} &\leq \|U_0\|_{\dot{H}^{s_c}_p} + \|L(U)(t)\|_{\dot{H}^{s_c}_p} \\
&\leq \|U_0\|_{H^{s_c}_p} + \int_0^t \|e^{(t-\tau)\Delta} P(D) F(U(\tau))\|_{\dot{H}^{s_c}_p}\, d\tau \\
&\leq \|U_0\|_{H^{s_c}_p} + C \int_0^t (t-\tau)^{-(s_c+d)/2}\, \|F(U(\tau))\|_{L^p}\, d\tau \qquad (54)\\
&\leq \|U_0\|_{H^{s_c}_p} + C \int_0^t (t-\tau)^{-\lambda(q)}\, \|F(U(\tau))\|_{L^{q/\alpha}}\, d\tau \\
&\leq \|U_0\|_{H^{s_c}_p} + C \int_0^t (t-\tau)^{-\lambda(q)}\, \|U(\tau)\|^{\alpha}_{L^q}\, d\tau\,,
\end{aligned}$$

where

$$\lambda(q) = \frac{s_c + d}{2} + \frac{n}{2}\Big(\frac{\alpha}{q} - \frac{1}{p}\Big)\,, \qquad q \in \,]p_c, p\,\alpha]\,. \tag{55}$$

and where, in the third inequality, we used the hypothesis of homogeneity on $P(D)$.

Since $p > p_c/\alpha$, one can check that $s_c < 2 - d$, and so taking $q \approx p\,\alpha$, one can always choose q such that $0 < \lambda(q) < 1$. Then, for this choice of q we obtain

$$\begin{aligned}\|U(t)\|_{\dot H_p^{s_c}} &\le \|U_0\|_{H_p^{s_c}} \\ &\qquad + C\big(\sup_{\mathbb{R}+} t^{\gamma(q)}\,\|U(t)\|_{L^q}\big)^\alpha \int_0^t (t-\tau)^{-\lambda(q)}\,\tau^{-\alpha\gamma(q)}\,d\tau \\ &\le \|U_0\|_{H_p^{s_c}} + C\big(\sup_{\mathbb{R}+} t^{\gamma(q)}\,\|U(t)\|_{L^q}\big)^\alpha\,,\end{aligned}$$

since $\lambda(q) + \alpha\,\gamma(q) = 1$. But, by Lemma 1.2, we know that $t^\gamma\|U(t)\|_{L^q}$ remains bounded for all $t \ge 0$ and so U belongs to $BC(\mathbb{R}^+, \dot H_p^{s_c})$. Thus we have proved that U belongs to $BC(\mathbb{R}^+, H_p^{s_c})$.

Now, let $U_0 \in H_p^s(\mathbb{R}^n)$ such that $\|U_0\|_{H_p^{s_c}} \le A'$. Then, according to Part a) of Theorem 1.3 and to Part a) of Theorem 1.5, there exists a unique solution of (2) in $C([0,T[\,, H_p^s) \cap BC(\mathbb{R}^+, H_p^{s_c})$ and so, to prove Part b) of Theorem 1.5, we must show that blow up in $H_p^s(\mathbb{R}^n)$ norm cannot occur. But, like in Part b) of Theorem 1.3, one can easily show that smoothing effects occur namely that

$$\|U(t) - e^{t\Delta}U_0\|_{H_p^{s_c+\theta}} \le C\,\|U(t)\|_{H_p^{s_c}}\ ,$$

this, as long as $\lambda(q) + \theta < 1$, where $\lambda(q)$ is given by (55). Hence, if blow up holds in $H_p^{s_c+\theta}(\mathbb{R}^n)$ norm, it holds also in $H_p^{s_c}(\mathbb{R}^n)$ norm: this contradicts Part a) of Theorem 1.5. Now, since $s > s_c$ is arbitrary, we have just to iterate this proof to obtain the required result.

4. Composition on $H_p^s(\mathbb{R}^n)$ spaces.

4.1. Introduction.

In this section we prove the nonlinear estimate

$$\|F(u)\|_{H_p^{s_\alpha}} \le C\,\|u\|_{H_p^s}^\alpha$$

that we used in a crucial way in the proof of Theorem 1.3 (our result about local existence and uniqueness for Equation (2)). First we are going to consider the case H2) (*i.e.* when $s_\alpha \le 0$). Then, after recalling a few results about Littlewood-Paley analysis, we will prove Theorem 1.3 when H3) is fulfilled ($0 < s_\alpha < \alpha - 1$).

4.2. The case $s_\alpha \leq 0$.

Here, we suppose that $\max\{0, n/p - n/\alpha\} < s$ and that H2) is fulfilled, *i.e.* that $s_\alpha \leq 0$ and that $|F(x)| \leq C\,|x|^\alpha$. Now consider $u(x) \in H_p^s(\mathbb{R}^n)$. Then, by the Sobolev embedding Theorem ($s \geq 0$ and $p \in]1, +\infty[$) we have

$$H_p^s(\mathbb{R}^n) \hookrightarrow L^{(1/p - s/n)^{-1}}(\mathbb{R}^n)\,. \tag{56}$$

Now, since $s > n/p - n/\alpha$ we have $(1/p - s/n)^{-1} > \alpha$ and, on the other hand, we have $|F(x)| \leq C|x|^\alpha$. Thus, by (56)

$$\|F(u)\|_{L^{(\alpha/p-(s\alpha)/n)^{-1}}} \leq C\,\|u\|^\alpha_{L^{(1/p-s/n)^{-1}}} \leq C\,\|u\|^\alpha_{H_p^s}\,. \tag{57}$$

Next, to conclude, we remark that

$$\frac{1}{p} - \frac{s_\alpha}{n} = \frac{\alpha}{p} - \frac{s\,\alpha}{n}$$

and then, since $s_\alpha \leq 0$ and $(\alpha/p - (s\alpha)/n)^{-1} > 1$, we can use the Sobolev embedding

$$L^{(\alpha/p-(s\alpha)/n)^{-1}}(\mathbb{R}^n) \hookrightarrow H_p^{s_\alpha}(\mathbb{R}^n)\,, \tag{58}$$

which, with the estimate (57), gives

$$\|F(u)\|_{H_p^{s_\alpha}} \leq C\,\|u\|^\alpha_{H_p^s}$$

as we claim.

4.3. Littlewood-Paley analysis.

Let us first recall the Littlewood-Paley dyadic decomposition for a tempered distribution. Let φ_{-1} be a non-negative radial test function such that $\widehat{\varphi_{-1}}(\xi) = 1$ for $|\xi| \leq 3/4$ and such that $\widehat{\varphi_{-1}}(\xi) = 0$ for $|\xi| \geq 1$.

Let $\varphi_j(x) = 2^{nj}\,\varphi_{-1}(2^j x)$ and let us consider the partial sum operators S_j associated with the φ_j and defined by

$$S_j(f)(x) = \varphi_j \star f(x)\,. \tag{59}$$

Now define $\psi_{-1}(x) = \varphi_{-1}(x)$ and $\psi_j(x) = \varphi_j(x) - \varphi_{j-1}(x)$ and, in the same way as previously, consider the operators Δ_j defined by

$$\Delta_j(f)(x) = S_j(f)(x) - S_{j-1}(f)(x) = \psi_j \star f(x)\,. \tag{60}$$

Thus,

$$f = \lim_{j\to\infty} S_j(f) = \Delta_{-1}(f) + \sum_{j=0}^{\infty} \Delta_j(f)\;. \tag{61}$$

More precisely one can prove the following result (see [Tr]).

Proposition 4.1. *The convergence in* (61) *occurs in* $H^s_p(\mathbb{R}^n)$ *for all* p *in* $]1,\infty[$ *and for all* s *in* $\mathbb{R}$. *Furthermore, for all* f *in* $H^s_p(\mathbb{R}^n)$,

$$\|f\|_{H^s_p} \sim \|\Delta_{-1}(f)\|_{L^p} + \Big\| \Big(\sum_{j=0}^{\infty} 4^{js}\,|\Delta_j(f)|^2 \Big)^{1/2} \Big\|_{L^p}\;.$$

Now we give some classical Lemmas which will be of great use in the sequel.

Lemma 4.1 (Bernstein's inequalities)**.** *Let* $p \in [1,\infty]$.

a) *If* f *has its spectrum in the ball* $B(0,r)$ *then there exists a constant* C *independent of* f *and* r *such that*

$$\|\Lambda_s f\|_{L^p} \le C\,r^s\,\|f\|_{L^p}\;, \qquad \textit{for all } s > 0\,.$$

b) *If* f *has its spectrum in the ring* $C(0,Ar,Br) = \{\xi : A\,r \le |\xi| \le B\,r\}$ *then there exists some constants* C_1 *and* C_2 *independent of* f *and* r *such that*

$$C_1\,r^s\,\|f\|_{L^p} \le \|\Lambda_s f\|_{L^p} \le C_2\,r^s\,\|f\|_{L^p}\;, \qquad \textit{for all } s > 0\,.$$

For a proof see [AG]. The second Lemma describes the behavior of $S_j(u)$ and $\Delta_j(u)$ in $L^\infty(\mathbb{R}^n)$ norm when u belongs to $H^s_p(\mathbb{R}^n)$ spaces.

Lemma 4.2. *Let*

$$s_n = s - \frac{n}{p}\,. \tag{62}$$

Then,

a) *For all s in $\mathbb{R}$, $\|\Delta_k(u)\|_{L^\infty} \le C\,2^{-ks_n}\,\|u\|_{H^s_p}$.*

b) *If $s < n/p$ then, $\|S_k(u)\|_{L^\infty} \le C\,2^{-ks_n}\,\|u\|_{H^s_p}$.*

The proof is left to the reader (hint: use Bernstein's inequalities).

Lemma 4.3. *Let $\{f_k\}_{k=0}^{\infty}$ be a sequence of functions in $\mathcal{S}'(\mathbb{R}^n)$ such that*

$$\operatorname{supp}(\hat{f}_k) \subset B(0, C\,2^k)\,.$$

Then there exists a constant C such that

$$\Big\|\Big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}|\Delta_j(f_k)|^2\Big)^{1/2}\Big\|_{L^p} \le C\Big\|\Big(\sum_{k=0}^{\infty}|f_k|^2\Big)^{1/2}\Big\|_{L^p}\,.$$

For a proof see [Me].

4.4. The paracomposition formula.

To prove Theorem 1.2 we use the paracomposition technique (see [Me], [Ta], [AG], [Co], ...) which generalizes the paraproduct technique introduced by J. M. Bony. We rewrite $F(u)$ as the serie

$$\begin{aligned} F(u) = F(S_0(u)) &+ (F(S_1(u)) - F(S_0(u))) + \cdots \\ &+ (F(S_{k+1}(u)) - F(S_k(u))) + \cdots \end{aligned}$$

and since F is C^1 at least

$$F(u) = F(S_0(u)) + \sum_{k=0}^{\infty}\Delta_k(u)\,m_k(u)\,, \tag{63}$$

where

$$m_k(u) = \int_0^1 F'(S_k(u) + t\,\Delta_k(u))\,dt\,. \tag{64}$$

To relocate the $m_k(u)$ spectrums we introduce a second Littlewood-Paley's partition of unity

$$\widehat{\varphi_{-1}}\Big(\frac{\xi}{A\,2^k}\Big) + \sum_{p=0}^{\infty} \widehat{\psi}\Big(\frac{\xi}{A\,2^{k+p}}\Big) = 1$$

and so,

$$m_k(u) = m_{k,-1}(u) + \sum_{p=0}^{\infty} m_{k,p}(u)\,, \tag{65}$$

where

$$\begin{cases} m_{k,-1}(u) = \mathcal{F}^{-1}\Big(\hat{\varphi}_{-1}\Big(\frac{\xi}{A\,2^k}\Big)\Big) \star m_k(u)\,, \\ m_{k,p}(u) = \mathcal{F}^{-1}\Big(\hat{\psi}_{-1}\Big(\frac{\xi}{A\,2^k}\Big)\Big) \star m_k(u)\,. \end{cases} \tag{66}$$

So, by (63) and (66),

$$F(u) = F(S_0(u)) + \sum_{k=0}^{\infty} \Delta_k(u)\,m_{k,-1}(u) + \sum_{k,p=0}^{\infty} \Delta_k(u)\,m_{k,p}(u) \tag{67}$$

and we want to prove that each of those terms belongs to $H_p^{s_\alpha}(\mathbb{R}^n)$ where $s_\alpha > 0$ is given by (9).

For the term $F(S_0(u))$ we refer to [Co] (one uses bounds on the maximal function of $F(S_0(U))$ to get the proof).

4.4.1. The series $\sum_{k=0}^{\infty} \Delta_k(u)\,m_{k,-1}(u)$ belongs to $H_p^{s_\alpha}(\mathbb{R}^n)$.

We begin with the following Lemma.

Lemma 4.4. *Under* H3)*,*

$$\|m_{k,-1}(u)\|_{L^\infty} \le C\,2^{-ks_n(\alpha-1)}\,\|u\|_{H_p^s}^{\alpha-1}\,, \qquad \textit{for all } k \in \mathbb{N}\,.$$

Lemma 4.4 follows from Lemma 4.2 since

$$\begin{aligned}\|m_{k,-1}(u)\|_{L^\infty} &= \|\mathcal{F}^{-1}(\hat{\varphi}_{-1}(\xi\, A^{-1}\, 2^{-k})) \star m_k(u)\|_{L^\infty}\\ &\leq \|\mathcal{F}^{-1}(\hat{\varphi}_{-1}(\xi\, A^{-1}\, 2^{-k}))\|_{L^1}\, \|m_k(u)\|_{L^\infty}\\ &\leq C\, \|m_k(u)\|_{L^\infty}\ .\end{aligned}$$

Now $|F'(x)| \leq C|x|^{\alpha-1}$ and so

$$|m_k(u)| \leq \int_0^1 C\, |S_k(u) + t\, \Delta_k(u)|^{\alpha-1}\, dt$$

and the estimates of Lemma 4.2 for $S_k(u)$ and $\Delta_k(u)$ in $L^\infty(\mathbb{R}^n)$ norm lead to the proof.

To prove that the series belongs to $H_p^{s_\alpha}(\mathbb{R}^n)$ by Proposition 4.1 it is then sufficient to show that the function

$$\sigma(x) = \Big(\sum_{j=0}^{\infty} 4^{j s_\alpha} \Big|\Delta_j\Big(\sum_{k=0}^{\infty} \Delta_k(u) m_{k,-1}(u)\Big)\Big|^2\Big)^{1/2}$$

belongs to $L^p(\mathbb{R}^n)$. By construction the $m_{k,-1}(u)$ spectrums are in the balls $B(0, A\,2^k)$ and the $\Delta_k(u)$ spectrums are in the rings $C(0, 2^{-1}A\,2^k, 2\,A\,2^k)$. Taking $A = 50$ (for instance) then the $m_{k,-1}(u)\Delta_k(u)$ spectrums are in some extended balls $B(0, A'2^k)$ and so, there exists an integer N such that $\Delta_j(m_{k,-1}(u)\Delta_k(u)) = 0$ for $j > k+N$ since the spectrums of φ_j and $\Delta_k(u)\, m_{k,-1}(u)$ are disjointed. So,

$$\begin{aligned}&\Big|\Delta_j\Big(\sum_{k=0}^{\infty} \Delta_k(u)\, m_{k,-1}(u)\Big)\Big|^2\\ &\qquad = \Big|\Delta_j\Big(\sum_{k=j+N}^{\infty} \Delta_k(u)\, m_{k,-1}(u)\Big)\Big|^2\\ &\qquad \leq C\, 4^{-j s_\alpha}\Big(\sum_{k=j+N}^{\infty} 4^{k s_\alpha}\, |\Delta_j(\Delta_k(u)\, m_{k,-1}(u))|^2\Big)\end{aligned}$$

by Cauchy-Schwartz inequality applied to the sequences

$$\{2^{-k s_\alpha}\, \mathbf{1}_{k\geq j+N}\} \qquad \text{and} \qquad \{2^{k s_\alpha}\Delta_k(u)\, m_{k,-1}(u)\, \mathbf{1}_{k\geq j+N}\}$$

(note that $s_\alpha > 0$ is needed). Then by definition of $\sigma(x)$ we get

$$\sigma(x) \le C\Big(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\Delta_j(2^{ks_\alpha}\Delta_k(u)\, m_{k,-1}(u))|^2\Big)^{1/2}$$

and, Lemma 4.3 applied to the sequence $\{2^{ks_\alpha}\Delta_k(u)\, m_{k,-1}(u)\}$ leads to

$$\|\sigma(x)\|_{L^p} \le \Big\|\Big(\sum_{k=0}^{\infty} 4^{ks_\alpha} |\Delta_k(u)\, m_{k,-1}(u)|^2\Big)^{1/2}\Big\|_{L^p}.$$

Now, using Lemma 4.4

$$\begin{aligned}|\Delta_k(u)\, m_{k,-1}(u)|^2 &\le \|m_{k,-1}(u)\|_{L^\infty}^2 \, |\Delta_k(u)|^2\\ &\le C\, 4^{-ks_n(\alpha-1)}\, \|u\|_{H_p^s}^{2(\alpha-1)}\, |\Delta_k(u)|^2\end{aligned}$$

and so,

$$\|\sigma(x)\|_{L^p} \le C\, \|u\|_{H_p^s}^{\alpha-1} \Big\|\Big(\sum_{k=0}^{\infty} 4^{k(s_\alpha - s_n(\alpha-1))}\, |\Delta_k(u)|^2\Big)^{1/2}\Big\|_{L^p}.$$

But, $s = s_\alpha - s_n(\alpha-1)$ and so

$$\|\sigma(x)\|_{L^p} \le C\, \|u\|_{H_p^s}^{\alpha-1} \Big\|\Big(\sum_{k=0}^{\infty} 4^{ks}\, |\Delta_k(u)|^2\Big)^{1/2}\Big\|_{L^p} \le C\, \|u\|_{H_p^s}^{\alpha}.$$

Thus the series belongs to $H_p^{s_\alpha}(\mathbb{R}^n)$ and its norm is bounded by $C\,\|u\|_{H_p^s}^{\alpha}$.

4.4.2. The series $\sum_{k=0}^{\infty}\big(\sum_{p=0}^{\infty}\Delta_k(u)\, m_{k,p}(u)\big)$ belongs to $H_p^{s_\alpha}(\mathbb{R}^n)$.

For fixed $p \ge 0$ we define

$$l_p(x) = \sum_{k=0}^{\infty} \Delta_k(u)\, m_{k,p}(u).$$

Taking the constant A large enough one can check that the $\Delta_k(u)\, m_{k,p}(u)$ spectrums are in some rings $\{\xi : C_1\, 2^{p+k} \le |\xi| \le C_2\, 2^{p+k}\}$. So, there exists an integer K (which does not depend of p) such that those rings are K to K disjointed. So we can use the Littlewood-Paley analysis on the K partial sums $l_p^r(x)$ defined by

$$l_p^r(x) = \sum_{k=r \bmod(K)} \Delta_k(u)\, m_{k,p}(u) \qquad \text{with } r \in \{0,\dots,K-1\} \tag{68}$$

and, by Proposition 4.1, we know that for all r in $\{0,\dots,K-1\}$,

$$\|l_p^r\|_{H_p^{s_\alpha}} \le C \Big\| \Big(\sum_{k=r \bmod(K)} 4^{(k+p)s_\alpha} \, |\Delta_k(u)\, m_{k,p}(u)|^2 \Big)^{1/2} \Big\|_{L^p} .$$

Let us assume that the following Lemma holds.

Lemma 4.5. *Under* H3),

$$\|m_{k,p}(u)\|_{L^\infty} \le C\, 2^{-(\alpha-1)p}\, 2^{-k(\alpha-1)s_n}\, \|u\|_{H_p^s}^{\alpha-1} , \qquad \textit{for all } k \in \mathbb{N} .$$

Then by Lemma 4.5,

$$\begin{aligned} \|l_p^r\|_{H_p^{s_\alpha}} &\le C\, 2^{p(s_\alpha-(\alpha-1))}\, \|u\|_{H_p^s}^{\alpha-1} \\ &\quad \cdot \Big\| \Big(\sum_{k=r \bmod(K)} 4^{k(s_\alpha - s_n(\alpha-1))}\, |\Delta_k(u)|^2 \Big)^{1/2} \Big\|_{L^p} \\ &\le C\, 2^{p(s_\alpha-(\alpha-1))}\, \|u\|_{H_p^s}^{\alpha-1} \Big\| \Big(\sum_{k=r \bmod(K)} 4^{ks}\, |\Delta_k(u)|^2 \Big)^{1/2} \Big\|_{L^p} \\ &\le C\, 2^{p(s_\alpha-(\alpha-1))}\, \|u\|_{H_p^s}^{\alpha} . \end{aligned}$$

Thus, for $s_\alpha < \alpha - 1$, the K series $\{l_p^r\}_{p\in\mathbb{N}}$ are uniformly convergent in $H_p^{s_\alpha}(\mathbb{R}^n)$ and furthermore, for $r \in \{0,\dots,K-1\}$,

$$\sum_p \|l_p^r\|_{H_p^{s_\alpha}} \le C\, \|u\|_{H_p^s}^{\alpha} ,$$

which ends the proof of Theorem 1.2.

So, to conclude, we have just to prove Lemma 4.5. Let us define

$$\theta = \alpha - 1 = N + \nu , \qquad \text{where } N = [\theta] \text{ and } \nu \in [0,1[. \tag{69}$$

and

$$P_k^t(x) = S_k(x) + t\,\Delta_k(x)\,. \tag{70}$$

By Lemma 4.1 applied with $p=\infty$,

$$\|m_{k,p}(u)\|_{L^\infty} \le C\,2^{-(k+p)\theta}\,\|m_k(u)\|_{C^\theta}\ , \tag{71}$$

where $C^\theta(\mathbb{R}^n)$ denotes the Hölder space of order θ endowed with the norm

$$\|h\|_{C^\theta} = \|h\|_{L^\infty} + \cdots + \|D^\theta h\|_{L^\infty}\ , \qquad \text{if } \theta \in \mathbb{N}, \tag{72}$$

and

$$\|h\|_{C^\theta} = \|h\|_{C^N} + \sup_{|x-y|<1} \frac{|D^N h(x) - D^N h(y)|}{|x-y|^\nu}\ , \qquad \text{if } \theta \notin \mathbb{N} \tag{73}$$

(for more details see [Tr] for instance).

So, by (71),

$$\begin{aligned}\|m_{k,p}(u)\|_{L^\infty} \le C\,2^{-(k+p)\theta}\Big(&\|m_k(u)\|_{L^\infty} + \cdots + \|D^N m_k(u)\|_{L^\infty}\\ &+ \sup_{|x-y|<1} \frac{|D^N m_k(u)(x) - D^N m_k(u)(y)|}{|x-y|^\nu}\Big)\,.\end{aligned} \tag{74}$$

The bound of $\|m_k(u)\|_{L^\infty}$ is easy to establish: we have just to argue as in the proof of Lemma 4.4 to get

$$\|m_k(u)\|_{L^\infty} \le C\,2^{-ks_n(\alpha-1)}\,\|u\|_{H_p^s}^{\alpha-1}\,. \tag{75}$$

Next we must bound $\|D^j m_k(u)\|_{L^\infty}$ for $j \in \{1,\dots,N\}$. Let γ be a multi-index such that $\gamma = \gamma_1 + \cdots + \gamma_n$ with total length $|\gamma| = |\gamma_1| + \cdots + |\gamma_n| = j$ then,

$$\partial^\gamma m_k(x) = \int_0^1 \sum_{q=1}^{j} \sum_{\gamma_1+\cdots+\gamma_q=\gamma} D^{q+1}F(P_k^t(x))\,\partial^{\gamma_1}P_k^t(x)\cdots\partial^{\gamma_q}P_k^t(x)\,dt\,,$$

where the second sum is taken on all the decompositions of $\gamma = \gamma_1 + \cdots + \gamma_q$. By Lemma 4.1 and Lemma 4.2,

$$\|\partial^{\gamma_i}P_k^t(x)\|_{L^\infty} \le C\,2^{|\gamma_i|k}\,\|P_k^t(x)\|_{L^\infty} \le C\,2^{|\gamma_i|k}\,2^{-ks_n}\,\|u\|_{H_p^s}$$

and so,

$$\begin{aligned}\|\partial^{\gamma_1} P_k^t(x) \cdots \partial^{\gamma_q} P_k^t(x)\|_{L^\infty} &\\ &\le C\,(2^{|\gamma_1|k}\, 2^{-ks_n}\, \|u\|_{H_p^s}) \cdots (2^{|\gamma_q|k}\, 2^{-ks_n}\, \|u\|_{H_p^s})\\ &\le C\, 2^{k(|\gamma_1|+\cdots+|\gamma_q|)}\, 2^{-kqs_n}\, \|u\|^q_{H_p^s}\,.\end{aligned}$$

Furthermore by Lemma 1.1 and Lemma 4.2

$$\begin{aligned}\|D^{q+1}F(P_k^t)(x)\|_{L^\infty} &\le C\, \|S_k(u) + t\,\Delta_k(u)\|^{\alpha-q-1}_{L^\infty}\\ &\le C\, 2^{-ks_n(\alpha-1-q)}\, \|u\|^{\alpha-1-q}_{H_p^s}\,.\end{aligned}$$

And so, for fixed q,

$$\begin{aligned}\Big\| \int_0^1 \sum_{\gamma_1+\cdots+\gamma_q=\gamma} D^{q+1}F(P_k^t(x))\, \partial^{\gamma_1} P_k^t(x) \cdots \partial^{\gamma_q} P_k^t(x)\, dt \Big\|_{L^\infty} &\\ \le C\, 2^{-ks_n(\alpha-1-q)}\, \|u\|^{\alpha-1-q}_{H_p^s}\, 2^{k(|\gamma_1|+\cdots+|\gamma_q|)}\, 2^{-kqs_n}\, \|u\|^q_{H_p^s} &\\ \le C\, 2^{-ks_n(\alpha-1)}\, 2^{jk}\, \|u\|^{(\alpha-1)}_{H_p^s}\,. &\end{aligned}$$

Thus, for $j \in \{1, \ldots, N\}$,

$$\|D^j m_k(u)\|_{L^\infty} \le C\, 2^{-ks_n(\alpha-1)}\, 2^{kj}\, \|u\|^{(\alpha-1)}_{H_p^s}\,. \tag{76}$$

To conclude we must estimate

$$\sup_{|x-y|<1} \frac{|D^{[N]}m_k(u)(x) - D^{[N]}m_k(u)(y)|}{|x-y|^\nu}\,.$$

Let γ be a multi-index of length N. Then,

$$\partial^\gamma m_k(x) = \int_0^1 \sum_{q=1}^N \sum_{\gamma_1+\cdots+\gamma_q=\gamma} D^{q+1}F(P_k^t(x))\, \partial^{\gamma_1} P_k^t(x) \cdots \partial^{\gamma_q} P_k^t(x)\, dt$$

and so, $\partial^\gamma m_k(x) - \partial^\gamma m_k(y) = I(x,y) + J(x,y)$ where

$$\begin{aligned}I(x,y) = \int_0^1 \sum_{q=1}^N \sum_{\gamma_1+\cdots+\gamma_q=\gamma} &(D^{q+1}F(P_k^t(x)) - D^{q+1}F(P_k^t(y)))\\ &\cdot \prod_{\gamma_i} \partial^{\gamma_i} P_k^t(x)\, dt\end{aligned}$$

and,

$$J(x,y) = \int_0^1 \sum_{q=1}^{N} \sum_{\gamma_1+\cdots+\gamma_q=\gamma} D^{q+1}F(P_k^t(y)) \cdot \Big(\prod_{\gamma_i} \partial^{\gamma_i} P_k^t(x) - \prod_{\gamma_i} \partial^{\gamma_i} P_k^t(y) \Big)\, dt\,.$$

We deal first with the term $I(x,y)$. For $q \in \{1,\ldots,N\}$ and $\{\gamma_i\}_{i=1,\ldots,q}$ a decomposition of γ we must estimate

$$I_k^q = \sup_{|x-y|<1} \Big\{ \frac{1}{|x-y|^\nu} \Big| \int_0^1 \sum_{\gamma_1+\cdots+\gamma_q=\gamma} (D^q F'(P_k^t(x)) - D^q F'(P_k^t(y))) \cdot \prod_{\gamma_i} \partial^{\gamma_i} P_k^t(x)\, dt \Big| \Big\}\,.$$

First suppose that $q \le N-1$. Then, by Lemmas 4.1 and 4.2

$$\Big\| \prod_{i=1}^{q} \partial^{\gamma_i} P_k^t(x) \Big\|_{L^\infty} \le C\, 2^{kN}\, 2^{-kqs_n}\, \|u\|_{H_p^s}^q\,. \tag{77}$$

Next we must bound

$$\sup_{|x-y|<1} \frac{|D^{q+1}F(P_k^t)(x) - D^{q+1}F(P_k^t)(y)|}{|x-y|^\nu}\,.$$

But, by Lemma 1.1, for $q \le N-1$,

$$|D^{q+1}F(x) - D^{q+1}F(y)| \le C\,|x-y|\,(|x|^{\alpha-q-2} + |y|^{\alpha-q-2})$$

and so

$$|D^{q+1}F(P_k^t)(x) - D^{q+1}F(P_k^t)(y)| \le C\,|P_k^t(x) - P_k^t(y)|\,(|P_k^t(x)|^{\alpha-q-2} + |P_k^t(y)|^{\alpha-q-2})\,.$$

But, by definition of the $C^\nu(\mathbb{R})$ norm,

$$\begin{aligned}\sup_{|x-y|<1} \frac{|P_k^t(x) - P_k^t(y)|}{|x-y|^\nu} &\le C\,\|P_k^t\|_{C^\nu} \\ &\le C\,2^{k\nu}\,\|P_k^t\|_{L^\infty} \\ &\le C\,2^{k\nu}\,2^{-ks_n}\,\|u\|_{H_p^s}\end{aligned}$$

by Lemmas 4.1 and 4.2. Thus, for all $x, y \in \mathbb{R}^n$ with $|x-y|<1$,

$$|P_k^t(x) - P_k^t(y)| \leq C\,|x-y|^\nu\, 2^{k\nu}\, 2^{-ks_n}\, \|u\|_{H_p^s}$$

and so

$$\sup_{|x-y|<1} \frac{|D^{q+1}F(P_k^t)(x) - D^{q+1}F(P_k^t)(y)|}{|x-y|^\nu} \leq C\, 2^{k\nu}\, 2^{-ks_n}\, \|u\|_{H_p^s}\, \|P_k^t\|_{L^\infty}^{\alpha-q-2}$$

and so, by Lemma 4.2, for all q in $\{0, \ldots, N-1\}$,

$$\sup_{|x-y|<1} \frac{|D^{q+1}F(P_k^t)(x) - D^{q+1}F(P_k^t)(y)|}{|x-y|^\nu} \leq C\, 2^{k\nu}\, 2^{-ks_n(\alpha-q-1)}\, \|u\|_{H_p^s}^{\alpha-q-1}\,. \tag{78}$$

Then, from (69), (77) and (78), we deduce that for all q in $\{0, \ldots, N-1\}$,

$$I_k^q \leq C\, 2^{k(\alpha-1)}\, 2^{-ks_n(\alpha-1)}\, \|u\|_{H_p^s}^{\alpha-1}\,. \tag{79}$$

Now we deal with the terms I_k^N. By lemmas 4.1 and 4.2,

$$\Big\| \prod_{i=1}^{N} \partial^{\gamma_i} P_k^t(x) \Big\|_{L^\infty} \leq C\, 2^{Nk}\, 2^{-kNs_n}\, \|u\|_{H_p^s}^N\,. \tag{80}$$

Now by H3)

$$|D^N F'(x) - D^N F'(y)| \leq C\,|x-y|^\nu$$

and so

$$\begin{aligned}\sup_{t\in[0,1]} |D^N F'(P_k^t(x)) - D^N F'(P_k^t(y))| &\leq C \sup_{t\in[0,1]} |P_k^t(x) - P_k^t(y)|^\nu \\ &\leq C \sup_{t\in[0,1]} |x-y|^\nu\, \|\nabla P_t^k\|_{L^\infty}^\nu \\ &\leq C\,|x-y|^\nu\, (2^k\, 2^{-ks_n}\, \|u\|_{H_p^s})^\nu\,,\end{aligned}$$

by Lemma 4.1 and Lemma 4.2. Combining these inequalities we get

$$\sup_{|x-y|<1} \frac{|P_k^t(x) - P_k^t(y)|}{|x-y|^\nu} \leq C\,(2^k\, 2^{-ks_n}\, \|u\|_{H_p^s})^\nu\,. \tag{81}$$

Then, by (80) and (81)

$$I_k^N \le C\,(2^k\,2^{-ks_n}\,\|u\|_{H_p^s})^{N+\nu}\,,$$

which leads to

$$I_k^N \le C\,2^{-ks_n(\alpha-1)}\,2^{k(\alpha-1)}\,\|u\|^{\alpha-1}_{H_p^s}\,. \tag{82}$$

Now by (79) and (82),

$$\sup_{|x-y|<1}\frac{|I(x,y)|}{|x-y|^\nu} \le C\,2^{-ks_n(\alpha-1)}\,2^{k(\alpha-1)}\,\|u\|^{\alpha-1}_{H_p^s}\,. \tag{83}$$

Now we deal with the term J. It can be rewritten as

$$\begin{aligned} J(x,y) = \sum_{q=1}^{N}\sum_{\gamma_1+\cdots+\gamma_q=\gamma}\sum_{j=1}^{q}\int_0^1 & D^{q+1}F(P_k^t(y))\,\partial^{\gamma_j}(P_k^t(x)-P_k^t(y)) \\ & \cdot\Big(\prod_{i>j}\partial^{\gamma_i}P_k^t(x)\Big)\Big(\prod_{i<j}\partial^{\gamma_i}P_k^t(y)\Big)\,dt\,. \end{aligned}$$

and we denote by $J_{q,\gamma_i,j}$ each term of the sum and we estimate them for all fixed triplet (q,γ_i,j). As previously, by Lemma 1.1

$$\|D^{q+1}F(P_k^t(y))\|_{L^\infty} \le C\,2^{-ks_n(\alpha-q-1)}\,\|u\|^{\alpha-q-1}_{H_p^s}\,. \tag{84}$$

Now, let $i\neq j$, then by Lemmas 4.1 and 4.2,

$$\|\partial_x^{\gamma_i}P_k^t(x)\| \le C\,2^{|\gamma_i|k}\,\|P_k^t(x)\|_{L^\infty} \le C\,2^{|\gamma_i|k}\,2^{-ks_n}\,\|u\|_{H_p^s}$$

and so

$$\begin{aligned} &\Big\|\prod_{i>j}\partial^{\gamma_i}P_k^t(x)\prod_{i<j}\partial^{\gamma_i}P_k^t(y)\Big\|_{L^\infty} \\ &\qquad\le C\,2^{k(\sum_{i\neq j}|\gamma_i|)}\,2^{-k(q-1)s_n}\,\|u\|^{q-1}_{H_p^s}\,. \end{aligned} \tag{85}$$

By definition of the $C^s(\mathbb{R})$ norm

$$\begin{aligned} \sup_{|x-y|<1}\frac{\partial^{\gamma_j}(P_k^t(x)-P_k^t(y))}{|x-y|^\nu} &\le \|P_k^t\|_{C^{|\gamma_j|+\nu}} \\ &\le C\,2^{k(|\gamma_j|+\nu)}\,\|P_k^t\|_{L^\infty} \\ &\le C\,2^{k(|\gamma_j|+\nu)}\,2^{-ks_n}\,\|u\|_{H_p^s}\,. \end{aligned}$$

And so,

$$\sup_{|x-y|<1} \frac{\partial^{\gamma_j}(P_k^t(x) - P_k^t(y))}{|x-y|^\nu} \le C\, 2^{k(|\gamma_j|+\nu)}\, 2^{-ks_n}\, \|u\|_{H_p^s} \,. \tag{86}$$

Then by (84), (85) and (86) we get

$$\begin{aligned}\sup_{|x-y|<1} \frac{|J(x,y)|}{|x-y|^\nu} &\le C\, 2^{-ks_n(\alpha-q-1)}\, \|u\|_{H_p^s}^{\alpha-q-1}\, 2^{k(|\gamma_j|+\nu)}\, 2^{-ks_n}\, \|u\|_{H_p^s} \\ &\quad \cdot 2^{k(\sum_{i\ne j}|\gamma_i|)}\, 2^{-k(q-1)s_n}\, \|u\|_{H_p^s}^{q-1} \\ &\le C\, 2^{k(N+\nu)}\, 2^{-ks_n(\alpha-1)}\, \|u\|_{H_p^s}^{\alpha-1}\end{aligned}$$

and so, since $N + \nu = \alpha - 1$,

$$\sup_{|x-y|<1} \frac{|J(x,y)|}{|x-y|^\nu} \le C\, 2^{k(\alpha-1)}\, 2^{-ks_n(\alpha-1)}\, \|u\|_{H_p^s}^{\alpha-1} \,. \tag{87}$$

Thus by (83) and (87)

$$\sup_{|x-y|<1} \left| \frac{D^{[N]}m_k(x) - D^{[N]}m_k(y)}{(x-y)^\nu} \right| \le 2^{-ks_n(\alpha-1)}\, 2^{k(\alpha-1)}\, \|u\|_{H_p^s}^{\alpha-1} \,. \tag{88}$$

Now by (69), (71), (75), (76) and (88) we see that

$$\begin{aligned}&\|m_{k,p}\|_{L^\infty} \\ &\quad \le C\, 2^{-(k+p)\theta} \Big(\sum_{j=0}^{N} 2^{-ks_n(\alpha-1)}\, 2^{jk} + 2^{-ks_n(\alpha-1)}\, 2^{k(\alpha-1)} \Big) \|u\|_{H_p^s}^{\alpha-1} \\ &\quad \le C\, 2^{-p(\alpha-1)}\, 2^{-ks_n(\alpha-1)}\, \|u\|_{H_p^s}^{\alpha-1} \Big(\sum_{j=0}^{N} 2^{k(j-(\alpha-1))} + 1 \Big) .\end{aligned}$$

And for $j \in \{0, \dots, N\}$, $j - \alpha + 1 \le 0$ from which we deduce that

$$\|m_{k,p}(u)\|_{L^\infty} \le C\, 2^{-ks_n(\alpha-1)}\, 2^{-p(\alpha-1)}\, \|u\|_{H_p^s}^{\alpha-1} \,,$$

which ends the proof of Lemma 4.5.

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