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A note on eigenvalues of ordinary differential operators

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In this follow-up on the work of [FS] an improved condition for the discrete eigenvalues of the operator $-d^2/dx^2 + V(x)$ is established for $V(x)$ satisfying certain hypotheses. The eigenvalue condition in [FS] establishes eigenvalues of this operator to within a small error. Through an observation due to C. Fefferman, the order of accuracy can be improved if a certain condition is true. This paper improves on the result obtained in [FS] by showing that this condition does indeed hold.

The theorem proven here relies on a version of WKB theory developed in [FS] and applies to operators with large slowly varying potentials. For example, it applies to potentials of the form $V(x) = \lambda^2 V_1(x)$ for fixed, smooth V_1, with $V'' > 0$, V having a local minimum, and $\lambda \gg 1$. The theorem applies to more general potentials as well.

Standard WKB theory yields the statement that all eigenvalues E of the differential operator $-d^2/dx^2 + V(x)$ satisfy

$$(1) \quad \int_{x_{\mathrm{left}}}^{x_{\mathrm{right}}} (E - V(x))^{1/2}\, dx = \pi \Big(k + \frac{1}{2}\Big) + O\left(\lambda^{-1}\right), \quad \text{for some } k \in \mathbb{Z},$$

where x_{left} and x_{right} are the two solutions of $E - V(x) = 0$.

[FS] shows that this condition for eigenvalues can be improved so that given $N > 0$, there exists $N' > 0$ and complex functions $h_l(E)$ defined in [FS] so that (1) becomes

$$\int_{x_{\mathrm{left}}}^{x_{\mathrm{right}}} (E - V(x))^{1/2}\, dx + \operatorname{Im} \log \Big(1 + \sum_{l=1}^{N'} h_l(E)\Big)$$

$$= \pi\Big(k + \frac{1}{2}\Big) + O\left(\Lambda^{-N}\right), \tag{2}$$

where Λ, which will be defined precisely in the theorem, plays a role analogous to λ. h_1 is explicitly given in [FS] and is purely imaginary. For the moment however, the critical property of h_l is that $h_l(E) = O(\Lambda^{-l})$, and the quantity $\sum h_l(E)$ is $O(\Lambda^{-1})$ in absolute value, and hence the Taylor series of log gives

$$\int_{x_{\text{left}}}^{x_{\text{right}}} (E - V(x))^{1/2}\, dx + i\, h_1(E) = \pi\Big(k + \frac{1}{2}\Big) + O\left(\Lambda^{-2}\right). \tag{3}$$

But if we were to carry out the same calculation to order Λ^{-3}, then since

$$\log\Big(1 + \sum_{l=1}^{N'} h_l(E)\Big) = h_1(E) + h_2(E) - \frac{1}{2}\, h_1^2(E) + O\left(\Lambda^{-3}\right), \tag{4}$$

we have

$$\begin{aligned}\int_{x_{\text{left}}}^{x_{\text{right}}} (E - V(x))^{1/2}\, dx + \operatorname{Im}\left(h_1(E) + h_2(E)\right)& \\ = \pi\Big(k + \frac{1}{2}\Big) + O\left(\Lambda^{-3}\right).&\end{aligned} \tag{5}$$

Note h_1^2 is real and therefore makes no contribution to the left-hand side of (5). Moreover, we shall show that h_k is purely imaginary whenever k is odd and real whenever k is even. This reduces the left-hand side of (5) to the simpler left-hand side of (3). This improves upon (3) since (5) holds to $O\left(\Lambda^{-3}\right)$ instead of $O\left(\Lambda^{-2}\right)$. Using the above fact we obtain an improved version of part of the WKB Eigenvalue Theorem. (*cf.* [FS, p. 239]). For the reader's convenience and for completeness we repeat the hypotheses here.

Theorem. *Suppose we are given positive functions $S(x)$ and $B(x)$ on I and a potential $V(x)$ supported on a possibly unbounded interval I_{BVP} with $I \subset I_{\text{BVP}}$. Furthermore, suppose we are given two real numbers $E_0 \le E_\infty$, positive numbers $\varepsilon < 1/100$, $K > 1$ and $N > K\varepsilon^{-10}$. Define $N' = [\varepsilon N/500]$ and $N'' = 3\,\varepsilon\, N'/2 - K - 33$. And suppose we have the following hypotheses*:

Hyp0) *If $x, y \in I$ and $|x-y| < c\,B(x)$, then*

$$c < \frac{B(y)}{B(x)} < C \qquad \text{and} \qquad c < \frac{S(y)}{S(x)} < C\,.$$

Hyp1) *For $x \in I$ and $\alpha \geq 0$ we have*

$$\left|\left(\frac{d}{dx}\right)^{\alpha} V(x)\right| \leq C_{\alpha}\, S(x) B^{\alpha}(x)\,.$$

Hyp2) *The equation $V(x) = E_0$ has two solutions $x_{\text{left}} < x_{\text{right}}$ in I, and they satisfy*

$$\operatorname{dist}(x_{\text{left}}, \partial I) > c\,B(x_{\text{left}})\,, \qquad \operatorname{dist}(x_{\text{right}}, \partial I) > c\,B(x_{\text{right}})\,.$$

Hyp3)

$$-V'(x) > c\,S(x_{\text{left}}) B^{-1}(x_{\text{left}})\,, \qquad \text{for } x \in [x_{\text{left}}, x_{\text{left}} + c_1 B(x_{\text{left}})]$$

and

$$V'(x) > c\,S(x_{\text{right}}) B^{-1}(x_{\text{right}})\,, \qquad \text{for } x \in [x_{\text{right}} - c_1 B(x_{\text{right}}), x_{\text{right}}]\,.$$

Hyp4)

$$c\,S(x) < E_0 - V(x) < CS(x)$$

for $x \in [x_{\text{left}} + c_1 B(x_{\text{left}}), x_{\text{right}} - c_1 B(x_{\text{right}})]$.

To state the remaining hypotheses, it is convenient to establish some notation. Set $\lambda(x) = S^{1/2}(x)B(x)$ for $x \in I$, and set

$$B_{\text{left}} = B(x_{\text{left}})\,, \qquad S_{\text{left}} = S(x_{\text{left}})\,, \qquad \lambda_{\text{left}} = \lambda(x_{\text{left}})\,.$$
$$B_{\text{right}} = B(x_{\text{right}})\,, \qquad S_{\text{right}} = S(x_{\text{right}})\,, \qquad \lambda_{\text{right}} = \lambda(x_{\text{right}})\,.$$

For $|E - E_0| < c\,S_{\text{left}}$, let $x_{\text{left}}(E)$ be the solution of $V(x) = E$ nearest to x_{left}, and for $|E - E_0| < cS_{\text{right}}$, let $x_{\text{right}}(E)$ be the solution of $V(x) = E$ nearest to x_{right}. Define

$$S_{\min} = \int_{x_{\text{left}} < x < x_{\text{right}}} S(x)\,dx$$

and

$$\Lambda = \int_{x_{\rm left}}^{x_{\rm right}} (S^{1/2}(x)B^2(x))^{-1}\, dx\,.$$

Our remaining hypotheses are as follows.

- *Assumptions on* $V(x)$ *in all of* $I_{\rm BVP}$:

Hyp5) *If* $|E - E_0| < c_2 S_{\rm min}$ *and* $E \le E_\infty$, *then* $V(x) > E$ *for all* $x \in I_{\rm BVP} - [x_{\rm left}(E), x_{\rm right}(E)]$.

Hyp6) *If* $x \in I_{\rm BVP}$ *satisfies* $x < x_{\rm left} - \lambda_{\rm left}^K B_{\rm left}/2$ *then* $V(x) \ge E_\infty + 100/|x - x_{\rm left}|^2$, *and if* $x \in I_{\rm BVP}$ *satisfies* $x > x_{\rm right} + \lambda_{\rm right}^K B_{\rm right}/2$, *then* $V(x) \ge E_\infty + 100/|x - x_{\rm right}|^2$.

- *Technical Assumptions*:

Hyp7) $\max_{x\in I} S(x) \le \lambda_{\rm left}^K S_{\rm left}$ *and* $\max_{x\in I} S(x) \le \lambda_{\rm right}^K S_{\rm right}$.

Hyp8)

$$\int_{x_{\rm left}}^{x_{\rm right}} \Big(\frac{dx}{S^{1/2}(x)}\Big) \le \Lambda^K \min\{S_{\rm left}^{-1/2} B_{\rm left}, S_{\rm right}^{-1/2} B_{\rm right}\}\,.$$

Hyp9)

$$\Big(\int_{x_{\rm left}}^{x_{\rm right}} \frac{dx}{S^{1/2}(x)\, B^4(x)}\Big)\Big(\int_{x_{\rm left}}^{x_{\rm right}} \frac{dx}{S^{1/2}(x)}\Big) \le \Lambda^K\,.$$

- *WKB Condition*:

Hyp10) Λ *is bounded below by a positive constant depending only on* ε, K *and* N, *and on* c, C, c_1, c_2, C_α *in* Hyp0)-Hyp4).

Then if E *is an eigenvalue of* $-d^2/dx^2 + V(x)$, *we have that*

$$\int_{x_{\rm left}}^{x_{\rm right}} (E - V(x))^{1/2}\, dx + i\, h_1(E) = \pi\Big(k + \frac{1}{2}\Big) + \phi_{\rm error}(E)\,,$$

with $|\phi_{\rm error}| \le C\Lambda^{-3}$ *and*

$$h_1(E) = \frac{i}{48} \lim_{\delta\to 0} \Big(\int_{x_{\rm left}+\delta}^{x_{\rm right}-\delta} V''(x)\, (E - V(x))^{-(3/2)}\, dx - q(E)\, \delta^{-1/2}\Big)$$

with $q(E)$ *uniquely specified by demanding the finiteness of the limit.*

PROOF. Let us say that a complex function f_l has the *alternating parity property on the index* l if it is real-valued for l even and purely imaginary for l odd. It suffices to show that h_l has the alternating parity property on the index l. Recall that the h_l's are inductively determined by

$$u_k^{\text{left}}(x,E) = \sum_{l=0}^{k} h_l(E)\, u_{k-l}^{\text{right}}(x,E)\,, \tag{6}$$

where u_k is the canonical solution of the transport equations

$$u_0 \equiv 1\,,$$

$$2\,i\,u'_{k+1} + \Big(\frac{5}{16}(p')^2 p^{-5/2} - \frac{1}{4}\,p'' p^{-3/2}\Big) u_k - \frac{1}{2}\,p'\,p^{-3/2}\,u'_k + p^{-1/2}\,u''_k = 0\,, \qquad 0 \le k < N'\,.$$

In particular, since $u_0^{\text{left}} = u_0^{\text{right}} = 1$,

$$h_2(E) = u_2^{\text{left}}(x,E) - h_1(E)\,u_1^{\text{right}}(x,E)\,.$$

Since h_1 is known to be purely imaginary, it suffices to show u_k^{left} and u_k^{right} each have the alternating parity property on the index k. Let us show u_k^{left} has the alternating parity property; the proof for u_k^{right} is totally analogous.

Lemma 10 of [FS] relate the canonical solution to the elementary solution of the transport equations in the following manner: if $u = (u_0(x), u_1(x), \ldots, u_{N''}(x))$ is the canonical solution of the transport equations, and if $\tilde{u} = (\tilde{u}_0, \ldots, \tilde{u}_{N''}(x))$ is the elementary solution, then

$$u_k(x) = \sum_{l=0}^{k} w_{k-l,0}\,\tilde{u}_l(x)\,,$$

where w_{kl} will be investigated in more detail below. Since the construction of the elementary solutions in [FS] makes it clear $\tilde{u}_l$ has the alternating parity property on the index l, we have reduced the problem to showing w_{kl} has the alternating parity property on the index k. Equivalently, letting $w_k(x) = \sum_{-3k \le l} w_{kl}\,x^{l/2}$, it suffices to show w_k has the alternating parity property on the index k.

Now all that is needed is to take account of the real and purely imaginary quantities that arise in the construction of w_k. (*cf.* [FS,

p. 155-162, 171]). We proceed as follows: $w_k(x)$ can be written in terms of $h^{\#}_{kl}, q^{\#}_{kl}$ and $\hat{h}_{kl}$ via the equation

$$\begin{aligned}\Big(1+\sum_{k=1}^{N}\lambda^{-k}w_k(x)\Big) &= \Big(\Big(1+\sum_{k=1}^{2N}\sum_{l=2-k}^{3N} h^{\#}_{kl}\,x^{l/2}\lambda^{-k}+O\left(\lambda^{-\varepsilon N/4}\right)\Big)\\ &\quad\cdot\Big(1+\sum_{k=1}^{N}\sum_{l=-k}^{N} q^{\#}_{kl}\,x^{l}\lambda^{-2k}+O\left(\lambda^{-\varepsilon N/4}\right)\Big)\\ &\quad\cdot\Big(1+\sum_{k=1}^{N}\sum_{l=-3k}^{N} \hat{h}_{kl}\,x^{l/2}\lambda^{-k}+O(\lambda^{-\varepsilon N/4})\Big)\Big).\end{aligned}\tag{7}$$

To prove w_k has the alternating parity property on the index k, we will want to show both $h^{\#}_{kl}$ and $\hat{h}_{kl}$ have this property on the index k and $q^{\#}_{kl}$ is real. Let us first look at $h^{\#}_{kl}$. [FS] shows

$$\begin{aligned}&\exp\Big(\sum_{k=1}^{N}\sum_{l=-k}^{N} h_{kl}\,x^{l+3/2}\lambda^{-(2k-1)}\Big)\\ &\qquad= \Big(1+\sum_{k=1}^{2N}\sum_{l=2-k}^{3N} h^{\#}_{kl}\,x^{l/2}\lambda^{-k}+O\left(\lambda^{-\varepsilon N/4}\right)\Big),\end{aligned}\tag{8}$$

where the right-hand side is a high-order Taylor expansion with remainder. Let us consider more carefully how $h^{\#}_{kl}$ depends on h_{kl}. Note that

$$\begin{aligned}&\frac{2\,i}{3}\,\lambda(y_0(x))^{3/2}\sum_{k=1}^{N}\sum_{l=-k}^{N} f^{\#\#}_{kl}x^{l}\lambda^{-2k}\\ &\qquad= \sum_{k=1}^{N}\sum_{l=-k}^{N} h_{kl}\,x^{l+3/2}\lambda^{-(2k-1)}+O\left(\lambda^{-\varepsilon N/4}\right).\end{aligned}\tag{9}$$

Since $y_0(x)$ and $f^{\#\#}_{kl}$ are real, h_{kl} is purely imaginary since it depends only on these quantities multiplied by i. Now set

$$X=\sum_{k=1}^{N}\sum_{l=-k}^{N} h_{kl}\,x^{l+3/2}\lambda^{-(2k-1)}.$$

A sufficiently high power of X will be $O\left(\lambda^{-\varepsilon N/4}\right)$, so the left-hand side of (8) has a Taylor expansion with remainder. Note that X^s is purely imaginary if and only if s is odd. Since X contains nothing but odd powers of λ, one finds upon collecting terms of the Taylor expansion with respect to λ that the coefficients are purely imaginary for all odd powers of λ, real for all even powers of λ. This says precisely that $h^{\#}_{kl}$ has the alternating parity property on the index k.

Now let us consider $q^{\#}_{kl}$. Quite simply, $q^{\#}_{kl}$ is real since all the other quantities in the following equation are real.

$$\Big(\frac{\partial y_N(x,\lambda)}{\partial x}\Big)^{-1/2}(y_N(x,\lambda))^{-1/4} = (p(x))^{-1/4}\Big(1+\sum_{k=1}^{N}\sum_{l=-k}^{N} q^{\#}_{kl}\lambda^{-2k} + O\left(\lambda^{-\varepsilon N/4}\right)\Big)\,. \tag{10}$$

Finally, let us consider $\hat{h}_{kl}$. We have that

$$\Big(1+\sum_{s=1}^{M} c_s\,\lambda^{-s}x^{-3s/2}\Big(\sum_{k=0}^{N}\sum_{l=-k}^{N} h^{s}_{kl}\,x^{l}\lambda^{-2k} + O\left(\lambda^{-\varepsilon N/5}\right)\Big)\Big) = \Big(1+\sum_{k=1}^{N}\sum_{l=-3k}^{N}\hat{h}_{kl}\,x^{l/2}\lambda^{-k} + O\left(\lambda^{-\varepsilon N/6}\right)\Big)\,, \tag{11}$$

where h^{s}_{kl} is real, and c_s has the alternating parity property on the index s. This is a consequence of the recurrence relation one finds upon substituting the asymptotic form of the Airey function

$$A(t) = \mathrm{Re}\Big(\frac{e^{\pm i\pi/4}e^{2it^{3/2}/3}}{t^{1/4}}\Big(1+\sum_{s=1}^{\infty} c_s\,t^{-(3/2)s}\Big)\Big)$$

into the Airey equation

$$\frac{d^2}{dy^2}A(y,\lambda) + \lambda^2 yA(y,\lambda) = 0\,.$$

Collecting the even and odd powers of λ on the left-hand side of (11) shows that $\hat{h}_{kl}$ has the alternating parity property on the index k.

Putting what we know about $h^{\#}_{kl}, q^{\#}_{kl}$ and $\hat{h}_{kl}$ into (7) reveals that coefficients of *even* powers of λ must involve products with *even* total numbers of $h^{\#}_{kl}$'s and $\hat{h}_{kl}$'s. Note also that the $q^{\#}_{kl}$ are always accompanied by *even* powers of λ. Therefore the coefficients of even powers of λ

on the left-hand side of (7) are real. On the other hand the coefficients of odd powers of λ are purely imaginary. Hence w_k has the alternating parity property on the index k.

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References.

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