Revista Matemática Iberoamericana
Vol. 12, N.° 2, 1996

On the uniqueness problem for quasilinear elliptic equations involving measures

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Abstract. We discuss the uniqueness of solutions to problems like

$$\begin{cases} \lambda\,|u|^{s-1}u - \operatorname{div}(|\nabla u|^{p-2}\,\nabla u) = \mu & \text{on } \Omega\,, \\ u = 0 & \text{in } \partial\Omega\,, \end{cases}$$

where $\lambda \geq 0$ and μ is a signed Radon measure.

1. Introduction.

Throughout this paper we let Ω be a bounded open set in $\mathbb{R}^n$ and $1 < p \leq n$ a fixed number with $p > 2 - 1/n$.[1] Suppose that μ is a signed Radon measure in Ω with finite total variation. We consider the solutions $u \in W^{1,1}_{\text{loc}}(\Omega)$ of the equation

$$B(u) - \operatorname{div}\mathcal{A}(x, \nabla u) = \mu\,,$$

[1] The restriction $p>2-1/n$ could be removed by using a generalized derivative as in [5] or a different concept of a solution as *e.g.* in [1] or [9].

where $B\colon \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous increasing function with $B(0) = 0$ and $\mathcal{A}\colon \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a mapping that satisfies the following assumptions for some numbers $0 < \alpha \le \beta < \infty$:

(1.1) the function $x \mapsto \mathcal{A}(x,\xi)$ is measurable for all $\xi \in \mathbf{R}^n$, and the function $\xi \mapsto \mathcal{A}(x,\xi)$ is continuous for a.e. $x \in \mathbf{R}^n$;

for all $\xi \in \mathbf{R}^n$ and almost every $x \in \mathbf{R}^n$

$$\mathcal{A}(x,\xi)\cdot\xi \ge \alpha\,|\xi|^p\,, \tag{1.2}$$

$$|\mathcal{A}(x,\xi)| \le \beta\,|\xi|^{p-1}\,, \tag{1.3}$$

$$(\mathcal{A}(x,\xi) - \mathcal{A}(x,\zeta))\cdot(\xi-\zeta) > 0\,, \tag{1.4}$$

whenever $\xi \ne \zeta$.

Solutions are understood in the sense of distributions, and we fix weak zero boundary values. More precisely, we consider the problem

$$\begin{cases} B(u) - \operatorname{div}\mathcal{A}(x,\nabla u) = \mu\,, \\ B(u) \in L^1(\Omega)\,, \\ u \in W^{1,\max\{p-1,1\}}_{\mathrm{loc}}(\Omega)\,, \\ T_k(u) \in W^{1,p}_0(\Omega) \text{ for } k > 0\,, \end{cases} \tag{1.5}$$

where T_k is the double side truncating operator at the level k,

$$T_k(t) = \max\{\,\min\{t,k\}, -k\}\,.$$

Here the first line in (1.5) means that

$$\int_\Omega B(u)\,\varphi\,dx + \int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla\varphi\,dx = \int_\Omega \varphi\,d\mu$$

for each $\varphi \in C^\infty_0(\Omega)$.

The prime examples of such equations arise from the p-Laplacian operator

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\,\nabla u)\,.$$

Keeping this example in mind one easily convinces oneself that, for an arbitrary measure μ, there is no hope to find a solution from the "natural" Sobolev space $W^{1,p}_0(\Omega)$. Indeed, existence of a solution in this space automatically implies that μ is in the dual of $W^{1,p}_0(\Omega)$. Moreover, it is well known that this dual does not contain point measures for $1 < p \leq n$ (see *e.g.* the discussion before Theorem 3.5 below).

Therefore we only require that the truncations of a solution be in $W_0^{1,p}(\Omega)$. Then, using compactness arguments we find that the solution itself lies in $W_0^{1,q(p-1)}(\Omega)$ for each $1 \leq q < n/(n-1)$.

There are several papers, where the authors discuss the existence of problems like (1.5) in different senses, see *e.g.* [7], [2], [5]. In the nonlinear case, there are a few results aiming at the treatment of the question of uniqueness: Lions and Murat have announced an existence and uniqueness result for renormalized solutions in the case when $p = 2$ and $\mu \in L^1$ (see [8]); unfortunately, we haven't seen their proof. Two different approaches to the general case with $\mu \in L^1$ are given in [1] and in [9]. Rakotoson [9] uses renormalized solutions, and Bénilan *et. al.* [1] an "entropy condition" which we shall adopt and modify. We shall consider measures μ that are absolutely continuous with respect to p-capacity (see 2.1 below), in particular, L^1-functions are particular cases of our consideration. We prove:

1.6. Theorem. *Let μ be a finite signed measure in Ω that is absolutely continuous with respect to p-capacity. Then there is a unique solution u of* (1.5) *such that for $\sigma \in \{+,-\}$*

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot \nabla T_k^\sigma(u-\varphi)\,dx + \int_\Omega B(u)\,T_k^{6\sigma}(u-\varphi)\,dx = \int_\Omega T_k^\sigma(u-\varphi)\,d\mu\,,$$

whenever $\varphi \in C_0^\infty(\Omega)$ and $k > 0$. Moreover, $u \in W_0^{1,q(p-1)}(\Omega)$ for each $1 \leq q < n/(n-1)$.

Here

$$T_k^+(t) = \max\big\{\min\{t,k\},0\big\}$$

and

$$T_k^-(t) = \min\big\{\max\{t,-k\},0\big\}$$

are the positive and negative truncating operators. Take notice that here and in what follows we always take the quasicontinuous, hence Borel, representatives of Sobolev functions; hence there are no problems with measurability.

To display a simple example that motivates the use of a constraint for the solutions, consider the p-Laplacian

$$\Delta_p u = 0$$

in the punctured ball $\Omega = B(0,1)\setminus\{0\}$. Then the identical zero function is a trivial solution of (1.5) in Ω (there $B = 0$, $\mu = 0$ and $\mathcal{A}(x,\xi) = |\xi|^{p-2}\xi$). Another solution is given by

$$u(x) = \begin{cases} |x|^{(p-n)/(p-1)} - 1\,, & \text{if } p < n\,, \\ \log|x|\,, & \text{if } p = n\,. \end{cases}$$

Observe that these functions both are SOLAs (solutions obtained as limits of approximations) in the sense of [3].

Note that the assumption that $p \leq n$ is no restriction, for any finite Radon measure belongs to the dual of the Sobolev space $W_0^{1,q}(\Omega)$ if $q > n$ and then the unique solvability of (1.5) is well known. On the contrary, the assumption that $p > 2 - 1/n$ is partly essential and partly purely technical. It is a simple matter to construct measures μ for which there cannot be any solutions with locally integrable distributional derivatives if $p \leq 2 - 1/n$. There are at least two different ways out of this trouble: either one could consider a generalized gradient as it was done in [5], or to leave distributional solutions and work with renormalized solutions as in [9] or [11]. We leave these technicalities to the interested reader.

2. Uniqueness.

To begin with, we recall that the Sobolev space $W^{1,q}(\Omega)$, $1 \leq q < \infty$, consists of all q-integrable functions u whose first distributional derivative ∇u is also q-integrable in Ω; equipped with the norm

$$\|u\|_{1,q} = \Big(\int_\Omega (|u|^q + |\nabla u|^q)\,dx\Big)^{1/q},$$

$W^{1,q}(\Omega)$ is a Banach space. The corresponding local space is marked as $W^{1,q}_{\mathrm{loc}}(\Omega)$. Moreover, $W_0^{1,q}(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ in $W^{1,q}(\Omega)$.

Next we define the *p-capacity of the set* $E \subset \mathbb{R}^n$ to be the number

$$\mathrm{cap}_p(E) = \inf \int_{\mathbb{R}^n} (|\varphi|^p + |\nabla\varphi|^p)\,dx\,,$$

where the infimum is taken over all $\varphi \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ such that $\varphi = 1$ on an open set containing E. Then cap_p defines an outer measure, but there are only very few measurable sets. The p-capacity is intimately connected with Sobolev spaces $W^{1,p}$ and with p-type equations (1.5), see *e.g.* [4], [12]. In particular each $u \in W^{1,p}(\Omega)$ has a quasicontinuous version, *i.e.* there is v such that $u = v$ almost everywhere and for each $\varepsilon > 0$ there is an open set G such that $\mathrm{cap}_p(G) < \varepsilon$ and the restriction to $\Omega \setminus G$ of v is continuous and real-valued.

We say that μ is *absolutely continuous with respect to p-capacity* if

$$\mu(E) = 0 \quad \text{whenever} \quad \mathrm{cap}_p(E) = 0\,. \tag{2.1}$$

Note that the Hausdorff dimension of a set of p-capacity zero is at most $n - p$, while a set with finite $n - p$ dimensional Hausdorff measure is of p-capacity zero, see *e.g.* [4].

In this section we establish uniqueness under a slightly weaker condition than was stated in Theorem 1.6. We say that a solution u of (1.5) satisfies the *entropy condition* if for $\sigma \in \{+,-\}$

$$\begin{aligned}\int_\Omega \mathcal{A}(x,\nabla u)\cdot \nabla T_k^\sigma(u-\varphi)\,dx + \int_\Omega B(u)\,T_k^\sigma(u-\varphi)\,dx \\ \leq \int_\Omega T_k^\sigma(u-\varphi)\,d\mu\end{aligned} \tag{2.2}$$

for all $\varphi \in C_0^\infty(\Omega)$ and $k > 0$;

In particular, we have that

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot \nabla T_k(u-\varphi)\,dx + \int_\Omega B(u)\,T_k(u-\varphi)\,dx \leq \int_\Omega T_k(u-\varphi)\,d\mu$$

whenever T_k is the double side truncating operator.

2.3. Lemma. *If u is a solution that satisfies the entropy condition* (2.2)*, then for each $M > 0$ and $k > 0$*

$$\int_{\{k\leq u\leq k+M\}} |\nabla u|^p\,dx \leq c\,M\,|\mu|(\{|u|>k\}) + c\,M \int_{\{|u|>k\}} |B(u)|\,dx\,.$$

PROOF. An easy approximation shows that one can replace φ in (2.2) by any bounded function from $W_0^{1,p}$ (see [1, Lemma 3.3]). In particular,

$$\begin{aligned} c \int_{\{k\le|u|\le k+M\}} |\nabla u|^p\,dx &\le \int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla T_M(u-T_k u)\,dx \\ &\le \int_\Omega T_M(u-T_k u)\,d\mu \\ &\quad - \int_\Omega B(u)\,T_M(u-T_k u)\,dx \\ &\le M\,|\mu|(\{|u|>k\}) \\ &\quad + M\int_{\{|u|>k\}} |B(u)|\,dx\,, \end{aligned}$$

as desired.

2.4. Corollary. *Let u be a solution that satisfies the entropy condition* (2.2). *If $|\mu|(\{|u|=\infty\})=0$, then*

$$\lim_{k\to\infty}\int_{\{k\le|u|\le k+M\}} |\nabla u|^p\,dx = 0\,.$$

Corollary 2.4 is in general false if $|\mu|(\{|u|=\infty\})>0$. Take, for instance, μ = the Dirac measure. Then if $\mathcal{A}(x,\xi)=|\xi|^{p-2}\xi$ is the p-Laplacian, we have

$$\lim_{k\to\infty}\int_{\{k\le u\le k+M\}} |\nabla u|^p\,dx = M\,.$$

In this paper, we restrict our consideration to measures which are absolutely continuous with respect to p-capacity. Then $|\mu|(\{|u|=\infty\})=0$ for p-quasicontinuous u.

2.5. Theorem. *Let μ_1 and μ_2 be finite signed Radon measures that are absolutely continuous with respect to p-capacity such that $\mu_1\le\mu_2$. If u and v are solutions of* (1.5) *with measures μ_1 and μ_2, respectively, that satisfy the entropy condition* (2.2)*, then $u\le v$.*

PROOF. By approximation,

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla T_k^+(u-T_l v)\,dx \\ \le \int_\Omega T_k^+(u-T_l v)\,d\mu_1 - \int_\Omega B(u)T_k^+(u-T_l v)\,dx$$

and

$$\int_\Omega \mathcal{A}(x,\nabla v)\cdot\nabla T_k^-(v-T_l u)\,dx \\ \le \int_\Omega T_k^-(v-T_l u)\,d\mu_2 - \int_\Omega B(v)T_k^-(v-T_l u)\,dx\,.$$

If we add these inequalities up and let $l\to\infty$, the right hand side is treated by the aid of the dominated convergence theorem and its limit is

$$\int_\Omega T_k^+(u-v)\,d\mu_1 - \int_\Omega T_k^+(u-v)\,d\mu_2 \\ - \int_\Omega \big(B(u)-B(v)\big)T_k^+(u-v)\,dx \le 0\,,$$

since $\mu_1 \le \mu_2$ and B is increasing. The set of integration on the left hand side is splitted into four parts:

$$\begin{aligned} G_1 &= \{|u-v|\le k\,,\ |v|\le l\,,\ \text{and } |u|\le l\}\,,\\ G_2 &= \{|u-v|>k\}\,,\\ B_1 &= \{|u-v|\le k\,,\ |v|\le l\,,\ \text{and } |u|> l\}\,,\\ B_2 &= \{|u-v|\le k\,,\ |v|> l\,,\ \text{and } |u|\le l\}\,. \end{aligned}$$

The parts B_1 and B_2 are symmetric and they tend to zero as is seen with an estimation like

$$\Big|\int_{B_1}\mathcal{A}(x,\nabla u)\cdot\nabla T_k^+(u-T_l v)\,dx\Big| \le c\int_{B_1}|\nabla u|^p\,dx + c\int_{B_1}|\nabla u|^{p-1}|\nabla v|\,dx \\ \le c\int_{\{l\le|u|\le l+k\}}|\nabla u|^p\,dx$$

$$+\,c\Big(\int\limits_{\{l\le|u|\le l+k\}}|\nabla u|^p\,dx\Big)^{p/(p-1)}$$
$$\cdot\Big(\int\limits_{\{l-k\le|v|\le l\}}|\nabla v|^p\,dx\Big)^{1/p}$$
$$\to 0+\,,$$

as $l\to\infty$ by Corollary 2.4. Further,

$$\Big|\int_{B_1}\mathcal{A}(x,\nabla v)\cdot\nabla T_k^-(v-T_lu)\,dx\Big|\le c\int\limits_{\{l-k\le|v|\le l\}}|\nabla v|^p\,dx\to 0\,,$$

as $l\to\infty$. Next we estimate the integrals over G_2. For instance

$$\Big|\int_{G_2}\mathcal{A}(x,\nabla u)\cdot\nabla T_k^+(u-T_lv)\,dx\Big|\le c\int\limits_{\{l-k\le|u|\le l+k\}}|\nabla u|^p\,dx\to 0$$

and the other integral is treated similarly.

Hence we conclude that the integral over G_1 tends to a nonpositive number as $l\to\infty$, and hence

$$\int\limits_{\{|u-v|\le k,\,u>v\}}\big(\mathcal{A}(x,\nabla u)-\mathcal{A}(x,\nabla v)\big)\cdot\big(\nabla u-\nabla v\big)\,dx\le 0\,.$$

Since this last integrand strictly positive if $\nabla u\ne\nabla v$, we have $\nabla u=\nabla v$ almost everywhere in the set where $|u-v|\le k$ and $u>v$. Letting $k\to\infty$ we find that $u\le v$ in Ω in the view of the weak boundary values. The proof is complete.

2.6. Corollary. *If μ is absolutely continuous with respect to p-capacity, then there is at most one solution u of* (1.5) *that satisfies the entropy condition.*

3. Existence.

There are various proofs for the existence of solutions to problem (1.5). Because we want that a particular solution satisfies the entropy condition, we have to give a proof that results in the entropy equality as well.

We start our investigation with a compactness lemma.

3.1. Lemma. *Let μ_j be a sequence of signed Radon measures that belong to the dual of $W_0^{1,p}(\Omega)$ such that*

$$|\mu_j|(\Omega) \le M < \infty$$

for each j. Let $u_j \in W_0^{1,p}(\Omega)$ be such that $B(u_j) \in L^1(\Omega)$ and

$$B(u_j) - \operatorname{div} \mathcal{A}(x, \nabla u_j) = \mu_j$$

in Ω. Then there is a subsequence u_j and a function u such that $u_j \to u$ pointwise almost everywhere and weakly in $W^{1,q(p-1)}$ whenever $1 \le q < n/(n-1) = n'$. Furthermore, $B(u_j)$ is bounded in $L^1(\Omega)$ and $\nabla u_j(x) \to \nabla u(x)$ for almost every x, $\mathcal{A}(x, \nabla u_j) \to \mathcal{A}(x, \nabla u)$ in $L^q(\Omega)$ and for each $k > 0$, the sequence of truncations $\nabla T_k(u_j)$ is bounded in $L^p(\Omega)$.

PROOF. By using the test functions $T_1(u_j/\varepsilon)$, $\varepsilon > 0$, we find that

$$\int_\Omega |B(u_j)|\, dx = \limsup_{\varepsilon \to 0} \left(\int_\Omega T_1(u_j/\varepsilon)\, d\mu_j - \frac{1}{\varepsilon} \int_{\{0<|u_j|<\varepsilon\}} \mathcal{A}(x, \nabla u_j) \cdot \nabla u_j\, dx \right)$$

$$\le |\mu_j|(\Omega) \le M < \infty\,. \tag{3.2}$$

Similarly, the use of the test function $T_k(u_j)$ shows that

$$\int_\Omega |\nabla T_k(u_j)|^p\, dx \le c\, k\, M\,, \tag{3.3}$$

so that, by the usual compactness arguments (see *e.g.* [4, 7.43]), the sequence $|\nabla u_j|^{p-1}$ is bounded in $L^q(\Omega)$ for all $1 \le q < n'$. Then there is $u \in W_0^{1,q(p-1)}(\Omega)$ such that $u_j \to u$ weakly in $W_0^{1,q(p-1)}(\Omega)$. By the aid of the Rellich compactness theorem we can extract a subsequence u_j that converges pointwise to u almost everywhere in Ω.

It remains to show that $\nabla u_j \to \nabla u$ pointwise almost everywhere. Fix $\varepsilon > 0$ and let

$$E_{j,k} = \{x \in \Omega\colon\ \big(\mathcal{A}(x, \nabla u_j) - \mathcal{A}(x, \nabla u_k)\big) \cdot \big(\nabla u_j - \nabla u_k\big) > \varepsilon\}\,.$$

We estimate the measure of $E_{j,k}$:

$$|E_{j,k}| \le |E_{j,k} \cap \{|u_j - u_k| \ge \varepsilon^2\}| + \frac{1}{\varepsilon} \int\limits_{E_{j,k}\cap\{|u_k-u_j|<\varepsilon^2\}} (\mathcal{A}(x,\nabla u_j) - \mathcal{A}(x,\nabla u_k)) \cdot (\nabla u_j - \nabla u_k)\, dx\,.$$

Using the test function $T_{\varepsilon^2}(u_j - u_k)$ we find the estimate

$$\begin{aligned}\int\limits_{E_{j,k}\cap\{|u_k-u_j|<\varepsilon^2\}} &(\mathcal{A}(x,\nabla u_j) - \mathcal{A}(x,\nabla u_k)) \cdot (\nabla u_j - \nabla u_k)\, dx \\ &\le \int_\Omega T_{\varepsilon^2}(u_j-u_k)\, d\mu_j - \int_\Omega T_{\varepsilon^2}(u_j-u_k)\, d\mu_k \\ &\quad - \int_\Omega B(u_j) T_{\varepsilon^2}(u_j-u_k)\, dx + \int_\Omega B(u_k) T_{\varepsilon^2}(u_j-u_k)\, dx \\ &\le c\,\varepsilon^2\end{aligned}$$

by what we proved above. Hence we arrive at the estimate

$$|E_{j,k}| \le c\,\varepsilon + |E_{j,k} \cap \{|u_j - u_k| \ge \varepsilon^2\}|\,, \tag{3.4}$$

where the constant c is independent of j, k, and ε.

Since $u_j \to u$ almost everywhere we easily infer from (3.4) and the monotonicity and continuity assumptions on $\mathcal{A}$ that ∇u_j converges pointwise almost everywhere to a function that must coincide with ∇u.

Now we consider a nonnegative finite Radon measure μ on Ω. We may as well assume that μ is defined on the whole of $\mathbb{R}^n$ with $\mu(\mathbb{R}^n\setminus\Omega) = 0$. Then $\mu \in \big(W_0^{1,p}(\Omega)\big)^*$ if and only if

$$\int_{\mathbb{R}^n} \mathbf{W}^\mu_{1,p}(x,1)\, d\mu < \infty\,,$$

where

$$\mathbf{W}^\mu_{1,p}(x,1) = \int_0^1 \left(\frac{\mu(B(x,r))}{r^{n-p}}\right)^{1/(p-1)} \frac{dr}{r}$$

is the Wolff potential (see *e.g.* [12, Theorem 4.7.5]).

Now we find a solution for which the entropy inequality (2.2) is an equality.

3.5. Theorem. *Let μ be a nonnegative finite measure in Ω with*

$$\mu(\{x\colon\ \mathbf{W}^{\mu}_{1,p}(x,1)=\infty\})=0\,.$$

Then there is a solution u of (1.5) *such that for $\sigma\in\{+,-\}$*

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla T_k^\sigma(u-\varphi)\,dx+\int_\Omega B(u)\,T_k^\sigma(u-\varphi)\,dx=\int_\Omega T_k^\sigma(u-\varphi)\,d\mu\,,$$

whenever $\varphi\in C_0^\infty(\Omega)$ and $k>0$.

PROOF. For a nonnegative integer j, let

$$E_j=\{x\colon\ \mathbf{W}^{\mu}_{1,p}(x,1)\le j\}\,,$$

and let μ_j be the restriction to E_j of μ,

$$\mu_j(E)=\mu(E\cap E_j)\,.$$

Then $0\le\mu_j\le\mu_{j+1}\le\mu$ and $\mu_j\to\mu$ weakly, for

$$\mu(\{x\colon\ \mathbf{W}^{\mu}_{1,p}(x,1)=\infty\})=0\,.$$

Since

$$\int_{\mathbb{R}^n}\mathbf{W}^{\mu_j}_{1,p}(x,1)\,d\mu_j\le\int_{\mathbb{R}^n} j\,d\mu_j\le j\,\mu(\Omega)<\infty\,,$$

we have $\mu_j\in\big(W_0^{1,p}(\Omega)\big)^*$. Hence there is a unique $u_j\in W_0^{1,p}(\Omega)$ such that $B(u_j)\in L^1(\Omega)$ and

$$B(u_j)-\operatorname{div}\mathcal{A}(x,\nabla u_j)=\mu_j \tag{3.6}$$

in Ω (see *e.g.* [10] or [7]). Using Lemma 3.1 we find a subsequence of u_j increasing to a function u such that $B(u)\in L^1(\Omega)$ and

$$B(u)-\operatorname{div}\mathcal{A}(x,\nabla u)=\mu$$

in Ω with weak boundary values.

The entropy equality for u is verified as follows: fix $\varphi\in C_0^\infty(\Omega)$. Then for each $k>0$ we have

$$\int_\Omega \mathcal{A}(x,\nabla u_j)\cdot\nabla T_k^\sigma(u-\varphi)\,dx+\int_\Omega B(u_j)T_k^\sigma(u-\varphi)\,dx$$
$$=\int_\Omega T_k^\sigma(u-\varphi)\,d\mu_j\,.$$

Letting $j \to \infty$ this gives us the desired equality. Indeed, the second integral does not cause any troubles, for $B(u_j) \to B(u)$ in L^1 since u_j increases to u. The first integral is treated by the aid of (3.3): for $M \geq k + \sup|\varphi|$ we have

$$\begin{aligned}\int_\Omega &\mathcal{A}(x, \nabla u_j) \cdot \nabla T_k^\sigma(u-\varphi)\,dx \\ &= \int_{\{u \le M\}} \mathcal{A}(x, \nabla T_M(u_j)) \cdot \nabla T_k^\sigma(u-\varphi)\,dx \\ &\to \int_{\{u \le M\}} \mathcal{A}(x, \nabla T_M(u)) \cdot \nabla T_k^\sigma(u-\varphi)\,dx \\ &= \int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla T_k^\sigma(u-\varphi)\,dx\,,\end{aligned}$$

since the sequence u_j is increasing and $\nabla u_j \to \nabla u$ pointwise almost everywhere. Finally,

$$\int_\Omega T_k^\sigma(u-\varphi)\,d\mu_j = \int_\Omega T_k^\sigma(u-\varphi)\,\chi_{E_j}\,d\mu \to \int_\Omega T_k^\sigma(u-\varphi)\,d\mu\,,$$

where χ_{E_j} stands for the characteristic function of the set E_j.

3.7. REMARK. If μ is in the dual of $W_0^{1,p}(\Omega)$, then

$$\mu(\{x:\ \mathbf{W}_{1,p}^\mu(x,1) = \infty\}) = 0\,.$$

Consequently, if μ is such that

$$\mu(\{x:\ \mathbf{W}_{1,p}^\mu(x,1) = \infty\}) > 0\,,$$

or equivalently[2], if μ is not absolutely continuous with respect to p-capacity, then there does not exist any increasing sequence of nonnegative Radon measures $\mu_j \in (W_0^{1,p}(\Omega))^*$ with $\mu_j \to \mu$ weakly.

[2] Indeed, the set where $\mathbf{W}_{1,p}^\mu(x,1)=\infty$ is of p-capacity zero by [6]. On the other hand, if $\mu(\{x:\ \mathbf{W}_{1,p}^\mu(x,1)=\infty\})=0$, then as in the previous proof, we find an increasing sequence μ_j of measures from the dual of $W_0^{1,p}(\Omega)$ such that $\mu_j \to \mu$ weakly. Then, since μ_j are absolutely continuous with respect to p-capacity, the same holds for the measure μ.

3.8. Corollary. *Let μ be a nonnegative finite measure in Ω that is absolutely continuous with respect to p-capacity. Then there is a unique solution u of* (1.5) *such that for $\sigma \in \{+,-\}$*

$$\int_\Omega \mathcal{A}(x,\nabla u)\cdot\nabla T_k^\sigma(u-\varphi)\,dx + \int_\Omega B(u)\,T_k^\sigma(u-\varphi)\,dx = \int_\Omega T_k^\sigma(u-\varphi)\,d\mu$$

whenever $\varphi \in C_0^\infty(\Omega)$ and $k>0$.

PROOF. The uniqueness follows from Corollary 2.6, the existence from Theorem 3.5, for the set

$$E = \{x\colon\ \mathbf{W}^\mu_{1,p}(x,1)=\infty\}$$

is of p-capacity zero (there is a p-superharmonic function u such that $u=\infty$ on E by [6]; thus $\operatorname{cap}_p(E)=0$ by [4, 10.1]).

Next we sketch the existence proof for signed measures.

PROOF OF THEOREM 1.6. The uniqueness was established in Corollary 2.6.

To prove the existence, let $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are nonnegative measures. Let $\sigma \in \{+,-\}$ and as in the proof of Theorem 3.5, write μ_j^σ for the restriction of μ to the set where $\mathbf{W}^{\mu^\sigma}_{1,p}(x,1)\le j$. Since, for fixed i the measure $\mu_{j,i} = \mu_j^+ - \mu_i^- \in \big(W_0^{1,p}(\Omega)\big)^*$, there is a unique $u_{j,i}\in W_0^{1,p}(\Omega)$ such that $B(u_{j,i})\in L^1(\Omega)$ and

$$B(u_{j,i}) - \operatorname{div}\mathcal{A}(x,\nabla u_{j,i}) = \mu_{j,i}$$

in Ω. By Lemma 3.1 there is $v_i \in W^{1,q(p-1)}(\Omega)$ such that the truncations $T_k(v_i)$ belong to $W_0^{1,p}(\Omega)$ and $u_{j,i}\to v_i$ weakly in $W^{1,q(p-1)}(\Omega)$ as $j\to\infty$. By the Rellich compactness theorem we have that (a subsequence of) $u_{j,i}$ converges to v_i a.e. and $\mathcal{A}(x,\nabla u_{j,i})\to\mathcal{A}(x,\nabla v_i)$ weakly in $L^q(\Omega)$. Then, since $u_{j,i}$ increases to v_i, we infer that v_i is a solution of

$$B(v_i) - \operatorname{div}\mathcal{A}(x,\nabla v_i) = \mu^+ - \mu_i^-$$

with $B(v_i)\in L^1(\Omega)$, and the entropy equality is proved almost verbatim as in Theorem 3.5.

Now Theorem 2.5 implies that the sequence v_i is decreasing. Repeating the analysis above one easily sees that the limit function $u = \lim_{i\to\infty} v_i$ is the desired solution; we leave the details to the reader.

3.9. REMARK. Suppose that u is a solution of (1.5). When does it automatically satisfy the entropy condition? The example we gave in the introduction shows that this is not always the case. Suppose that the sets

$$E_j = \{x \in \Omega : \ |u(x)| \geq j\}$$

are compact for j large enough (see the estimates in [6] for the pointwise behavior of u in terms of the potential $\mathbf{W}^{\mu}_{1,p}$). Then $T_k(u-\varphi)$ can be approximated in $W^{1,p}_0(\Omega)$ by $C^\infty_0(\Omega)$ functions whose gradients vanish on E_j. Thus it follows that we can plug $T_k(u-\varphi)$ in as a test function, and u therefore satisfies the entropy condition.

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Recibido: 20 de mayo de 1.995

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* Research supported partially by an NSF grant (DMS9424448)