

Piecewise Cubic Interpolation and Two-Point Boundary Value Problems

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1. Introduction

Cubic splines are of much use for approximating solutions of simple two-point boundary value problems for both linear and nonlinear ordinary differential equations. In the present paper, we shall give its mathematical foundation by the use of Urabe's method [5], which is quite universally applicable.

We consider the following two-point boundary value problem:

$$(1) \quad \ddot{x}(t) = f(t, x(t), \dot{x}(t)) \quad (0 \leq t \leq 1)$$

with boundary conditions

$$(2) \quad A_0 x(0) - B_0 \dot{x}(0) = a$$

$$(3) \quad A_1 x(1) + B_1 \dot{x}(1) = b$$

where $f(t, x, w)$ is defined and twice continuously differentiable in a region D of (t, x, w) -space intercepted by two hyperplanes $t=0$ and $t=1$.

We rewrite the problem (1)-(3) in the following form:

$$(4) \quad \dot{x}(t) = w(t) \quad (0 \leq t \leq 1)$$

$$(5) \quad \dot{w}(t) = f(t, x(t), w(t)) \quad (0 \leq t \leq 1)$$

$$(6) \quad A_0 x(0) - B_0 w(0) = a$$

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$$(7) \quad A_1x(1) + B_1w(1) = b.$$

Now making use of B -spline $Q_{m+1}(t) = \frac{1}{m!} \sum_{i=0}^{m+1} (-1)^i \binom{m+1}{i} (t-i)_+^m$, we consider spline functions of the form $x_h(t) = \sum_{p=-3}^{n-1} \alpha_p Q_4\left(\frac{t}{h} - p\right)$, $w_h(t) = \sum_{p=-2}^{n-1} \beta_p Q_3\left(\frac{t}{h} - p\right)$ ($nh=1$) with undetermined coefficients $\alpha_{-3}, \alpha_{-2}, \dots, \alpha_{n-1}$ and $\beta_{-2}, \beta_{-1}, \dots, \beta_{n-1}$. The above x_h and w_h will be an approximate solution to the problem (4)-(7) if they satisfy

$$(8) \quad \dot{x}_h(t) = w_h(t) \quad (0 \leq t \leq 1)$$

$$(9) \quad \dot{w}_h(t) = Pf(t, x_h(t), w_h(t)) \quad (0 \leq t \leq 1)$$

$$(10) \quad A_0x_h(0) - B_0w_h(0) = a$$

$$(11) \quad A_1x_h(1) + B_1w_h(1) = b.$$

Here P is an operator defined by $(Pf)(t) = \sum_{p=0}^n f(t_p) L_p(t)$, where $L_p(t)$ is a piecewise linear function with property $L_p(t_q) = \delta_{p,q}$ ($t_q = qh$). From a well known relation $\dot{Q}_{m+1}(t) = Q_m(t) - Q_m(t-1)$, we see that equation (8) is equivalent to the following system of $n+2$ equations:

$$(12) \quad \frac{\alpha_p - \alpha_{p-1}}{h} = \beta_p \quad (p = -2, -1, \dots, n-1).$$

Any two piecewise linear functions coincide with each other if and only if they coincide at the nodes, therefore we see that equation (9) is equivalent to the following system of $n+1$ equations:

$$(13) \quad \frac{\beta_{p-1} - \beta_{p-2}}{h} = f\left(t_p, \frac{\alpha_{p-3} + 4\alpha_{p-2} + \alpha_{p-1}}{6}, \frac{\beta_{p-2} + \beta_{p-1}}{2}\right) \quad (p=0, 1, \dots, n).$$

The boundary conditions (10) and (11) give two equations:

$$(14) \quad A_0 \frac{\alpha_{-3} + 4\alpha_{-2} + \alpha_{-1}}{6} - B_0 \frac{\beta_{-2} + \beta_{-1}}{2} = a$$

$$(15) \quad A_1 \frac{\alpha_{n-3} + 4\alpha_{n-2} + \alpha_{n-1}}{6} + B_1 \frac{\beta_{n-2} + \beta_{n-1}}{2} = b.$$

The number of undetermined coefficients is $2n + 5$ and the conditions (12)-(15) precisely give the requisite number of equations. For the convenience of the analysis, we rewrite (12)-(15) in the following form:

$$(16) \quad F_h^{(1)}(\alpha, \beta) = \left\{ \frac{\alpha_p - \alpha_{p-1}}{h} - \beta_p \right\}_{p=-2}^{n-1} = 0,$$

$$F_h^{(2)}(\alpha, \beta) = \left\{ \frac{\beta_{p-1} - \beta_{p-2}}{h} - f\left(t_p, \frac{\alpha_{p-3} + 4\alpha_{p-2} + \alpha_{p-1}}{6}, \frac{\beta_{p-2} + \beta_{p-1}}{2}\right) \right\}_{p=0}^n = 0,$$

$$F_h^{(3)}(\alpha, \beta) = \left\{ A_0 \frac{\alpha_{-3} + 4\alpha_{-2} + \alpha_{-1}}{6\sqrt{h}} - B_0 \frac{\beta_{-2} + \beta_{-1}}{2\sqrt{h}} - \frac{a}{\sqrt{h}}, \right. \\ \left. A_1 \frac{\alpha_{n-3} + 4\alpha_{n-2} + \alpha_{n-1}}{6\sqrt{h}} + B_1 \frac{\beta_{n-2} + \beta_{n-1}}{2\sqrt{h}} - \frac{b}{\sqrt{h}} \right\} = 0.$$

In what follows, the system of equations (16) will be called a determining equation for spline approximations. In the present paper we use (16) in order to facilitate the analysis, but clearly in practical computations it is more convenient to use the equations containing only $\alpha_p (p = -3, -2, \dots, n-1)$ which can be obtained from (12)-(15) by eliminating β_p .

In the present paper we assume that the problem (4)-(7) has an isolated solution $(\hat{x}(t), \hat{w}(t))$ satisfying the internality condition

$$U = \{(t, x, w) \mid [(x - \hat{x}(t))^2 + (w - \hat{w}(t))^2]^{\frac{1}{2}} \leq \delta, t \in [0, 1]\} \subset D$$

for some $\delta > 0$. By the definition in [4] (p. 46), a solution $(\hat{x}(t), \hat{w}(t))$ to the problem (4)-(7) is isolated if and only if

$$(17) \quad G = \begin{bmatrix} A_0 & -B_0 \\ A_1 y_1(1) + B_1 z_1(1) & A_1 y_2(1) + B_1 z_2(1) \end{bmatrix}$$

is non-singular, where

$$\begin{bmatrix} y_1(t) & y_2(t) \\ z_1(t) & z_2(t) \end{bmatrix} = \Phi(t)$$

is a fundamental matrix with property $\Phi(0) = E$ (E the unit matrix) of the first variation equation of (4)–(5) with respect to $(\hat{x}(t), \hat{w}(t))$, that is,

$$\dot{y}(t) = z(t),$$

$$\dot{z}(t) = f_x(t, \hat{x}(t), \hat{w}(t))y(t) + f_w(t, \hat{x}(t), \hat{w}(t))z(t).$$

Corresponding to $\hat{x}(t)$, as easily seen, one can determine uniquely a cubic spline function $\hat{x}_h(t)$ of the form

$$(18) \quad \hat{x}_h(t) = \sum_{p=-3}^{n-1} \hat{\alpha}_p Q_4\left(\frac{t}{h} - p\right)$$

so that

$$\hat{x}_h(t_p) = \hat{x}(t_p) \quad (p=0, 1, 2, \dots, n),$$

$$(19) \quad \dot{\hat{x}}_h(t_0) = \dot{\hat{x}}(t_0),$$

$$\dot{\hat{x}}_h(t_n) = \dot{\hat{x}}(t_n).$$

Since $\hat{x}(t) \in C^4[0, 1]$ due to the assumption $f(t, x, w) \in C_{t,x,w}^2(D)$, by Theorem 2.3.4 in [1] (p. 29) it is valid that

$$(20) \quad \left| \left(\frac{d}{dt} \right)^k [\hat{x}_h(t) - \hat{x}(t)] \right| = O(h^{4-k}) \quad (k=0, 1, 2, 3)$$

uniformly on $[0, 1]$ as $h \rightarrow 0$. From $\hat{x}_h(t)$, one can easily construct a quadratic spline function $\hat{w}_h(t)$ of the form

$$\hat{w}_h(t) = \sum_{p=-2}^{n-1} \hat{\beta}_p Q_3\left(\frac{t}{h} - p\right)$$

so that

$$\hat{w}_h(t) = \hat{x}_h(t) \quad (0 \leq t \leq 1).$$

By (20), it is then valid that

$$(21) \quad \begin{aligned} |\hat{w}_h(t) - \hat{w}(t)| &= |\hat{x}_h(t) - \hat{x}(t)| = O(h^3) \\ |\hat{w}'_h(t) - \hat{w}'(t)| &= |\hat{x}'_h(t) - \hat{x}'(t)| = O(h^2) \end{aligned}$$

uniformly on $[0, 1]$ as $h \rightarrow 0$.

2. Some Properties of Spline Functions

In what follows, for any $\varphi(t) \in L^2[0, 1]$, we shall denote $\left[\int_0^1 \varphi^2(t) dt \right]^{\frac{1}{2}}$ by $\|\varphi\|$, and for any finite dimensional vector c , we shall denote its Euclidean norm by $|c|$.

Lemma 1. *Let $\varphi(t) = \sum_{p=0}^n c_p L_p(t)$, then $\mu_1 \sqrt{h} |c| \leq \|\varphi\| \leq \lambda_1 \sqrt{h} |c|$, where μ_1 and λ_1 are positive constants independent of h , and $c = (c_0, c_1, c_2, \dots, c_n)$.*

Proof. For $t \in [t_p, t_{p+1}]$ ($p = 0, 1, \dots, n-1$), $\varphi(t) = c_p L_p(t) + c_{p+1} L_{p+1}(t)$. Therefore we easily get

$$\|\varphi\|^2 = \sum_{p=0}^{n-1} \int_{t_p}^{t_{p+1}} \varphi^2(t) dt = \frac{h}{3} \sum_{p=0}^{n-1} (c_p^2 + c_p c_{p+1} + c_{p+1}^2).$$

Since $\frac{1}{2}(c_p^2 + c_{p+1}^2) \leq c_p^2 + c_p c_{p+1} + c_{p+1}^2 \leq \frac{3}{2}(c_p^2 + c_{p+1}^2)$, we have $\frac{1}{6} h |c|^2 \leq \|\varphi\|^2 \leq h |c|^2$, which proves the lemma with $\mu_1 = \frac{1}{\sqrt{6}}$ and $\lambda_1 = 1$.

Lemma 2. *Let $\varphi(t) = \sum_{p=-1}^{n-1} c_p Q_3\left(\frac{t}{h} - p\right)$, then $\mu_2 \sqrt{h} |c| \leq \|\varphi\| \leq \lambda_2 \sqrt{h} |c|$, where μ_2 and λ_2 are positive constants independent of h , and $c = (c_{-2}, c_{-1}, \dots, c_{n-1})$.*

Proof. Since $\varphi(t)$ is a quadratic polynomial on $[t_p, t_{p+1}]$ ($p = 0, 1, 2, \dots, n-1$), it can be written on $[t_p, t_{p+1}]$ as follows:

$$\varphi(t) = \varphi(t_p) + \dot{\varphi}(t_p)(t - t_p) + \frac{\ddot{\varphi}(t_p)}{2!}(t - t_p)^2.$$

Here it is easily seen that $\varphi(t_p) = \frac{c_{p-2} + c_{p-1}}{2}$, $\dot{\varphi}(t_p) = \frac{c_{p-1} - c_{p-2}}{h}$, and

$\ddot{\varphi}(t_p+) = \frac{c_{p-2} - 2c_{p-1} + c_p}{h^2}$. Therefore it follows that

$$\begin{aligned} \|\varphi\|^2 &= \sum_{p=0}^{n-1} \int_{t_p}^{t_{p+1}} \varphi^2(t) dt \\ &= \sum_{p=0}^{n-1} h \int_0^1 \left(\varphi(t_p) + h\dot{\varphi}(t_p)t + \frac{h^2}{2}\ddot{\varphi}(t_p+)t^2 \right)^2 dt \\ &= \sum_{p=0}^{n-1} hI_p, \end{aligned}$$

where I_p is a quadratic form with respect to c_{p-2} , c_{p-1} and c_p independent of h . I_p is nonnegative, and $I_p=0$ implies that $\varphi(t_p)=\dot{\varphi}(t_p)=\ddot{\varphi}(t_p+)=0$, that is, $c_{p-2}=c_{p-1}=c_p=0$. Therefore I_p is a positive definite quadratic form, consequently there are positive constants μ_2 and λ_2 such that $\mu_2^2(c_{p-2}^2 + c_{p-1}^2 + c_p^2) \leq I_p \leq \frac{\lambda_2^2}{3}(c_{p-2}^2 + c_{p-1}^2 + c_p^2)$. From this readily follows the conclusion of the lemma.

Lemma 3. Let $\varphi(t) = \sum_{p=-3}^{n-1} c_p Q_4\left(\frac{t}{h} - p\right)$, then $\mu_3\sqrt{h}|c| \leq \|\varphi\| \leq \lambda_3\sqrt{h}|c|$, where μ_3 and λ_3 are positive constants independent of h , and $c = (c_{-3}, c_{-2}, \dots, c_{n-1})$.

Proof. Since $\varphi(t)$ is a cubic polynomial on $[t_p, t_{p+1}]$ ($p=0, 1, \dots, n-1$), it can be written on $[t_p, t_{p+1}]$ as follows:

$$\varphi(t) = \varphi(t_p) + \dot{\varphi}(t_p)(t - t_p) + \frac{\ddot{\varphi}(t_p)}{2!}(t - t_p)^2 + \frac{\ddot{\varphi}(t_p+)}{3!}(t - t_p)^3.$$

Here it is easily seen that $\varphi(t_p) = \frac{c_{p-3} + 4c_{p-2} + c_{p-1}}{6}$, $\dot{\varphi}(t_p) = \frac{c_{p-1} - c_{p-3}}{2h}$,

$$\ddot{\varphi}(t_p) = \frac{c_{p-1} - 2c_{p-2} + c_{p-3}}{h^2}, \text{ and } \ddot{\varphi}(t_p+) = \frac{c_{p-1} - 3c_{p-1} + 3c_{p-2} - c_{p-3}}{h^3}.$$

Hence likewise as the proof of lemma 2, we have easily the conclusion of the present lemma.

Lemma 4. Let $\varphi(t) = \sum_{p=-m}^{n-1} c_p Q_{m+1}\left(\frac{t}{h} - p\right)$, then $\|\varphi\|_{\infty} \leq |c|_{\infty}$ for $m \geq 1$, where $\|\varphi\|_{\infty} = \sup_{0 \leq t \leq 1} |\varphi(t)|$, $c = (c_{-m}, \dots, c_{n-1})$ and $|c|_{\infty} = \max_p |c_p|$.

Proof. As well known, *B*-splines can be characterized by convolution of characteristic functions as follows:

$$Q_{m+1}(t) = \underbrace{(x * \dots * x)}_{m+1}(t),$$

where

$$x(t) = \begin{cases} 1 & \text{for } t \in [0, 1] \\ 0 & \text{for } t \notin [0, 1]. \end{cases}$$

From this characterization of *B*-splines it readily follows that

- (i) $0 \leq Q_{m+1}(t) \leq 1$ $(m = 0, 1, 2, \dots)$,
- (ii) $\sum_{p=-\infty}^{\infty} Q_{m+1}(t-p) = 1$ $(m = 1, 2, \dots)$.

From these inequalities, we readily get the desired inequality.

Lemma 5. *If $g(t)$ is continuously differentiable, then*

$$\|(I-p)g\| \leq \frac{h}{\sqrt{6}} \|\dot{g}\| \quad (I \text{ the unit operator}).$$

Proof. One can easily verify that

$$(I-p)g(t) = \int_{t_p}^{t_{p+1}} k_p(t, s) \dot{g}(s) ds \quad \text{for any } t \in [t_p, t_{p+1}],$$

where $k_p(t, s)$ is a piecewise continuous function such that

$$k_p(t, s) = \begin{cases} 1 - \frac{t-t_p}{h} & \text{if } s \leq t, \\ -\frac{t-t_p}{h} & \text{if } t < s. \end{cases}$$

One then has

$$\begin{aligned} \int_{t_p}^{t_{p+1}} [(I-p)g(t)]^2 dt &\leq \int_{t_p}^{t_{p+1}} dt \int_{t_p}^{t_{p+1}} k_p^2(t, s) ds \int_{t_p}^{t_{p+1}} \dot{g}^2(s) ds \\ &\leq \frac{h^2}{6} \int_{t_p}^{t_{p+1}} \dot{g}^2(s) ds, \end{aligned}$$

from which readily follows the conclusion of the lemma.

3. The Jacobian Matrix of $(F_h^{(1)}(\alpha, \beta), F_h^{(2)}(\alpha, \beta), F_h^{(3)}(\alpha, \beta))$

Put $(F_h^{(1)}(\alpha, \beta), F_h^{(2)}(\alpha, \beta), F_h^{(3)}(\alpha, \beta)) = F_h(\alpha, \beta)$ and let $J_h(\alpha, \beta)$ be the Jacobian matrix of $F_h(\alpha, \beta)$ with respect to $\alpha_p (p = -3, -2, \dots, n-1)$ and $\beta_p (p = -2, -1, \dots, n-1)$. In order to investigate the properties of $J_h(\alpha, \beta)$, let us consider a linear system

$$(22) \quad J_h(\alpha, \beta)(\xi_1, \xi_2) + (\gamma_1, \gamma_2, \gamma_3) = 0$$

where $\xi_1 = (u_{-3}, u_{-2}, \dots, u_{n-1})$, $\xi_2 = (v_{-2}, v_{-1}, \dots, v_{n-1})$, $\gamma_1 = (c_{-2}, c_{-1}, \dots, c_{n-1})$, $\gamma_2 = (d_0, d_1, \dots, d_{n-1})$ and $\gamma_3 = (e_1, e_2)$.

Corresponding to ξ_1 and ξ_2 , we consider cubic and quadratic spline functions $y_1(t)$ and $y_2(t)$ defined by

$$y_1(t) = \sum_{p=-3}^{n-1} u_p Q_4\left(\frac{t}{h} - p\right) \quad \text{and} \quad y_2(t) = \sum_{p=-2}^{n-1} v_p Q_3\left(\frac{t}{h} - p\right),$$

and in a similar way, corresponding to γ_1 and γ_2 , we consider quadratic and linear spline functions $\varphi_1(t)$ and $\varphi_2(t)$ defined by

$$\varphi_1(t) = \sum_{p=-2}^{n-1} c_p Q_3\left(\frac{t}{h} - p\right) \quad \text{and} \quad \varphi_2(t) = \sum_{p=0}^n d_p L_p(t).$$

Since $J(\alpha, \beta)(\xi_1, \xi_2) = \lim_{\theta \rightarrow 0} \theta^{-1} [F(\alpha + \theta \xi_1, \beta + \theta \xi_2) - F(\alpha, \beta)]$, corresponding to (22), we have:

$$(23) \quad \dot{y}_1(t) = y_2(t) - \varphi_1(t) \quad (0 \leq t \leq 1),$$

$$(24) \quad \dot{y}_2(t_p) = f_x\left(t_p, \frac{\alpha_{p-3} + 4\alpha_{p-2} + \alpha_{p-1}}{6}, \frac{\beta_{p-2} + \beta_{p-1}}{2}\right) y_1(t_p) \\ + f_w\left(t_p, \frac{\alpha_{p-3} + 4\alpha_{p-2} + \alpha_{p-1}}{6}, \frac{\beta_{p-2} + \beta_{p-1}}{2}\right) y_2(t_p) \\ - \varphi_2(t_p) \quad (p = 0, 1, \dots, n)$$

$$(25) \quad A_0 y_1(0) - B_0 y_2(0) = -\sqrt{h} e_1,$$

$$(26) \quad A_1 y_1(1) + B_1 y_2(1) = -\sqrt{h} e_2.$$

Now substitute $(\hat{\alpha}, \hat{\beta})$ for (α, β) in (24), then we have

$$\dot{y}_2(t) = P[f_x(t, \hat{x}_h(t), \hat{w}_h(t))y_1(t) + f_w(t, \hat{x}_h(t), \hat{w}_h(t))y_2(t)] - \varphi_2(t) \quad (0 \leq t \leq 1),$$

since $\dot{y}_2(t)$ and $\varphi_2(t)$ are both piecewise linear. Equation (24) can be rewritten as follows:

$$(27) \quad \dot{y}_2(t) = f_x(t, \hat{x}_h(t), \hat{w}_h(t))y_1(t) + f_w(t, \hat{x}_h(t), \hat{w}_h(t))y_2(t) + R(t) - \varphi_2(t) \quad (0 \leq t \leq 1),$$

where $R = -(I - P)[f_x(t, \hat{x}_h(t), \hat{w}_h(t))y_1(t) + f_w(t, \hat{x}_h(t), \hat{w}_h(t))y_2(t)]$, (I the unit operator).

Now by (23) and (27) we have:

$$\begin{aligned} & \frac{d}{dt}[f_x(t, \hat{x}_h(t), \hat{w}_h(t))y_1(t) + f_w(t, \hat{x}_h(t), \hat{w}_h(t))y_2(t)] \\ &= f_x(t, \hat{x}_h(t), \hat{w}_h(t))(y_2(t) - \varphi_1(t)) + f_w(t, \hat{x}_h(t), \hat{w}_h(t)) \\ & \quad \times [f_x(t, \hat{x}_h(t), \hat{w}_h(t)) \cdot y_1(t) + f_w(t, \hat{x}_h(t), \hat{w}_h(t))y_2(t) + R(t) - \varphi_2(t)] \\ & \quad + \frac{d}{dt}f_x(t, \hat{x}_h(t), \hat{w}_h(t)) \cdot y_1(t) + \frac{d}{dt}f_w(t, \hat{x}_h(t), \hat{w}_h(t)) \cdot y_2(t) \end{aligned}$$

for any $t \in [0, 1]$. On the other hand, if h_0 is sufficiently small, by (20) and (21) there is $K_1 > 0$ such that

$$\begin{aligned} & |f_x(t, \hat{x}_h(t), \hat{w}_h(t))|, |f_w(t, \hat{x}_h(t), \hat{w}_h(t))|, \left| \frac{d}{dt}f_x(t, \hat{x}_h(t), \hat{w}_h(t)) \right|, \\ & \left| \frac{d}{dt}f_w(t, \hat{x}_h(t), \hat{w}_h(t)) \right| \leq K_1 \quad (0 \leq t \leq 1) \quad \text{for any } h \leq h_0. \end{aligned}$$

Therefore by Lemma 5 we have

$$\begin{aligned} \|R\| \leq & \frac{h}{\sqrt{6}} [K_1(\|y_2\| + \|\varphi_1\|) + K_1 \{K_1(\|y_1\| + \|y_2\|) + \|R\| + \|\varphi_2\|\}] \\ & + K_1(\|y_1\| + \|y_2\|), \end{aligned}$$

from which follows

$$\left(1 - \frac{h}{\sqrt{6}}K_1\right)\|R\| \leq \frac{h}{\sqrt{6}}K_1[(1 + K_1)\|y_1\| + (2 + K_1)\|y_2\| + \|\varphi_1\| + \|\varphi_2\|].$$

Hence we have the estimate of $\|R\|$ of the form

$$\|R\| \leq \frac{h}{\sqrt{2}}K[(\|y_1\| + \|y_2\|) + (\|\varphi_1\| + \|\varphi_2\|)],$$

or

$$(28) \quad \|R\| \leq hK[\|(y_1, y_2)\| + \|(\varphi_1, \varphi_2)\|],$$

where

$$\|(y_1, y_2)\| = \left[\int_0^1 |(y_1(t), y_2(t))|^2 dt \right]^{\frac{1}{2}} = [\|y_1\|^2 + \|y_2\|^2]^{\frac{1}{2}},$$

$$\|(\varphi_1, \varphi_2)\| = \left[\int_0^1 |(\varphi_1(t), \varphi_2(t))|^2 dt \right]^{\frac{1}{2}} = [\|\varphi_1\|^2 + \|\varphi_2\|^2]^{\frac{1}{2}}.$$

Now let $\Phi_h(t)$ be the fundamental matrix with property $\Phi_h(0) = E$ of the following linear homogeneous system:

$$(29) \quad \dot{y}(t) = z(t)$$

$$\dot{z}(t) = f_x(t, \hat{x}_h(t), \hat{u}_h(t))y(t) + f_w(t, \hat{x}_h(t), \hat{u}_h(t))z(t),$$

then comparing (29) with (18), by (20) and (21), we have

$$(30) \quad |\Phi_h(t) - \Phi(t)| = O(h^3) \quad (0 \leq t \leq 1),$$

therefore we have a non-singular matrix G_h associated with (29) corresponding to G associated with (18) by (17). Then applying Proposition 1 of [4] (p. 44) to the system of equations (23), (27), (25) and (26), we have

$$(31) \quad \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = -\Phi_h(t)G_h^{-1} \begin{pmatrix} \sqrt{h}e_1 \\ \sqrt{h}e_2 \end{pmatrix} + \int_0^1 H_h(t, s) \begin{pmatrix} -\varphi_1(s) \\ R(s) - \varphi_2(s) \end{pmatrix} ds,$$

where

$$(32) \quad H_h(t, s) = \begin{cases} \Phi_h(t) \left[E - G_h^{-1} \begin{pmatrix} 0 & 0 \\ A_1 & B_1 \end{pmatrix} \Phi_h(1) \right] \Phi_h^{-1}(s) & (s \leq t) \\ -\Phi_h(t) G_h^{-1} \begin{pmatrix} 0 & 0 \\ A_1 & B_1 \end{pmatrix} \Phi_h(1) \Phi_h^{-1}(s) & (t < s). \end{cases}$$

From (31), by (30) we have the inequality of the form

$$\|(y_1, y_2)\| \leq \sqrt{h} M_1 |(e_1, e_2)| + M_2 \|(-\varphi_1, R - \varphi_2)\|$$

for any $h \leq h_0$ provided h_0 is sufficiently small. Here M_1 and M_2 are positive constants independent of h . Since

$$\|(-\varphi_1, R - \varphi_2)\| \leq \|R\| + \|\varphi_1\| + \|\varphi_2\| \leq \|R\| + \sqrt{2} \|(\varphi_1, \varphi_2)\|,$$

by the use of (28) we then have

$$(1 - hKM_2) \|(y_1, y_2)\| \leq \sqrt{h} M_1 |(e_1, e_2)| + M_2(\sqrt{2} + hK) \|(\varphi_1, \varphi_2)\|,$$

from which we obtain the inequality of the form

$$\|(y_1, y_2)\| \leq \sqrt{h} M_3 |(e_1, e_2)| + M_4 \|(\varphi_1, \varphi_2)\|$$

for any $h \leq h_0$ provided h_0 is sufficiently small. Now by Lemmas 1, 2 and 3,

$$|\xi_1| \leq \frac{1}{\mu_3} \cdot \frac{1}{\sqrt{h}} \|y_1\|, \quad |\xi_2| \leq \frac{1}{\mu_2} \cdot \frac{1}{\sqrt{h}} \|y_2\|,$$

$$\|\varphi_1\| \leq \lambda_2 \sqrt{h} |\gamma_1|, \quad \|\varphi_2\| \leq \lambda_1 \sqrt{h} |\gamma_2|.$$

Therefore we finally have the inequality of the form

$$(33) \quad |(\xi_1, \xi_2)| \leq M |(\gamma_1, \gamma_2, \gamma_3)|$$

for any $h \leq h_0$ provided h_0 is sufficiently small. Here M is a positive constant independent of h . By (22), inequality (33) implies the non-singularity of $J_h(\hat{\alpha}, \hat{\beta})$ and in addition the inequality

$$(34) \quad |J_h^{-1}(\hat{\alpha}, \hat{\beta})| \leq M \quad \text{for any } h \leq h_0.$$

Let $\alpha = (\alpha_{-3}, \alpha_{-2}, \dots, \alpha_{n-1})$, $\beta = (\beta_{-2}, \beta_{-1}, \dots, \beta_{n-1})$, $\alpha' = (\alpha'_{-3}, \alpha'_{-2}, \dots, \alpha'_{n-1})$, $\beta' = (\beta'_{-2}, \beta'_{-1}, \dots, \beta'_{n-1})$ be arbitrary vectors that

$$(35) \quad \left\{ \left[\frac{1}{6}(\alpha_{p-3} + 4\alpha_{p-2} + \alpha_{p-1}) - \hat{x}(t_p) \right]^2 + \left[\frac{1}{2}(\beta_{p-2} + \beta_{p-1}) - \hat{w}(t_p) \right]^2 \right\}^{\frac{1}{2}} \leq \delta,$$

$$\left\{ \left[\frac{1}{6}(\alpha'_{p-3} + 4\alpha'_{p-2} + \alpha'_{p-1}) - \hat{x}(t_p) \right]^2 + \left[\frac{1}{2}(\beta'_{p-2} + \beta'_{p-1}) - \hat{w}(t_p) \right]^2 \right\}^{\frac{1}{2}} \leq \delta$$

($p=0, 1, \dots, n$).

Put

$$\frac{\partial}{\partial \alpha_{q_1}} f \left[t_p, \frac{1}{6}(\alpha_{p-3} + 4\alpha_{p-2} + \alpha_{p-1}), \frac{1}{2}(\beta_{p-2} + \beta_{p-1}) \right] = J_{p q_1}^{(2)}(\alpha, \beta),$$

$$\frac{\partial}{\partial \beta_{q_2}} f \left[t_p, \frac{1}{6}(\alpha_{p-3} + 4\alpha_{p-2} + \alpha_{p-1}), \frac{1}{2}(\beta_{p-2} + \beta_{p-1}) \right] = J_{p q_2}^{(2)}(\alpha, \beta),$$

then from (16) it is easily seen that

$$(36) \quad |J_h(\alpha, \beta) - J_h(\alpha', \beta')|^2 \leq \sum_{p=0}^n \left\{ \sum_{q_1} [J_{p q_1}^{(2)}(\alpha, \beta) - J_{p q_1}^{(2)}(\alpha', \beta')]^2 \right. \\ \left. + \sum_{q_2} [J_{p q_2}^{(2)}(\alpha, \beta) - J_{p q_2}^{(2)}(\alpha', \beta')]^2 \right\}.$$

Now

$$J_{p q_1}^{(2)}(\alpha, \beta) = f_x \left[t_p, \frac{1}{6}(\alpha_{p-3} + 4\alpha_{p-2} + \alpha_{p-1}), \frac{1}{2}(\beta_{p-2} + \beta_{p-1}) \right] \\ \frac{1}{6}(\delta_{q_1, p-3} + 4\delta_{q_1, p-2} + \delta_{q_1, p-1}),$$

$$J_{p q_2}^{(2)}(\alpha, \beta) = f_w \left[t_p, \frac{1}{6}(\alpha_{p-3} + 4\alpha_{p-2} + \alpha_{p-1}), \frac{1}{2}(\beta_{p-2} + \beta_{p-1}) \right] \\ \frac{1}{2}(\delta_{q_2, p-2} + \delta_{q_2, p-1}).$$

Hence by means of the mean-value theorem we have

$$(37) \quad \sum_{q_1} [J_{p q_1}^{(2)}(\alpha, \beta) - J_{p q_1}^{(2)}(\alpha', \beta')]^2 + \sum_{q_2} [J_{p q_2}^{(2)}(\alpha, \beta) - J_{p q_2}^{(2)}(\alpha', \beta')]^2 \\ \leq C^2 \left[\{(\alpha_{p-3} - \alpha'_{p-3})^2 + (\alpha_{p-2} - \alpha'_{p-2})^2 + (\alpha_{p-1} - \alpha'_{p-1})^2\} \right. \\ \left. + \{(\beta_{p-2} - \beta'_{p-2})^2 + (\beta_{p-1} - \beta'_{p-1})^2\} \right],$$

where C is positive constant such that

$$|f_{xx}(t, x, w)|, |f_{xw}(t, x, w)|, |f_{ww}(t, x, w)| \leq C$$

for all $(t, x, w) \in U$. By (36) and (37), we thus obtain

$$(38) \quad |J_h(\alpha, \beta) - J_h(\alpha', \beta')| \leq M' |(\alpha - \alpha', \beta - \beta')|$$

where $M' = \sqrt{3} C$. Clearly M' is independent of h .

4. Existence of Spline Approximations

First let us estimate $|F(\hat{\alpha}, \hat{\beta})|$. Since $\hat{w}_h(t) = \hat{x}_h(t)$ ($0 \leq t \leq 1$), we have

$$F_h^{(1)}(\hat{\alpha}, \hat{\beta}) = \left\{ \frac{\hat{\alpha}_p - \hat{\alpha}_{p-1}}{h} - \hat{\beta}_p \right\}_{p=-2}^{n-1} = 0.$$

Further, since

$$\hat{x}_h(t_p) = \hat{x}(t_p), \quad \hat{x}_h'(t_p) = \hat{x}'(t_p) \quad (p=0, n)$$

by (19), we have

$$F_h^{(3)}(\hat{\alpha}, \hat{\beta}) = \left\{ \frac{1}{\sqrt{h}} A_0 \hat{x}_h(0) - \frac{1}{\sqrt{h}} B_0 \hat{w}_h(0) - \frac{a}{\sqrt{h}}, \right. \\ \left. \frac{1}{\sqrt{h}} A_1 \hat{x}_h(1) + \frac{1}{\sqrt{h}} B_1 \hat{w}_h(1) - \frac{b}{\sqrt{h}} \right\} = 0.$$

For $F_h^{(2)}(\hat{\alpha}, \hat{\beta})$, by (20) and (21), we have

$$F_h^{(2)}(\hat{\alpha}, \hat{\beta}) = \{ \hat{w}_h(t_p) - f(t_p, \hat{x}_h(t_p), \hat{w}_h(t_p)) \}_{p=0}^n \\ = \{ [\hat{w}_h(t_p) - \hat{w}(t_p)] \\ + [f(t_p, \hat{x}(t_p), \hat{w}(t_p)) - f(t_p, \hat{x}_h(t_p), \hat{w}_h(t_p))] \}_{p=0}^n \\ = O(h^2) \quad (h \rightarrow 0).$$

Therefore we have

$$|F_h^{(2)}(\hat{\alpha}, \hat{\beta})| = \sqrt{n+1} O(h^2).$$

Since $n = \frac{1}{h}$, we then have

$$|F_h^{(2)}(\hat{\alpha}, \hat{\beta})| = O(h^{\frac{3}{2}}).$$

Thus we see that

$$(39) \quad |F_h(\hat{\alpha}, \hat{\beta})| \leq Lh^{\frac{3}{2}}$$

for any $h \leq h_1$ provided h_1 is sufficiently small. Here clearly L is a positive constant independent of h . Inequality (39) expresses that $(\alpha, \beta) = (\hat{\alpha}, \hat{\beta})$ is an approximate solution of $F_h(\alpha, \beta) = 0$. Hence we apply Proposition 2 of [3] (p. 123) to the determining equation $F_h(\alpha, \beta) = 0$ to prove the existence of its solution, that is, the existence of spline approximations to the solution of the problem (4)–(7). By (20) and (21), let us note that there is a positive constant N such that

$$(40) \quad \|(\hat{x}_h, \hat{w}_h) - (\hat{x}, \hat{w})\|_\infty = \sup_{0 \leq t \leq 1} |(\hat{x}_h(t), \hat{w}_h(t)) - (\hat{x}(t), \hat{w}(t))| \leq Nh^3$$

for any $h \leq h_2$ provided h_2 sufficiently small. We suppose that $Nh_2^3 < \delta$, and consider the set V_h defined by

$$V_h = \{(t, x, w) \mid |(x, w) - (\hat{x}_h(t), \hat{w}_h(t))| \leq \delta - Nh^3, t \in [0, 1]\}.$$

Then by (40) it is clear that

$$(41) \quad V_h \subset U \quad \text{for any } h \leq h_2.$$

Let Ω_h be the set defined by

$$\Omega_h = \{(\alpha, \beta) \mid |(\alpha, \beta) - (\hat{\alpha}, \hat{\beta})| \leq \delta - Nh^3\},$$

then for

$$x_h(t) = \sum_{p=-3}^{n-1} \alpha_p Q_4\left(\frac{t}{h} - p\right) \quad \text{and} \quad w_h(t) = \sum_{p=-2}^{n-1} \beta_p Q_3\left(\frac{t}{h} - p\right)$$

with $(\alpha, \beta) \in \Omega_h$, by Lemma 4 we have

$$|x_h(t) - \hat{x}_h(t)|^2 + |w_h(t) - \hat{w}_h(t)|^2 \leq |\alpha - \hat{\alpha}| + |\beta - \hat{\beta}| \quad (0 \leq t \leq 1),$$

consequently $(t, x_h(t), w_h(t)) \in V_h \subset U$ for any $t \in [0, 1]$. This means that $F_h(\alpha, \beta)$ is defined on the region Ω_h for any $h \leq h_2$.

Now by (34) and (38) it holds that

$$(42) \quad |J_h^{-1}(\hat{\alpha}, \hat{\beta})| \leq M \quad \text{for any } h \leq h_0,$$

and

$$(43) \quad |J_h(\alpha, \beta) - J_h(\hat{\alpha}, \hat{\beta})| \leq M |(\alpha, \beta) - (\hat{\alpha}, \hat{\beta})|$$

for any $h \leq h_2$ and any $(\alpha, \beta) \in \Omega_h$.

Take an arbitrary positive number $k < 1$ and put $\delta_1 = \min\left[\frac{k}{MM'}, \delta - Nh_2^3\right]$. If we take sufficiently small $h_3 \leq \min(h_0, h_1, h_2)$, then it is possible to take δ_h so that

$$(44) \quad \frac{MLh^{\frac{3}{2}}}{1-k} \leq \delta_h \leq \delta_1 \quad \text{for any } h \leq h_3.$$

Let Ω_{δ_h} be the set defined by

$$\Omega_{\delta_h} = \{(\alpha, \beta) \mid |(\alpha, \beta) - (\hat{\alpha}, \hat{\beta})| \leq \delta_h\}.$$

Then for any $h \leq h_3$ and any $(\alpha, \beta) \in \Omega_{\delta_h}$, we have $|(\alpha, \beta) - (\hat{\alpha}, \hat{\beta})| \leq \delta_h \leq \delta - Nh_2^3 \leq \delta - Nh^3$, which means $(\alpha, \beta) \in \Omega_h$. Hence we see that

$$(45) \quad \Omega_{\delta_h} \subset \Omega_h \quad \text{for any } h \leq h_3.$$

Now for any $h \leq h_3$ and any $(\alpha, \beta) \in \Omega_{\delta_h}$, by (43) we have

$$(46) \quad |J_h(\alpha, \beta) - J_h(\hat{\alpha}, \hat{\beta})| \leq M |(\alpha, \beta) - (\hat{\alpha}, \hat{\beta})| \leq M \delta_h \leq M' \delta_1 \leq \frac{k}{M}.$$

Moreover by (39) and (44) we have:

$$(47) \quad \frac{M |F_h(\hat{\alpha}, \hat{\beta})|}{1-k} \leq \frac{MLh^{\frac{3}{2}}}{1-k} \leq \delta_h \quad \text{for any } h \leq h_3.$$

The expressions (42), (45), (46) and (47) show that the conditions of Proposition 2 of [3] (p. 123) are all fulfilled. Thus we see that the determining equation $F_h(\alpha, \beta)=0$ has one and only one solution $(\alpha, \beta)=(\bar{\alpha}, \bar{\beta})$ in \mathcal{Q}_{δ_h} . This proves the existence of spline approximations to the solution of the problem (4)–(7).

5. Convergence of Spline Approximations

By Proposition 2 of [3] (p. 123), the solution $(\bar{\alpha}, \bar{\beta})$ of the determining equation $F_h(\alpha, \beta)=0$ satisfies the inequality

$$(48) \quad |(\bar{\alpha}, \bar{\beta}) - (\hat{\alpha}, \hat{\beta})| \leq \frac{MLh^{\frac{3}{2}}}{1-k} \quad \text{for any } h \leq h_3.$$

Let $\bar{\alpha}=(\bar{\alpha}_{-3}, \bar{\alpha}_{-2}, \dots, \bar{\alpha}_{n-1})$ and $\bar{\beta}=(\bar{\beta}_{-2}, \bar{\beta}_{-1}, \dots, \bar{\beta}_{n-1})$, and put

$$\bar{x}_h(t) = \sum_{p=-3}^{n-1} \bar{\alpha}_p Q_4\left(\frac{t}{h} - p\right),$$

$$\bar{w}_h(t) = \sum_{p=-2}^{n-1} \bar{\beta}_p Q_3\left(\frac{t}{h} - p\right).$$

Since $F_h^{(1)}(\bar{\alpha}, \bar{\beta})=0$, it is clear that

$$(49) \quad \dot{\bar{x}}_h(t) = \bar{w}_h(t) \quad (0 \leq t \leq 1).$$

From (48) follows

$$|\bar{\alpha} - \hat{\alpha}|, |\bar{\beta} - \hat{\beta}| = O\left(h^{\frac{3}{2}}\right) \quad (h \rightarrow 0).$$

Therefore by Lemma 3 and 2 we have

$$(50) \quad \|\bar{x}_h - \hat{x}_h\|, \|\bar{w}_h - \hat{w}_h\| = O(h^2) \quad (h \rightarrow 0),$$

and by Lemma 4 we have

$$(51) \quad \|\bar{x}_h - \hat{x}_h\|_{\infty}, \|\bar{w}_h - \hat{w}_h\|_{\infty} = O\left(h^{\frac{3}{2}}\right) \quad (h \rightarrow 0).$$

On the other hand,

$$\|\hat{x}_h - \hat{x}\|_{\infty} = O(h^4) \quad (h \rightarrow 0)$$

by (20), and

$$\|\hat{w}_h - \hat{w}\|_\infty = O(h^3) \quad (h \rightarrow 0)$$

by (21). Hence from (50) and (51) we have:

$$\|\bar{x}_h - \hat{x}\|, \|\bar{w}_h - \hat{w}\| = O(h^2) \quad (h \rightarrow 0)$$

and

$$\|\bar{x}_h - \hat{x}\|_\infty, \|\bar{w}_h - \hat{w}\|_\infty = O(h^{\frac{3}{2}}) \quad (h \rightarrow 0).$$

Since $\dot{\bar{x}}_h(t) = \bar{w}_h(t)$ ($0 \leq t \leq 1$) by (49) and $\hat{w}(t) = \dot{\hat{x}}(t)$ by the definition, we thus have

Theorem. *In a sufficiently small neighborhood of the isolated solution $\hat{x}(t)$ of the problem (1)-(3), there is a cubic spline function $\bar{x}_h(t)$ of the form*

$$\bar{x}_h(t) = \sum_{p=-3}^{n-1} \bar{\alpha}_p Q_4\left(\frac{t}{h} - p\right)$$

such that

(i) *the coefficient $\bar{\alpha} = (\bar{\alpha}_{-3}, \bar{\alpha}_{-2}, \dots, \bar{\alpha}_{n-1})$ satisfies the determining equation $F_h(\alpha, \beta) = 0$ together with the coefficient $\bar{\beta} = (\bar{\beta}_{-2}, \bar{\beta}_{-1}, \dots, \bar{\beta}_{n-1})$ of the quadratic function defined by*

$$\bar{w}_h(t) = \dot{\bar{x}}_h(t) = \sum_{p=-2}^{n-1} \bar{\beta}_p Q_3\left(\frac{t}{h} - p\right) \quad (0 \leq t \leq 1),$$

(ii)

$$(52) \quad \|\bar{x}_h - \hat{x}\|, \|\dot{\bar{x}}_h - \dot{\hat{x}}\| = O(h^2) \quad (h \rightarrow 0)$$

and

$$(53) \quad \|\bar{x}_h - \hat{x}\|_\infty, \|\dot{\bar{x}}_h - \dot{\hat{x}}\|_\infty = O(h^{\frac{3}{2}}) \quad (h \rightarrow 0).$$

The expression (53) clearly implies the uniform convergence of $\bar{x}_h(t)$ to the exact solution $\hat{x}(t)$ on $[0, 1]$ together with its first order derivative.

Remark. If $B_0 = 0$, then $\bar{x}_h(0) = \hat{x}(0)$ by (14). Then applying

Schwarz's inequality to the equality

$$\bar{x}_h(t) - \hat{x}(t) = \int_0^1 [\dot{\bar{x}}_h(s) - \dot{\hat{x}}(s)] ds,$$

we have

$$|\bar{x}_h(t) - \hat{x}(t)| \leq \|\dot{\bar{x}}_h - \dot{\hat{x}}\| = O(h^2) \quad (0 \leq t \leq 1),$$

that is,

$$(54) \quad \|\bar{x}_h - \hat{x}\|_\infty = O(h^2) \quad (h \rightarrow 0).$$

One can see in a similar way equality (54) is valid also when $B_1 = 0$. Thus if $B_0 = 0$ or $B_1 = 0$, we always have (54).

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