Publ. RIMS Kyoto Univ. **50** (2014), 695–721
DOI 10.4171/PRIMS/147

Normalizers inside Amalgamated Free Product von Neumann Algebras

by

Stefaan VAES

Abstract

Recently, Adrian Ioana proved that all crossed products $L^\infty(X) \rtimes (\Gamma_1 * \Gamma_2)$ by free ergodic probability measure preserving actions of a nontrivial free product group $\Gamma_1 * \Gamma_2$ have a unique Cartan subalgebra up to unitary conjugacy. Ioana deduced this result from a more general dichotomy theorem on the normalizer $\mathcal{N}_M(A)''$ of an amenable subalgebra A of an amalgamated free product von Neumann algebra $M = M_1 *_B M_2$. We improve this dichotomy theorem by removing the spectral gap assumptions and obtain in particular a simpler proof for the uniqueness of the Cartan subalgebra in $L^\infty(X) \rtimes (\Gamma_1 * \Gamma_2)$.

2010 Mathematics Subject Classification: Primary 46L36; Secondary 46L54, 37A40.
Keywords: amalgamated free products, Cartan subalgebras, deformation/rigidity theory, nonsingular group actions.

§1. Introduction and main results

Each free ergodic nonsingular group action $\Gamma \curvearrowright (X, \mu)$ on a standard probability space gives rise to a crossed product von Neumann algebra $L^\infty(X) \rtimes \Gamma$, in which $L^\infty(X)$ is a *Cartan subalgebra.* More generally, Cartan subalgebras arise as $L^\infty(X) \subset L(\mathcal{R})$ where $\mathcal{R}$ is a countable nonsingular Borel equivalence relation on (X, μ). One of the main questions in the classification of these von Neumann algebras $L^\infty(X) \rtimes \Gamma$ and $L(\mathcal{R})$ is whether or not $L^\infty(X)$ is their unique Cartan subalgebra up to unitary conjugacy. Indeed, if uniqueness holds, the classification problem is reduced to classifying the underlying (orbit) equivalence relations.

Within Popa's deformation/rigidity theory, there has been a lot of recent progress on the uniqueness of Cartan subalgebras in II_1 factors, starting with [OP07] where it was shown that all crossed products $L^\infty(X) \rtimes \mathbb{F}_n$ by free ergodic probability measure preserving (pmp) *profinite* actions of the free groups $\mathbb{F}_n$ have

Communicated by N. Ozawa. Received October 3, 2013.

S. Vaes: Department of Mathematics, KU Leuven, Leuven, Belgium;
e-mail: `stefaan.vaes@wis.kuleuven.be`

a unique Cartan subalgebra. Note that this provided the first uniqueness theorem for Cartan subalgebras up to unitary conjugacy. The result of [OP07] was gradually extended to profinite actions of larger classes of groups Γ in [OP08, CS11, CSU11], but all relied on profiniteness of the action and weak amenability of the group Γ. At the same time, it was conjectured that crossed products $L^\infty(X) \rtimes \mathbb{F}_n$ by actions of the free groups could have a unique Cartan subalgebra without any profiniteness assumptions on $\mathbb{F}_n \curvearrowright (X, \mu)$.

In a joint work with Popa [PV11, PV12], we solved this conjecture and proved that the free groups $\Gamma = \mathbb{F}_n$ and all nonelementary hyperbolic groups Γ are $\mathcal{C}$-rigid (Cartan-rigid), i.e. for every free ergodic pmp action $\Gamma \curvearrowright (X, \mu)$, the II_1 factor $L^\infty(X) \rtimes \Gamma$ has a unique Cartan subalgebra up to unitary conjugacy. We obtained this result as a consequence of a general dichotomy theorem about normalizers of amenable subalgebras in crossed product von Neumann algebras $N \rtimes \Gamma$, arising from trace preserving actions of such groups Γ on arbitrary tracial (N, τ).

Then in [Io12], the general dichotomy result of [PV11] was exploited to establish $\mathcal{C}$-rigidity for arbitrary nontrivial free products $\Gamma = \Gamma_1 * \Gamma_2$ and large classes of amalgamated free products $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$. This provided in particular the first non-weakly amenable $\mathcal{C}$-rigid groups. The main idea of [Io12] is to use the free malleable deformation from [IPP05] of a crossed product $B \rtimes (\Gamma_1 * \Gamma_2)$, providing a 1-parameter family of embeddings $\theta_t : B \rtimes (\Gamma_1 * \Gamma_2) \to N \rtimes \mathbb{F}_2$ into some crossed product by the free group $\mathbb{F}_2$. Then the main result of [PV11] is applied to this crossed product $N \rtimes \mathbb{F}_2$ and a very careful and delicate analysis is needed to "come back" and deduce results about the original crossed product $B \rtimes (\Gamma_1 * \Gamma_2)$.

The purpose of this article is to give a simpler approach to this "come back" procedure and, at the same time, prove a more general result removing the spectral gap assumptions of [Io12]. As a result, we obtain a simpler proof of the $\mathcal{C}$-rigidity of amalgamated free product groups.

Our method allows us to prove a more generic theorem about the normalizer of a subalgebra inside an amalgamated free product of von Neumann algebras—see Theorem A below. This theorem has the advantage of immediately implying a similar result for HNN extensions of von Neumann algebras (see Theorem 4.1). Thus we obtain, without extra effort, $\mathcal{C}$-rigidity for a large class of HNN extensions $\Gamma = \mathrm{HNN}(\Gamma_1, \Sigma, \theta)$, established before in [DI12] using more involved methods.

As we explain below, following the strategy of [HV12], we also prove a uniqueness theorem for Cartan subalgebras in type III factors. This then allows us to give first examples of type III actions $\Gamma \curvearrowright (X, \mu)$ that are W*-superrigid, i.e. such that the group Γ and its action $\Gamma \curvearrowright (X, \mu)$ can be recovered from $L^\infty(X) \rtimes \Gamma$, up to induction of actions.

To state the main result of the article, we first recall Popa's theory of *intertwining-by-bimodules* from [Po01, Po03]. When (M, τ) is a tracial von Neumann algebra and $A \subset pMp$ and $B \subset M$ are von Neumann subalgebras, we say that $A \prec_M B$ (A embeds into B inside M) if $L^2(pM)$ admits a nonzero A-B-subbimodule that is finitely generated as a right Hilbert B-module. This is "almost" equivalent to the existence of a partial isometry $v \in B$ such that $vAv^* \subset B$. By [Po03, Theorem 2.1 and Corollary 2.3], the negation $A \not\prec_M B$ is equivalent to the existence of a net $(a_i)_{i \in I}$ of unitaries in $\mathcal{U}(A)$ satisfying $\lim_i \|E_B(xu_iy)\|_2 = 0$ for all $x, y \in M$.

Also recall from [OP07, Definition 2.2] that A is said to be *amenable relative to B inside M* if there exists an A-central state Ω on Jones' basic construction von Neumann algebra $p\langle M, e_B\rangle p$ satisfying $\Omega(x) = \tau(x)$ for all $x \in pMp$. When B is amenable, this is equivalent to A being amenable. When $M = D \rtimes \Gamma$ and $\Lambda, \Sigma < \Gamma$ are subgroups, then the relative amenability of $D \rtimes \Lambda$ with respect to $D \rtimes \Sigma$ is equivalent to the relative amenability of Λ with respect to Σ inside Γ, i.e. to the existence of a Λ-invariant mean on Γ/Σ.

The following is the main result of the article. The same result was proven in [Io12, Theorem 1.6] under the extra assumption that the normalizer $\mathcal{N}_{pMp}(A) = \{u \in \mathcal{U}(pMp) \mid uAu^* = A\}$ of A inside pMp has spectral gap.

Theorem A. *Let (M_i, τ_i) be tracial von Neumann algebras with a common von Neumann subalgebra $B \subset M_i$ satisfying $\tau_{1|B} = \tau_{2|B}$. Denote by $M = M_1 *_B M_2$ the amalgamated free product with respect to the unique trace preserving conditional expectations. Let $p \in M$ be a nonzero projection and $A \subset pMp$ a von Neumann subalgebra that is amenable relative to one of the M_i inside M. Then at least one of the following statements holds:*

- *$A \prec_M B$.*
- *There is an $i \in \{1, 2\}$ such that $\mathcal{N}_{pMp}(A)'' \prec_M M_i$.*
- *$\mathcal{N}_{pMp}(A)''$ is amenable relative to B inside M.*

As in [Io12], several uniqueness theorems for Cartan subalgebras can be deduced from Theorem A. This is in particular the case for II_1 factors $M = L(\mathcal{R})$ that arise from a countable pmp equivalence relation $\mathcal{R}$ that can be decomposed as a free product $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ of subequivalence relations $\mathcal{R}_i \subset \mathcal{R}$. Since we now no longer need to prove the spectral gap assumption, we can directly deduce from Theorem A the following improvement of [Io12, Corollary 1.4] and [BHR12, Theorem 6.3].

Corollary B. *Let $\mathcal{R}$ be a countable ergodic pmp equivalence relation on the standard probability space (X, μ). Assume that $\mathcal{R} = \mathcal{R}_1 * \mathcal{R}_2$ for two subequivalence* *relations* $\mathcal{R}_i \subset \mathcal{R}$. *Assume that* $|\mathcal{R}_1 \cdot x| \geq 3$ *and* $|\mathcal{R}_2 \cdot x| \geq 2$ *for a.e.* $x \in X$. *Then* $L^\infty(X)$ *is the unique Cartan subalgebra of* $L(\mathcal{R})$ *up to unitary conjugacy.*

A tracial von Neumann algebra (M,τ) is called *strongly solid* if for every diffuse amenable von Neumann subalgebra $A \subset M$, the normalizer $\mathcal{N}_M(A)''$ is still amenable. For completeness, we also show how to deduce from Theorem A the following stability result for strong solidity under amalgamated free products, slightly improving on [Io12, Theorem 1.8].

For the formulation of the result, recall from [Po03, Section 3] that an inclusion $B \subset (M_1,\tau)$ of tracial von Neumann algebras is called *mixing* if for every sequence $b_n \in B$ with $\|b_n\| \leq 1$ for all n and $b_n \to 0$ weakly, we have $\lim_n \|E_B(xb_ny)\|_2 = 0$ for all $x, y \in M_1 \ominus B$. Typical examples of mixing inclusions arise as $L(\Sigma) \subset L(\Gamma)$ when $\Sigma < \Gamma$ is a subgroup such that $g\Sigma g^{-1} \cap \Sigma$ is finite for all $g \in \Gamma - \Sigma$, or as $L(\Sigma) \subset B \rtimes \Sigma$ whenever Σ acts in a mixing and trace preserving way on (B,τ).

Corollary C. *Let* (M_i,τ_i) *be strongly solid von Neumann algebras with a common amenable von Neumann subalgebra* $B \subset M_i$ *satisfying* $\tau_{1|B} = \tau_{2|B}$. *Assume that the inclusion* $B \subset M_1$ *is mixing. Denote by* $M = M_1 *_B M_2$ *the amalgamated free product with respect to the unique trace preserving conditional expectations. Then* M *is strongly solid.*

On the level of tracial von Neumann algebras, by [Ue07], amalgamated free products and HNN extensions are one and the same thing, up to amplifications. Therefore, Theorem A has an immediate counterpart for HNN extensions that we formulate as Theorem 4.1 below.

As a consequence, we can reprove [Io12, Theorem 1.1] and [DI12, Corollary 1.7], showing $\mathcal{C}$-rigidity for amalgamated free product groups, HNN extensions and their direct products. We refer to Theorem 5.1 for a precise statement.

Finally in Section 8, we use the methods of [HV12] to deduce from Theorem A a uniqueness theorem for Cartan subalgebras in type III factors $L^\infty(X) \rtimes \Gamma$ arising from nonsingular free ergodic actions of amalgamated free product groups (see Theorem 8.1), generalizing [BHR12, Theorem D]. As a consequence, we can provide the following first nonsingular actions *of type III* that are W*-superrigid.

Proposition D. *Consider the linear action of* $\mathrm{SL}(5,\mathbb{Z})$ *on* $\mathbb{R}^5$ *and define the subgroup* $\Sigma < \mathrm{SL}(5,\mathbb{Z})$ *of matrices* A *satisfying* $Ae_i = e_i$ *for* $i = 1, 2$. *Put* $\Gamma = \mathrm{SL}(5,\mathbb{Z}) *_\Sigma (\Sigma \times \mathbb{Z})$ *and denote by* $\pi : \Gamma \to \mathrm{SL}(5,\mathbb{Z})$ *the natural quotient homomorphism. The diagonal action* $\Gamma \curvearrowright \mathbb{R}^5/\mathbb{R}_+ \times [0,1]^\Gamma$ *given by*

$$g \cdot (x,y) = (\pi(g) \cdot x, g \cdot y),$$

where $g \cdot y$ is given by the Bernoulli shift, is a nonsingular free ergodic action of type III_1 that is W^*-superrigid.

This means that for any nonsingular free action $\Gamma' \curvearrowright (X', \mu')$, the following two statements are equivalent:

- $L^\infty(X) \rtimes \Gamma \cong L^\infty(X') \rtimes \Gamma'$.
- *There exists an embedding of Γ into Γ' such that $\Gamma' \curvearrowright X'$ is conjugate to the induction of $\Gamma \curvearrowright X$ to a Γ'-action.*

To clarify the statement of Proposition D, one should make the following observations. In contrast to the case of probability measure preserving actions, it is not relevant to consider stable isomorphisms, since the type III factor $M = L^\infty(X) \rtimes \Gamma$ is isomorphic to $B(H) \mathbin{\overline{\otimes}} M$ for every separable Hilbert space H. For the same reason, it is unavoidable that $\Gamma' \curvearrowright (X', \mu')$ can be any induction of $\Gamma \curvearrowright (X, \mu)$ and need not be conjugate to $\Gamma \curvearrowright (X, \mu)$ itself.

It is also possible to prove that for $0 < \lambda < 1$, the analogous action of Γ on $\mathbb{R}^5/\lambda^{\mathbb{Z}} \times [0,1]^\Gamma$ is of type III_λ and W*-superrigid in the appropriate sense. The correct formulation is necessarily more intricate because the action is by construction orbit equivalent to the action of $\Gamma \times \mathbb{Z}$ on $\mathbb{R}^5 \times [0,1]^\Gamma$. More generally for a type III_λ free ergodic action $\Gamma \curvearrowright (X, \mu)$, there is always a canonically orbit equivalent action $\Gamma \times \mathbb{Z} \curvearrowright (X', \mu')$ where the Γ-action preserves the infinite measure μ' and the $\mathbb{Z}$-action scales μ' by powers of λ.

§2. Preliminaries

In the proof of our main technical result (Theorem 3.4), we make use of the following criterion for relative amenability due to [OP07] (see also [PV11, Section 2.5]). We copy the formulation of [Io12, Lemma 2.3].

Lemma 2.1 ([OP07, Corollary 2.3]). *Let (M, τ) be a tracial von Neumann algebra and $p \in M$ a nonzero projection. Let $A \subset pMp$ and $B \subset M$ be von Neumann subalgebras. Let $\mathcal{L}$ be any B-M-bimodule. Assume that there exists a net $(\xi_i)_{i \in I}$ of vectors in $pL^2(M) \otimes_B \mathcal{L}$ with the following properties:*

- $\limsup_{i \in I} \|x\xi_i\|_2 \leq \|x\|_2$ *for every $x \in pMp$.*
- $\limsup_{i \in I} \|\xi_i\|_2 > 0$.
- $\lim_{i \in I} \|a\xi_i - \xi_i a\|_2 = 0$ *for every $a \in \mathcal{U}(A)$.*

Then there exists a nonzero projection q in the center of $A' \cap pMp$ such that Aq is amenable relative to B inside M.

§3. Key technical theorem

Throughout this section, we fix tracial von Neumann algebras (M_i, τ_i) with a common von Neumann subalgebra $B \subset M_i$ satisfying $\tau_{1|B} = \tau_{2|B}$. We denote by $M = M_1 *_B M_2$ the amalgamated free product with respect to the unique trace preserving conditional expectations and denote its canonical trace by τ.

§3.1. The malleable deformation of an amalgamated free product

We recall from [IPP05, Section 2.2] the construction of Popa's malleable deformation of M. We denote $G = \mathbb{F}_2$, with free generators $a, b \in G$. Write $G_1 = a^{\mathbb{Z}}$ and $G_2 = b^{\mathbb{Z}}$. We define $\widetilde{M} = M *_B (B \mathbin{\overline{\otimes}} L(G))$. Writing $\widetilde{M}_i = M_i *_B (B \mathbin{\overline{\otimes}} L(G_i))$, we can also view $\widetilde{M} = \widetilde{M}_1 *_B \widetilde{M}_2$. Define the self-adjoint elements $h_j \in L(G_j)$ with spectrum $[-\pi, \pi]$ such that $u_a = \exp(ih_1)$ and $u_b = \exp(ih_2)$. Consider the 1-parameter groups $(u_{j,t})_{t\in\mathbb{R}}$ of unitaries in $L(G_j)$ given by $u_{j,t} = \exp(ith_j)$. Finally define the 1-parameter group $(\theta_t)_{t\in\mathbb{R}}$ of automorphisms of $\widetilde{M}$ by

$$\theta_t(x) = u_{j,t} x u_{j,t}^* \quad \text{for all } x \in \widetilde{M}_j.$$

Note that θ_t is well defined because $u_{j,t} b u_{j,t}^* = b$ for all $b \in B$ and $j \in \{1, 2\}$.

We define $\mathcal{S}$ as the set of all finite alternating sequences of 1's and 2's, including the empty sequence $\emptyset$. So the elements of $\mathcal{S}$ are the finite sequences of the form $(1, 2, 1, 2, \dots)$ or $(2, 1, 2, 1, \dots)$. The length of an alternating sequence $\mathcal{I} \in \mathcal{S}$ is denoted by $|\mathcal{I}|$. For every $(i_1, \dots, i_n) \in \mathcal{S}$, we define $\mathcal{H}_{(i_1,\dots,i_n)} \subset L^2(M)$ as the closed linear span of $(M_{i_1} \ominus B) \cdots (M_{i_n} \ominus B)$. By convention, we put $\mathcal{H}_\emptyset = L^2(B)$. So we have the orthogonal decomposition

$$L^2(M) = \bigoplus_{\mathcal{I}\in\mathcal{S}} \mathcal{H}_{\mathcal{I}}.$$

We denote by $P_{\mathcal{I}}$ the orthogonal projection of $L^2(M)$ onto $\mathcal{H}_{\mathcal{I}}$.

Denote $\rho_t = |\sin(\pi t)/\pi t|^2$. A direct computation shows that for all $x \in L^2(M)$ and all $t \in \mathbb{R}$,

$$\begin{aligned} \|E_M(\theta_t(x))\|_2^2 &= \sum_{\mathcal{I}\in\mathcal{S}} \rho_t^{2|\mathcal{I}|} \|P_{\mathcal{I}}(x)\|_2^2, \\ \|x - \theta_t(x)\|_2^2 &= \sum_{\mathcal{I}\in\mathcal{S}} 2(1 - \rho_t^{|\mathcal{I}|}) \|P_{\mathcal{I}}(x)\|_2^2, \\ \|\theta_t(x) - E_M(\theta_t(x))\|_2^2 &= \sum_{\mathcal{I}\in\mathcal{S}} (1 - \rho_t^{2|\mathcal{I}|}) \|P_{\mathcal{I}}(x)\|_2^2. \end{aligned} \tag{3.1}$$

The last two equalities imply the following transversality property in the sense of [Po06, Lemma 2.1]:

$$\|x - \theta_t(x)\|_2 \le \sqrt{2}\|\theta_t(x) - E_M(\theta_t(x))\|_2 \quad \text{for all } x \in L^2(M),\ t \in \mathbb{R}. \tag{3.2}$$

The following is the main technical result of [IPP05]. For a proof of the version we state here, we refer to [Ho07, Section 5] and [PV09, Theorem 5.4].

Theorem 3.1 ([IPP05, Theorem 3.1]). *Let $p \in M$ be a nonzero projection and $A \subset pMp$ a von Neumann subalgebra. Assume that there exists an $\varepsilon > 0$ and a $t > 0$ such that $\|E_M(\theta_t(a))\|_2 \geq \varepsilon$ for all $a \in \mathcal{U}(A)$. Then at least one of the following statements holds:*

- *$A \prec_M B$.*
- *There exists an $i \in \{1,2\}$ such that $\mathcal{N}_{pMp}(A)'' \prec_M M_i$.*

§3.2. The algebra $\widetilde{M}$ as a crossed product with $\mathbb{F}_2$ and random walks on $\mathbb{F}_2$

We recall here the fundamental idea of [Io12] to consider $\widetilde{M}$ as a crossed product with the free group $\mathbb{F}_2$ and to exploit the spectral gap of random walks on the nonamenable group $G = \mathbb{F}_2$. As in [Io06, Remark 4.5] and [Io12, Section 3], we decompose $M = N \rtimes G$, where N is defined as the von Neumann subalgebra of $\widetilde{M}$ generated by $\{u_g M u_g^* \mid g \in G\}$ and normalized by the unitaries $(u_g)_{g\in G}$. Note that N is the infinite amalgamated free product of the subalgebras $u_g M u_g^*$, $g \in G$, over the common subalgebra B. From this point of view, the action of G on N is the free Bernoulli action.

For every $i \in \{1,2\}$ and $t \in (0,1)$, we define the maps $\beta_{i,t} : G_i \to \mathbb{R}$ by

$$\beta_{i,t}(g) = \tau(u_{i,t} u_g^*) \quad \text{for all } g \in G_i.$$

We then denote by $\gamma_{i,t}$ and $\mu_{i,t}$ the probability measures on G given by

$$\gamma_{i,t}(g) = \begin{cases} |\beta_{i,t}(g)|^2 & \text{if } g \in G_i, \\ 0 & \text{if } g \notin G_i, \end{cases} \qquad \mu_{i,t} = \gamma_{i,t} * \gamma_{i,t},$$

where we have used the usual convolution product between probability measures on G:

$$(\gamma * \gamma')(g) = \sum_{h,k\in G,\, hk=g} \gamma(h)\gamma'(k).$$

For $\mathcal{I} \in \mathcal{S}$, we finally denote by $\mu_{\mathcal{I},t}$ the probability measure on G given by

$$\mu_{\emptyset,t}(g) = \delta_{g,e} \quad \text{and} \quad \mu_{(i_1,\ldots,i_n),t} = \mu_{i_1,t} * \mu_{i_2,t} * \cdots * \mu_{i_n,t}.$$

The probability measures $\mu_{\mathcal{I},t}$ give rise to the Markov operators $T_{\mathcal{I},t}$ on $\ell^2(G)$ given by

$$T_{\mathcal{I},t} = \sum_{g\in G} \mu_{\mathcal{I},t}(g)\lambda_g.$$

The support of the probability measures $\gamma_{i,t}$ and $\mu_{i,t}$ equals G_i. So the support S of the probability measure $\mu_{(1,2),t}$ equals $G_1 G_2$. Since SS^{-1} generates the group $\mathbb{F}_2$ and since $\mathbb{F}_2$ is nonamenable, it follows from Kesten's criterion (see e.g. [Pi84, Corollary 18.5]) that $\|T_{(1,2),t}\| < 1$ for all $t \in (0,1)$. Writing $c_t = \|T_{(1,2),t}\|^{1/2}$, we have found numbers $0 < c_t < 1$ such that

$$\|T_{\mathcal{I},t}\| \leq c_t^{|\mathcal{I}|-1} \quad \text{for all } \mathcal{I} \in \mathcal{S} \text{ and all } 0 < t < 1.$$

For every $x \in \widetilde{M}$ and $h \in G$, we define $(x)_h = E_N(x u_h^*)$. So with $\|\cdot\|_2$-convergence, we have $x = \sum_{h \in G} (x)_h u_h$. We recall the following result of [Io12].

Lemma 3.2 ([Io12, formula (3.5)]). *For all $t \in (0,1)$, $h \in G$ and $x, y \in L^2(M)$,*

$$\langle (\theta_t(x))_h, (\theta_t(y))_h \rangle = \sum_{\mathcal{I} \in \mathcal{S}} \langle P_{\mathcal{I}}(x), y \rangle \, \mu_{\mathcal{I},t}(h).$$

Also recall from [Io12] that Lemma 3.2 yields the following result.

Theorem 3.3 ([Io12, Theorem 3.2]). *Let $p \in M$ be a nonzero projection and $A \subset pMp$ a von Neumann subalgebra. Assume that $\theta_t(A) \prec_{\widetilde{M}} N$ for some $t \in (0,1)$. Then at least one of the following statements holds:*

- *$A \prec_M B$.*
- *There exists an $i \in \{1,2\}$ such that $\mathcal{N}_{pMp}(A)'' \prec_M M_i$.*

Proof. Assume that the conclusion fails. By Theorem 3.1, we find a net $(a_i)_{i \in I}$ of unitaries in $\mathcal{U}(A)$ such that $\lim_{i \in I} \|E_M(\theta_s(a_i))\|_2 = 0$ for all $s \in (0,1)$. We will prove that $\theta_t(A) \not\prec_{\widetilde{M}} N$ for all $t \in (0,1)$. So fix $t \in (0,1)$. It suffices to prove that $\lim_{i \in I} \|(\theta_t(a_i))_h\|_2 = 0$ for all $h \in G$.

Fix $h \in G$ and fix $\varepsilon > 0$. Take a large enough integer n_0 such that $c_t^{n_0 - 1} < \varepsilon$. So, for all $\mathcal{I} \in \mathcal{S}$ with $|\mathcal{I}| \geq n_0$, we have $\|T_{\mathcal{I},t}\| < \varepsilon$ and, in particular,

$$\mu_{\mathcal{I},t}(h) = \langle T_{\mathcal{I},t} \delta_e, \delta_h \rangle < \varepsilon.$$

Denote by

$$P_0 = \sum_{\mathcal{I} \in \mathcal{S},\, |\mathcal{I}| < n_0} P_{\mathcal{I}}$$

the projection onto the closed linear span of "all words of length $< n_0$". Using Lemma 3.2, we see that for all $i \in I$,

$$\|(\theta_t(a_i))_h\|_2^2 \leq \|P_0(a_i)\|_2^2 + \varepsilon.$$

By (3.1), we can take $s > 0$ small enough such that $\|P_0(a_i)\|_2 \leq 2\|E_M(\theta_s(a_i))\|_2$ for all $i \in I$. Since $\lim_{i \in I} \|E_M(\theta_s(a_i))\|_2 = 0$, it follows that

$$\limsup_{i \in I} \|(\theta_t(a_i))_h\|_2^2 \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it indeed follows that $\lim_{i\in I} \|(\theta_t(a_i))_h\|_2 = 0$ for all $h \in G$. □

§3.3. Relative amenability and the malleable deformation

The following is our main technical result. The same statement was proven in [Io12, Theorem 5.1] under the additional assumption that $A' \cap (pMp)^\omega = \mathbb{C}1$ for some free ultrafilter ω, i.e. under the assumption that there are no nontrivial bounded sequences in pMp that asymptotically commute with A.

Theorem 3.4. *Let $p \in M$ be a nonzero projection and $A \subset pMp$ a von Neumann subalgebra. Assume that for all $t \in (0,1)$, $\theta_t(A)$ is amenable relative to N inside $\widetilde{M}$. Then at least one of the following statements holds:*

- *There exists $i \in \{1,2\}$ such that $A \prec_M M_i$.*
- *A is amenable relative to B inside M.*

Proof. Assume that $A \not\prec_M M_1$ and $A \not\prec_M M_2$. Denote by z the maximal projection in the center of $A' \cap pMp$ such that Az is amenable relative to B inside M. If $z = p$, then the theorem is proven. If $z < p$, we replace p by $p - z$ and we replace A by $A(p - z)$. So, $A \not\prec_M M_i$ for all $i \in \{1,2\}$, and Aq is amenable relative to B for no nonzero projection $q \in \mathcal{Z}(A' \cap pMp)$. We refer to this last property by saying that "no corner of A is amenable relative to B inside M." We will derive a contradiction.

Exactly as in the proof of [Io12, Theorem 5.1], we define the index set I to consist of all quadruplets $i = (X, Y, \delta, t)$ where $X \subset \widetilde{M}$ and $Y \subset \mathcal{U}(A)$ are finite subsets, $\delta \in (0,1)$ and $t \in (0,1)$. We turn I into a directed set by putting $(X, Y, \delta, t) \leq (X', Y', \delta', t')$ if and only if $X \subset X'$, $Y \subset Y'$, $\delta' \leq \delta$ and $t' \leq t$. Since $\theta_t(A)$ is amenable relative to N inside $\widetilde{M}$ for all $t \in (0,1)$, we can choose, for every $i = (X, Y, \delta, t)$ in I, a vector $\xi_i \in L^2(\langle \widetilde{M}, e_N \rangle)$ such that $\|\xi_i\|_2 \leq 1$ and

$$
\begin{aligned}
|\langle x\xi_i, \xi_i\rangle - \tau(x)| \leq \delta \quad & \text{whenever } x \in X \text{ or} \\
& x = (\theta_t(y) - y)^*(\theta_t(y) - y) \text{ with } y \in Y, \\
\|\theta_t(y)\xi_i - \xi_i\theta_t(y)\|_2 \leq \delta \quad & \text{whenever } y \in Y.
\end{aligned}
$$

It follows that $\lim_{i\in I} \langle x\xi_i, \xi_i\rangle = \tau(x)$ for all $x \in \widetilde{M}$. Since $\lim_{t\to 0} \|\theta_t(y) - y\|_2 = 0$ for all $y \in \mathcal{U}(A)$, it follows that $\lim_{i\in I} \|y\xi_i - \xi_i y\|_2 = 0$ for all $y \in \mathcal{U}(A)$.

Denote by $\mathcal{K}$ the closed linear span of $\{x u_g e_N u_g^* \mid x \in M, g \in G\}$ inside $L^2(\langle \widetilde{M}, e_N \rangle)$. Since $u_g^* M u_g \subset N$, we see that $u_g e_N u_g^*$ commutes with M for all $g \in G$. Therefore, $\mathcal{K}$ is an M-M-bimodule. Denote by e the orthogonal projection onto $\mathcal{K}$. The net of vectors $\xi_i' = p(1 - e)(\xi_i)$ satisfies $\limsup_{i\in I} \|x\xi_i'\|_2 \leq \|x\|_2$ for all $x \in pMp$ and $\lim_{i\in I} \|a\xi_i' - \xi_i' a\|_2 = 0$ for all $a \in A$. By [Io12, Lemma 4.2], the

M-M-bimodule $L^2(\langle \widetilde{M}, e_N\rangle) \ominus \mathcal{K}$ is isomorphic to $L^2(M) \otimes_B \mathcal{L}$ for some B-M-bimodule $\mathcal{L}$. Since no corner of A is amenable relative to B inside M, it follows from Lemma 2.1 that $\lim_{i\in I} \|\xi_i'\|_2 = 0$. So,

$$\lim_{i\in I} \|p\xi_i - e(p\xi_i)\|_2 = 0.$$

Define the isometry

$$U : L^2(M) \otimes \ell^2(G) \to L^2(\langle \widetilde{M}, e_N\rangle) : U(x \otimes \delta_g) = x\, u_g e_N u_g^*.$$

Note that $UU^* = e$ and

$$U((x \otimes 1)\eta(y \otimes 1)) = xU(\eta)y \quad \text{for all } x, y \in M,\ \eta \in L^2(M) \otimes \ell^2(G).$$

We define the net $(\zeta_i)_{i\in I}$ of vectors in $pL^2(M) \otimes \ell^2(G)$ by $\zeta_i = U^*(p\xi_i)$. Note that $\|\zeta_i\|_2 \leq 1$. The properties of $(\xi_i)_{i\in I}$ imply that

$$\begin{aligned}
&\lim_{i\in I} \|p\xi_i - U(\zeta_i)\|_2 = 0,\\
&\lim_{i\in I} \langle (x \otimes 1)\zeta_i, \zeta_i\rangle = \tau(pxp) \quad \text{for all } x \in M,\\
&\lim_{i\in I} \|(a \otimes 1)\zeta_i - \zeta_i(a \otimes 1)\|_2 = 0 \quad \text{for all } a \in \mathcal{U}(A).
\end{aligned}$$

We view $pL^2(M) \otimes \ell^2(G)$ as a closed subspace of $L^2(\widetilde{M}) \otimes \ell^2(G)$. Hence, the following claim makes sense.

Claim. *For every $\varepsilon > 0$, there exists an $s_0 \in (0,1)$ and an $i_0 \in I$ such that*

$$\|\zeta_i - (\theta_s \otimes \mathrm{id})(\zeta_i)\|_2 < \varepsilon \quad \textit{for all } s \in [0, s_0] \textit{ and all } i \geq i_0.$$

Proof of the claim. Assume the contrary. Using (3.2), we then find an $\varepsilon > 0$ such that for every $s \in (0,1)$, we have

$$\limsup_{i\in I} \|(\theta_s \otimes \mathrm{id})(\zeta_i) - (E_M \circ \theta_s \otimes \mathrm{id})(\zeta_i)\|_2 \geq \varepsilon.$$

Since $\lim_{s\to 0} \|\theta_s(a) - a\|_2 = 0$ for every $a \in \mathcal{U}(A)$, we can choose a subnet (μ_k) of the net of vectors

$$\big((p \otimes 1)((\mathrm{id} - E_M) \circ \theta_s \otimes \mathrm{id})(\zeta_i)\big)_{(i,s)\in I\times(0,1)}$$

with the properties that

$$\begin{aligned}
&\limsup_k \|(x \otimes 1)\mu_k\|_2 \leq \|x\|_2 \quad \text{for all } x \in pMp,\\
&\liminf_k \|\mu_k\|_2 \geq \varepsilon,\\
&\lim_k \|(a \otimes 1)\mu_k - \mu_k(a \otimes 1)\|_2 = 0 \quad \text{for all } a \in \mathcal{U}(A).
\end{aligned}$$

The M-M-bimodule $L^2(\widetilde{M} \ominus M) \otimes \ell^2(G)$ is isomorphic to $L^2(M) \otimes_B \mathcal{L}$ for some B-M-bimodule $\mathcal{L}$. By Lemma 2.1, we have reached a contradiction with the assumption that no corner of A is amenable relative to B inside M. This proves the claim.

Put $\varepsilon = \tau(p)/14$. Fix $i_0 \in I$ and $s_0 \in (0,1)$ such that for all $i \geq i_0$ and all $s \in [0, s_0]$, we have

$$\|p\xi_i - U(\zeta_i)\|_2 < \varepsilon \quad \text{and} \quad \|\zeta_i - (\theta_s \otimes \mathrm{id})(\zeta_i)\|_2 < \varepsilon.$$

Write $i_0 = (X_0, Y_0, \delta_0, t_0)$. Enlarging i_0 if necessary, we may assume that $p \in X_0$, $p \in Y_0$ (note that p is the unit element of $\mathcal{U}(A)$), $\delta_0 < \varepsilon^2/2$, $t_0 \leq s_0$ and $\|\theta_{t_0}(p) - p\|_2 < \varepsilon/2$.

Denote by J the index set consisting of all triplets $j = (X, Y, \delta)$, where $X \subset pMp$ and $Y \subset \mathcal{U}(A)$ are finite subsets and $\delta \in (0, \delta_0)$. We turn J into a directed set in a similar way to I above. For every $j = (X, Y, \delta)$, we put

$$\eta_j = \zeta_{(X_0 \cup X, Y_0 \cup Y, \delta, t_0)}.$$

Note that we use here the fixed index t_0. In particular, $(\eta_j)_{j \in J}$ is not a subnet of $(\zeta_i)_{i \in I}$. Also note that $\|\eta_j\|_2 \leq 1$. We claim that the net $(\eta_j)_{j \in J}$ of vectors in $pL^2(M) \otimes \ell^2(G)$ has the following properties:

$$\limsup_{j \in J} \|(x \otimes 1)\eta_j\|_2 \leq \|x\|_2 \quad \text{for all } x \in M, \tag{3.3}$$

$$\liminf_{j \in J} |\langle \theta_{t_0}(a)\, U(\eta_j), U(\eta_j)\, \theta_{t_0}(a) \rangle| \geq \tau(p) - 6\varepsilon \quad \text{for all } a \in \mathcal{U}(A), \tag{3.4}$$

$$\|\eta_j - (\theta_s \otimes \mathrm{id})(\eta_j)\|_2 \leq \varepsilon \quad \text{for all } s \in [0, t_0],\ j \in J. \tag{3.5}$$

To prove (3.3), fix $x \in M$ and fix $j = (X, Y, \delta)$ with $px^*xp \in X$. It suffices to prove that

$$\|(x \otimes 1)\eta_j\|_2^2 \leq \|x\|_2^2 + \delta. \tag{3.6}$$

Put $i = (X_0 \cup X, Y_0 \cup Y, \delta, t_0)$. We get

$$\|(x \otimes 1)\eta_j\|_2 = \|(x \otimes 1)\zeta_i\|_2 = \|x\, U(\zeta_i)\|_2 = \|xe(p\xi_i)\|_2 = \|e(xp\xi_i)\|_2 \leq \|xp\xi_i\|_2.$$

But also

$$\|xp\xi_i\|_2^2 = \langle px^*xp\xi_i, \xi_i \rangle \leq \tau(px^*xp) + \delta \leq \|x\|_2^2 + \delta$$

because $px^*xp \in X \subset X_0 \cup X$. So (3.6) follows and (3.3) is proven.

To prove (3.4), fix $a \in \mathcal{U}(A)$ and fix $j = (X, Y, \delta)$ with $a \in Y$. It suffices to prove that

$$|\langle \theta_{t_0}(a)\, U(\eta_j), U(\eta_j)\, \theta_{t_0}(a) \rangle| \geq \tau(p) - 6\varepsilon - 2\delta. \tag{3.7}$$

Put $i = (X_0 \cup X, Y_0 \cup Y, \delta, t_0)$. Since $p \in Y_0 \subset Y_0 \cup Y$, we have

$$\|\theta_{t_0}(p)\xi_i - p\xi_i\|_2^2 \leq \|\theta_{t_0}(p) - p\|_2^2 + \delta \leq \varepsilon^2/2 + \delta_0 \leq \varepsilon^2.$$

So $\|\theta_{t_0}(p)\xi_i - p\xi_i\|_2 \leq \varepsilon$. Since $\|p\xi_i - U(\zeta_i)\|_2 \leq \varepsilon$, we get

$$\|\theta_{t_0}(p)\xi_i - U(\eta_j)\|_2 \leq 2\varepsilon.$$

Since $p \in Y_0 \subset Y_0 \cup Y$, we also have $\|\theta_{t_0}(p)\xi_i - \xi_i\theta_{t_0}(p)\|_2 \leq \delta \leq \delta_0 \leq \varepsilon$. In combination with the previous inequality, this gives

$$\|\xi_i\theta_{t_0}(p) - U(\eta_j)\|_2 \leq 3\varepsilon.$$

In the following computation, we write $y \approx_\varepsilon z$ when $y, z \in \mathbb{C}$ with $|y - z| \leq \varepsilon$. We also use throughout that $\|\zeta_i\|_2 \leq 1$ and $\|\eta_j\|_2 \leq 1$ for all $i \in I$ and $j \in J$. So,

$$\begin{aligned}
&\langle \theta_{t_0}(a)U(\eta_j), U(\eta_j)\,\theta_{t_0}(a)\rangle \\
&\quad \approx_{2\varepsilon} \langle \theta_{t_0}(a)\xi_i, U(\eta_j)\,\theta_{t_0}(a)\rangle && \text{because } \|U(\eta_j) - \theta_{t_0}(p)\xi_i\|_2 \leq 2\varepsilon, \\
&\quad \approx_{3\varepsilon} \langle \theta_{t_0}(a)\xi_i, \xi_i\theta_{t_0}(a)\rangle && \text{because } \|U(\eta_j) - \xi_i\theta_{t_0}(p)\|_2 \leq 3\varepsilon, \\
&\quad \approx_{\delta} \langle \theta_{t_0}(a)\xi_i, \theta_{t_0}(a)\xi_i\rangle && \text{because } \|\xi_i\theta_{t_0}(a) - \theta_{t_0}(a)\xi_i\|_2 \leq \delta \text{ since } a \in Y, \\
&\quad = \langle \theta_{t_0}(p)\xi_i, \xi_i\rangle && \\
&\quad \approx_{\varepsilon} \langle p\xi_i, \xi_i\rangle && \text{because } \|\theta_{t_0}(p)\xi_i - p\xi_i\|_2 \leq \varepsilon, \\
&\quad \approx_{\delta} \tau(p) && \text{because } p \in X_0 \subset X_0 \cup X.
\end{aligned}$$

From this computation, (3.7) follows immediately. So also (3.4) is proven.

Finally (3.5) follows because $\|\zeta_i - (\theta_s \otimes \mathrm{id})(\zeta_i)\|_2 \leq \varepsilon$ for all $s \in [0, t_0]$ and all $i \geq i_0$.

Denote $\eta_j = \sum_{g \in G} \eta_{j,g} \otimes \delta_g$, where $\eta_{j,g} \in L^2(M)$ and where $(\delta_g)_{g \in G}$ is the canonical orthonormal basis of $\ell^2(G)$. Recall that for every $x \in L^2(\widetilde{M})$ and $h \in G$, we denote $(x)_h = E_N(xu_h^*)$.

For every $a \in \mathcal{U}(A)$, we have

$$\sum_{g,h \in G} \|(\theta_{t_0}(a\eta_{j,g}))_h\|_2^2 = \sum_{g \in G} \|\theta_{t_0}(a\eta_{j,g})\|_2^2 = \sum_{g \in G} \|a\eta_{j,g}\|_2^2 = \|(a \otimes 1)\eta_j\|_2^2 \leq 1.$$

Because the subspaces $(L^2(N)u_{hg}e_N u_g^*)_{h,g \in G}$ of $L^2(\langle \widetilde{M}, e_N\rangle)$ are orthogonal, the formula

$$\xi(a, j) = \sum_{g,h \in G} (\theta_{t_0}(a\eta_{j,g}))_h u_{hg} e_N u_g^*$$

provides a well defined vector in $L^2(\langle \widetilde{M}, e_N\rangle)$ with $\|\xi(a, j)\|_2 \leq 1$. We claim that for every $a \in \mathcal{U}(A)$ and all $j \in J$, we have

$$\|\theta_{t_0}(a)U(\eta_j) - \xi(a, j)\|_2 \leq \varepsilon. \tag{3.8}$$

To prove (3.8), first note that

$$\theta_{t_0}(a)U(\eta_j) = \sum_{g\in G} \theta_{t_0}(a)\eta_{j,g}u_g e_N u_g^* = \sum_{g,h\in G} (\theta_{t_0}(a)\eta_{j,g})_h u_{hg} e_N u_g^*.$$

It then follows that

$$\begin{aligned}\|\theta_{t_0}(a)U(\eta_j) - \xi(a,j)\|_2^2 &= \sum_{g,h\in G} \|(\theta_{t_0}(a)\eta_{j,g})_h - (\theta_{t_0}(a\eta_{j,g}))_h\|_2^2 \\ &= \sum_{g\in G} \|\theta_{t_0}(a)\eta_{j,g} - \theta_{t_0}(a\eta_{j,g})\|_2^2 = \sum_{g\in G} \|\eta_{j,g} - \theta_{t_0}(\eta_{j,g})\|_2^2 \\ &= \|\eta_j - (\theta_{t_0}\otimes \mathrm{id})(\eta_j)\|_2^2 \leq \varepsilon^2.\end{aligned}$$

So (3.8) is proven.

We similarly define the vectors $\xi'(a,j) \in L^2(\langle \widetilde{M}, e_N\rangle)$ by the formula

$$\xi'(a,j) = \sum_{g,h\in G} (\theta_{t_0}(\eta_{j,g}a))_h u_g e_N u_g^* u_h = \sum_{g,h\in G} (\theta_{t_0}(\eta_{j,hg}a))_h u_{hg} e_N u_g^*$$

and deduce that $\|\xi'(a,j)\|_2 \leq 1$ and

$$\|U(\eta_j)\theta_{t_0}(a) - \xi'(a,j)\|_2 \leq \varepsilon \tag{3.9}$$

for all $a \in \mathcal{U}(A)$ and all $j \in J$.

Combining (3.7)–(3.9), we find that for all $a \in \mathcal{U}(A)$,

$$\limsup_{j\in J} |\langle \xi(a,j), \xi'(a,j)\rangle| \geq \tau(p) - 8\varepsilon.$$

We now apply Lemma 3.2 and the notation introduced before its formulation. For every $a \in \mathcal{U}(A)$ and $j \in J$, we have

$$\begin{aligned}\langle \xi(a,j), \xi'(a,j)\rangle &= \sum_{g,h\in G} \langle (\theta_{t_0}(a\eta_{j,g}))_h, (\theta_{t_0}(\eta_{j,hg}a))_h\rangle \\ &= \sum_{g,h\in G}\sum_{\mathcal{I}\in\mathcal{S}} \langle P_{\mathcal{I}}(a\eta_{j,g}), \eta_{j,hg}a\rangle \mu_{\mathcal{I},t_0}(h) \\ &= \langle Q_{t_0}((a\otimes 1)\eta_j), \eta_j(a\otimes 1)\rangle,\end{aligned}$$

where $Q_{t_0} \in B(L^2(M)\otimes \ell^2(G))$ is defined by

$$Q_{t_0} = \sum_{\mathcal{I}\in\mathcal{S}} P_{\mathcal{I}} \otimes T_{\mathcal{I},t_0}.$$

So for all $a \in \mathcal{U}(A)$,

$$\limsup_{j\in J} |\langle Q_{t_0}((a\otimes 1)\eta_j), \eta_j(a\otimes 1)\rangle| \geq \tau(p) - 8\varepsilon. \tag{3.10}$$

Fix a large enough integer n_0 such that $c_{t_0}^{n_0-1} \leq \varepsilon$. So, $\|T_{\mathcal{I},t_0}\| \leq \varepsilon$ whenever $|\mathcal{I}| \geq n_0$. Denote by

$$P_0 = \sum_{\mathcal{I}\in\mathcal{S},\, |\mathcal{I}|<n_0} P_{\mathcal{I}}$$

the projection onto the closed linear span of "all words of length $< n_0$".

We claim that there exists a unitary $a \in \mathcal{U}(A)$ such that

$$\limsup_{j\in J} \|(P_0 \otimes 1)((a \otimes 1)\eta_j)\|_2 \leq 4\varepsilon. \tag{3.11}$$

To prove this claim, we first use (3.1) to fix $0 < s \leq t_0$ close enough to zero such that

$$\|(P_0 \otimes 1)(\eta)\|_2 \leq 2\|(E_M \otimes \mathrm{id})((\theta_s \otimes \mathrm{id})(\eta))\|_2 \quad \text{for all } \eta \in L^2(M) \otimes \ell^2(G).$$

Since $A \not\prec M_1$ and $A \not\prec M_2$, it follows from Theorem 3.1 that we can choose $a \in \mathcal{U}(A)$ such that $\|E_M(\theta_s(a))\|_2 \leq \varepsilon$. We will prove that this unitary $a \in \mathcal{U}(A)$ satisfies (3.11).

From (3.5), we know that $\|\eta_j - (\theta_s \otimes \mathrm{id})(\eta_j)\|_2 \leq \varepsilon$ for all $j \in J$. It follows that

$$\|(\theta_s \otimes \mathrm{id})((a \otimes 1)\eta_j) - (\theta_s(a) \otimes 1)\eta_j\|_2 \leq \varepsilon \quad \text{for all } j \in J.$$

So for all $j \in J$, we get

$$\begin{aligned}\|(P_0 \otimes 1)((a \otimes 1)\eta_j)\|_2 &\leq 2\|(E_M \otimes \mathrm{id})((\theta_s \otimes \mathrm{id})((a \otimes 1)\eta_j))\|_2 \\ &\leq 2\|(E_M \otimes \mathrm{id})((\theta_s(a) \otimes 1)\eta_j)\|_2 + 2\varepsilon \\ &= 2\|(E_M(\theta_s(a)) \otimes 1)\eta_j\|_2 + 2\varepsilon.\end{aligned}$$

Using (3.3), we get

$$\limsup_{j\in J} \|(P_0 \otimes 1)((a \otimes 1)\eta_j)\|_2 \leq 2\|E_M(\theta_s(a))\|_2 + 2\varepsilon \leq 4\varepsilon.$$

So the claim in (3.11) is proven and we fix the unitary $a \in \mathcal{U}(A)$ satisfying (3.11).

We will now deduce that

$$\limsup_{j\in J} \|Q_{t_0}((a \otimes 1)\eta_j)\|_2 \leq 5\varepsilon. \tag{3.12}$$

Indeed, since $\|T_{\mathcal{I},t_0}\| \leq 1$ for all $\mathcal{I} \in \mathcal{S}$ and $\|T_{\mathcal{I},t_0}\| \leq \varepsilon$ for all $\mathcal{I} \in \mathcal{S}$ with $|\mathcal{I}| \geq n_0$, we get

$$\begin{aligned}\|Q_{t_0}((a\otimes 1)\eta_j)\|_2^2 &= \sum_{\mathcal{I}\in\mathcal{S}} \|(P_{\mathcal{I}}\otimes T_{\mathcal{I},t_0})((a\otimes 1)\eta_j)\|_2^2 \\ &\leq \sum_{\mathcal{I}\in\mathcal{S},\,|\mathcal{I}|<n_0} \|(P_{\mathcal{I}}\otimes 1)((a\otimes 1)\eta_j)\|_2^2 + \varepsilon^2 \sum_{\mathcal{I}\in\mathcal{S},\,|\mathcal{I}|\geq n_0} \|(P_{\mathcal{I}}\otimes 1)((a\otimes 1)\eta_j)\|_2^2 \\ &\leq \|(P_0\otimes 1)((a\otimes 1)\eta_j)\|_2^2 + \varepsilon^2\|(a\otimes 1)\eta_j\|_2^2.\end{aligned}$$

Taking the lim sup over $j \in J$ and using (3.11) and (3.3), we arrive at

$$\limsup_{j\in J} \|Q_{t_0}((a\otimes 1)\eta_j)\|_2^2 \leq 17\varepsilon^2,$$

and (3.12) follows. But (3.12) implies that

$$\limsup_{j\in J} |\langle Q_{t_0}((a\otimes 1)\eta_j), \eta_j(a\otimes 1)\rangle| \leq 5\varepsilon.$$

Since $\varepsilon = \tau(p)/14$, we have $5\varepsilon < \tau(p) - 8\varepsilon$ and so we have obtained a contradiction with (3.10). □

§4. Proof of Theorem A and a version for HNN extensions

Proof of Theorem A. We use the malleable deformation θ_t of $M \subset \widetilde{M}$ as explained in Section 3.1. Write $G = \mathbb{F}_2$ and $\widetilde{M} = N \rtimes G$ as in Section 3.2. By assumption, A is amenable relative to one of the M_i inside M. A fortiori, A is amenable relative to M_i inside $\widetilde{M}$. Fix $t \in (0,1)$. Applying θ_t, we see that $\theta_t(A)$ is amenable relative to $\theta_t(M_i)$ inside $\widetilde{M}$. Since $\theta_t(M_i)$ is unitarily conjugate to M_i and $M_i \subset N$, it follows that $\theta_t(A)$ is amenable relative to N inside $\widetilde{M}$.

Put $P := \mathcal{N}_{pMp}(A)''$. We apply [PV11, Theorem 1.6 and Remark 6.3] to the crossed product $\widetilde{M} = N \rtimes G$ and the subalgebra $\theta_t(A)$ of this crossed product. We conclude that at least one of the following statements holds: $\theta_t(A) \prec_{\widetilde{M}} N$ or $\theta_t(P)$ is amenable relative to N inside $\widetilde{M}$. Since this holds for every $t \in (0,1)$, we see that at least one of the following is true:

- There exists a $t \in (0,1)$ such that $\theta_t(A) \prec_{\widetilde{M}} N$.
- $\theta_t(P)$ is amenable relative to N inside $\widetilde{M}$ for every $t \in (0,1)$.

In the first case, Theorem 3.3 implies that $A \prec_M B$ or $P \prec_M M_i$ for some $i \in \{1,2\}$. In the latter case, Theorem 3.4 implies that $P \prec_M M_i$ for some $i \in \{1,2\}$, or that P is amenable relative to B inside M. □

By [Ue07], HNN extensions can be viewed as corners of amalgamated free products. Since Theorem A has no particular assumptions on the inclusions $B \subset M_i$, we can immediately deduce the following result.

Theorem 4.1. *Let $M = \mathrm{HNN}(M_0, B, \theta)$ be the HNN extension of the tracial von Neumann algebra (M_0, τ) with von Neumann subalgebra $B \subset M_0$ and trace preserving embedding $\theta : B \to M_0$. Let $p \in M$ be a nonzero projection and $A \subset pMp$ a von Neumann subalgebra that is amenable relative to M_0 inside M. Then at least one of the following statements holds:*

- *$A \prec_M B$.*
- *$\mathcal{N}_{pMp}(A)'' \prec_M M_0$.*
- *$\mathcal{N}_{pMp}(A)''$ is amenable relative to B inside M.*

Proof. By [Ue07, Proposition 3.1], we can view $M = \mathrm{HNN}(M_0, B, \theta)$ as a corner of an amalgamated free product. More precisely, we put $M_1 = \mathrm{M}_2(\mathbb{C}) \otimes M_0$ and $M_2 = \mathrm{M}_2(\mathbb{C}) \otimes B$. We consider $B_0 = B \oplus B$ as a subalgebra of both M_1 and M_2, where the embedding $B_0 \hookrightarrow M_2$ is diagonal and the embedding $B_0 \hookrightarrow M_1$ is given by $b \oplus d \mapsto b \oplus \theta(d)$. We denote by e_{ij} the matrix units in M_1 and by f_{ij} the matrix units in M_2. The HNN extension M is generated by M_0 and the stable unitary u. There is a unique surjective $*$-isomorphism

$$\Psi : \mathrm{HNN}(M_0, B, \theta) \to e_{11}\big(M_1 *_{B_0} M_2\big)e_{11} : \begin{cases} \Psi(x) = e_{11}x & \text{for all } x \in M_0, \\ \Psi(u) = e_{12}f_{21}. \end{cases}$$

Note that in the amalgamated free product, $e_{11} = f_{11}$ and $e_{22} = f_{22}$. Therefore $e_{12}f_{21}$ is really a unitary.

Denote $\mathcal{M} := M_1 *_{B_0} M_2$. Whenever $Q \subset pMp$ is a von Neumann subalgebra, one checks that:

- $Q \prec_M B$ iff $\Psi(Q) \prec_{\mathcal{M}} B_0$ iff $\Psi(Q) \prec_{\mathcal{M}} M_2$.
- $Q \prec_M M_0$ iff $\Psi(Q) \prec_M M_1$.
- Q is amenable relative to B inside M iff $\Psi(Q)$ is amenable relative to B_0 inside $\mathcal{M}$.

So Theorem 4.1 is a direct consequence of Theorem A. □

§5. Cartan-rigidity for amalgamated free product groups and HNN extensions

Recall from [PV11] that a countable group Γ is called *$\mathcal{C}$-rigid* if for every free ergodic pmp action $\Gamma \curvearrowright (X, \mu)$, $L^\infty(X)$ is the unique Cartan subalgebra of

$L^\infty(X) \rtimes \Gamma$ up to unitary conjugacy. By [Po01, Theorem A.1], $\mathcal{C}$-rigidity is an immediate consequence of the following stronger property $(*)$:

$(*)$ For every trace preserving action $\Gamma \curvearrowright (B, \tau)$, projection $p \in M = B \rtimes \Gamma$ and amenable von Neumann subalgebra $A \subset pMp$ with $\mathcal{N}_{pMp}(A)'' = pMp$, we have $A \prec B$.

As was shown in the proof of [PV12, Theorem 1.1], a direct product $\Gamma_1 \times \cdots \times \Gamma_n$ of finitely many groups Γ_i with property $(*)$ is $\mathcal{C}$-rigid.

Property $(*)$ was shown to hold, among other groups, for all weakly amenable Γ with $\beta_1^{(2)}(\Gamma) > 0$ in [PV11, Theorem 7.1] and for all nonelementary hyperbolic Γ in [PV12, Theorem 1.4]. In [Io12, Theorem 7.1], property $(*)$ was proven for a large class of amalgamated free products, and in [DI12, Proof of Theorem 8.1] for a large class of HNN extensions. For completeness, we show how to deduce these last two results from Theorem A, resp. Theorem 4.1.

Theorem 5.1. *The following groups have property $(*)$ and, in particular, are $\mathcal{C}$-rigid:*

1. ([Io12, Theorem 7.1]) *Amalgamated free products $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ such that $[\Gamma_1 : \Sigma] \geq 3$, $[\Gamma_2 : \Sigma] \geq 2$ and there are $g_1, \dots, g_n \in \Gamma$ with $|\bigcap_{k=1}^n g_k \Sigma g_k^{-1}| < \infty$.*
2. ([DI12, Proof of Theorem 8.1]) *HNN extensions $\Gamma = \mathrm{HNN}(\Gamma_1, \Sigma, \theta)$, given by a subgroup $\Sigma < \Gamma_1$ and an injective group homomorphism $\theta : \Sigma \to \Gamma_1$, such that $\Sigma \neq \Gamma_1 \neq \theta(\Sigma)$ and there are $g_1, \dots, g_n \in \Gamma$ with $|\bigcap_{k=1}^n g_k \Sigma g_k^{-1}| < \infty$.*

Proof. Let $\Gamma \curvearrowright (B, \tau)$ be a trace preserving action and put $M = B \rtimes \Gamma$. Let $p \in M$ be a projection and $A \subset pMp$ an amenable von Neumann subalgebra with $\mathcal{N}_{pMp}(A)'' = pMp$. In the first case, M is the amalgamated free product of $B \rtimes \Gamma_1$ and $B \rtimes \Gamma_2$ over $B \rtimes \Sigma$. In the second case, M is the HNN extension of $B \rtimes \Gamma_1$ over $B \rtimes \Sigma$. In both cases, $\Gamma_i < \Gamma$ has infinite index and $\Sigma < \Gamma$ is not co-amenable (see e.g. the final paragraphs of the proof of [Io12, Theorem 7.1] and [DI12, Lemma 7.2]). So it follows from Theorem A and Theorem 4.1 that $A \prec B \rtimes \Sigma$.

Define the projection $z(\Sigma) \in M \cap (B \rtimes \Sigma)'$ as in [HPV10, Section 4]. Since $A \prec B \rtimes \Sigma$, we see that $z(\Sigma) \neq 0$. From [HPV10, Proposition 8], we know that $z(\Sigma)$ belongs to the center of M. Take $g_1, \dots, g_n \in \Gamma$ such that $\Sigma_0 = \bigcap_{k=1}^n g_k \Sigma g_k^{-1}$ is a finite group. We have $z(g_k \Sigma g_k^{-1}) = u_{g_k} z(\Sigma) u_{g_k}^* = z(\Sigma)$, because $z(\Sigma)$ belongs to the center of M. It then follows from [HPV10, Proposition 6] that

$$z(\Sigma_0) = z(g_1 \Sigma g_1^{-1}) \cdots z(g_n \Sigma g_n^{-1}) = z(\Sigma) \neq 0.$$

So $A \prec B \rtimes \Sigma_0$. Since Σ_0 is finite, we conclude that $A \prec B$. □

§6. Proof of Corollary B

Write $M_i = L(\mathcal{R}_i)$ and $B = L^\infty(X)$. Note that $L(\mathcal{R}) = M_1 *_B M_2$. Corollary B is a direct consequence of Theorem A, provided that we prove the following two statements:

1. $M \not\prec_M M_i$.
2. M is not amenable relative to B, i.e. M is not amenable itself.

Since $|\mathcal{R}_1 \cdot x| \geq 3$ for a.e. $x \in X$ and using e.g. [IKT08, Lemma 2.6], we can take unitaries $u, v \in \mathcal{U}(M_1)$ such that $E_B(u) = E_B(v) = E_B(u^* v) = 0$. We similarly find a unitary $w \in \mathcal{U}(M_2)$ with $E_B(w) = 0$.

Proof of 1. Define $w_n \in \mathcal{U}(M)$ by $w_n = (uw)^n$. Denote by $X_m \subset M$ the linear span of all products of at most m elements from $M_1 \ominus B$ and $M_2 \ominus B$. Whenever $2n > 2m + 1$ and $x, y \in X_m$, a direct computation shows that $E_{M_i}(x w_n y) = 0$. So it follows that $\lim_n \|E_{M_i}(x w_n y)\|_2 = 0$ for all $x, y \in M$, and statement 1 follows.

Proof of 2. Assume that M is amenable and take an M-central state Ω on $B(L^2(M))$. Define K_1 as the closed linear span of B and all products of the form $x_1 x_2 \cdots x_n$ with $x_1 \in M_1 \ominus B$, $x_2 \in M_2 \ominus B$, $x_3 \in M_1 \ominus B$, etc. Define K_2 as the closed linear span of all products of the form $y_1 y_2 \cdots y_n$ with $y_1 \in M_2 \ominus B$, $y_2 \in M_1 \ominus B$, $y_3 \in M_2 \ominus B$, etc. By construction, $L^2(M) = K_1 \oplus K_2$. Denote by e_i the orthogonal projection of $L^2(M)$ onto K_i. It follows that $u e_2 u^*$ and $v e_2 v^*$ are orthogonal and lie under e_1. Hence, $2\Omega(e_2) = \Omega(u e_2 u^*) + \Omega(v e_2 v^*) \leq \Omega(e_1)$. On the other hand, $w e_1 w^* \leq e_2$, implying that $\Omega(e_1) = \Omega(w e_1 w^*) \leq \Omega(e_2)$. Altogether it follows that $\Omega(e_1) = \Omega(e_2) = 0$. Since $1 = e_1 + e_2$ and $\Omega(1) = 1$, we have reached a contradiction. □

§7. Proof of Corollary C

Let $A \subset M$ be a diffuse amenable von Neumann subalgebra. Denote $P = \mathcal{N}_M(A)''$ and assume that P is not amenable. Take a nonzero central projection $z \in \mathcal{Z}(P)$ such that Pz has no amenable direct summand. Since $Pz \subset \mathcal{N}_{zMz}(Az)''$, it follows from Theorem A that one of the following statements holds:

1. $Az \prec_M B$.
2. $Pz \prec_M M_i$ for some $i \in \{1, 2\}$.
3. Pz is amenable relative to B inside M.

It suffices to prove that each of the three statements is false.

1. Observe that the inclusion $M_2 \subset M$ is mixing. To prove this, fix a sequence b_n in the unit ball of M_2 such that $b_n \to 0$ weakly. We must show that

$\lim_n \|E_{M_2}(x^* b_n y)\|_2 = 0$ for all $x, y \in M \ominus M_2$. It suffices to prove this when $x = x_1 x_2 \cdots x_n$ and $y = y_1 y_2 \cdots y_m$ with $n, m \geq 2$, $x_1, y_1 \in M_2$, $x_2, y_2 \in M_1 \ominus B$, $x_3, y_3 \in M_2 \ominus B$, etc. But then

$$E_{M_2}(x^* b_n y) = E_{M_2}(x_n^* \cdots x_3^* E_B(x_2^* E_B(x_1^* b_n y_1) y_2) y_3 \cdots y_n),$$

and the conclusion follows because $E_B(x_1^* b_n y_1) \to 0$ weakly and the inclusion $B \subset M_1$ is mixing.

Assume that statement 1 holds. Then certainly $Az \prec_M M_2$. Since the inclusion $M_2 \subset M$ is mixing, it follows from [Io12, Lemma 9.4] that $Pz \prec_M M_2$. So statement 2 holds and we proceed to the next point.

2. Assume that statement 2 holds. We then find a nonzero projection $p \in M_n(\mathbb{C}) \otimes M_i$ and a normal unital $*$-homomorphism $\varphi : Pz \to p(M_n(\mathbb{C}) \otimes M_i)p$. Then $\varphi(Az)$ is a diffuse von Neumann subalgebra of $p(M_n(\mathbb{C}) \otimes M_i)p$ whose normalizer contains $\varphi(Pz)$. Since Pz has no amenable direct summand, $\varphi(Pz)$ is nonamenable. Hence $p(M_n(\mathbb{C}) \otimes M_i)p$ is not strongly solid. Since M_i is strongly solid, this contradicts the stability of strong solidity under amplifications as proven in [Ho09, Proposition 5.2].

3. Since B is amenable, statement 3 implies that Pz is amenable, contradicting our assumptions. □

§8. W*-superrigid actions of type III

In the same way as [HV12, Theorem A] was deduced from the results in [PV12], we can deduce from Theorem A the following type III uniqueness statement for Cartan subalgebras. Our theorem is a generalization of [BHR12, Theorem D], where the same result was proven under the assumption that Σ is a finite group.

Rather than looking for the most general statement possible, we provide a more ad hoc formulation that suffices to prove the W*-superrigidity of the type III_1 actions in Proposition D (see also Remark 8.3 below). Recall that a nonsingular action $\Lambda \curvearrowright (X, \mu)$ is said to be *recurrent* if there is no Borel subset $\mathcal{U} \subset X$ such that $\mu(\mathcal{U}) > 0$ and $\mu(g \cdot \mathcal{U} \cap \mathcal{U}) = 0$ for all $g \in \Lambda - \{e\}$.

Theorem 8.1. *Let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ be an amalgamated free product group and assume that there exist $g_1, \ldots, g_n \in \Gamma$ such that $\bigcap_{k=1}^n g_k \Sigma g_k^{-1}$ is finite. Let $\Gamma \curvearrowright (X, \mu)$ be any nonsingular free ergodic action. Assume that each Γ_i admits a subgroup Λ_i such that the restricted action $\Lambda_i \curvearrowright (X, \mu)$ is recurrent and $\Lambda_i \cap \Sigma$ is finite. Then $L^\infty(X)$ is the unique Cartan subalgebra of $L^\infty(X) \rtimes \Gamma$ up to unitary conjugacy.*

For Theorem 8.1 to hold, it is essential to impose some recurrence of $\Gamma_i \curvearrowright (X, \mu)$ relative to Σ. Indeed, otherwise the action $\Gamma \curvearrowright (X, \mu)$ could simply be the induction of an action $\Gamma_i \curvearrowright (Z, \eta)$ so that $L^\infty(X) \rtimes \Gamma \cong B(H) \mathbin{\overline{\otimes}} (L^\infty(Z) \rtimes \Gamma_i)$ and we cannot expect uniqueness of the Cartan subalgebra.

Before proving Theorem 8.1, we provide a semifinite variant of the machinery developed in [HPV10, Sections 4 and 5]. We start from the following elementary lemma, leaving the proof to the reader.

Lemma 8.2. *Let (N, Tr) be a von Neumann algebra equipped with a normal semifinite faithful trace. Let H be a right Hilbert N-module and $p \in N$ a projection. We consider dimensions using the trace Tr and its restrictions to subalgebras of N and pNp.*

(i) $\dim_{pNp}(Hp) \leq \dim_N(H)$.
(ii) $\dim_N(\mathrm{closure}(KN)) = \dim_{pNp}(K)$ *for all closed pNp-submodules $K \subset Hp$.*
(iii) *Let $P \subset N$ be a von Neumann subalgebra such that $\mathrm{Tr}_{|P}$ is semifinite. Let $K \subset H$ be a closed P-submodule. Then $\dim_N(\mathrm{closure}(KN)) \leq \dim_P(K)$.*

Assume that Γ is a countable group and $\Gamma \curvearrowright (B, \mathrm{Tr})$ a trace preserving action on a von Neumann algebra B equipped with a normal semifinite faithful trace Tr. Denote $\mathcal{M} = B \rtimes \Gamma$ and use the canonical trace Tr on $\mathcal{M}$. Let $p \in \mathcal{M}$ be a projection with $\mathrm{Tr}(p) < \infty$ and $A \subset p\mathcal{M}p$ a von Neumann subalgebra with $\mathcal{N}_{p\mathcal{M}p}(A)'' = p\mathcal{M}p$. Whenever $\Lambda < \Gamma$ is a subgroup, we consider

$$\mathcal{E}_\Lambda = \{H \mid H \text{ is an } A\text{-}(B \rtimes \Lambda)\text{-subbimodule of } L^2(p\mathcal{M}) \text{ with } \dim_{B \rtimes \Lambda}(H) < \infty\}.$$

If $H \in \mathcal{E}_\Lambda$, $u \in \mathcal{N}_{p\mathcal{M}p}(A)$ and $v \in \mathcal{U}(B \rtimes \Lambda)$, then uHv again belongs to $\mathcal{E}_\Lambda$. So the closed linear span of all $H \in \mathcal{E}_\Lambda$ is of the form $L^2(p\mathcal{M}z(\Lambda))$, where $z(\Lambda)$ is a projection in $\mathcal{M} \cap (B \rtimes \Lambda)'$. We make $z(\Lambda)$ uniquely determined by requiring that $z(\Lambda)$ is smaller than or equal to the central support of p in $\mathcal{M}$.

If $\Lambda < \Lambda' < \Gamma$ are subgroups, we have $z(\Lambda) \leq z(\Lambda')$. Indeed, whenever $H \subset L^2(p\mathcal{M})$ is an A-$(B \rtimes \Lambda)$-subbimodule with $\dim_{B \rtimes \Lambda}(H) < \infty$, we define K as the closed linear span of $H(B \rtimes \Lambda')$. By Lemma 8.2, we see that $\dim_{B \rtimes \Lambda'}(K) < \infty$. Since $H \subset K$ and since this works for all choices of H, we conclude that $z(\Lambda) \leq z(\Lambda')$.

The basic construction $\langle \mathcal{M}, e_{B \rtimes \Lambda} \rangle$ carries a natural semifinite trace Tr satisfying $\mathrm{Tr}(x e_{B \rtimes \Lambda} x^*) = \mathrm{Tr}(xx^*)$ for all $x \in \mathcal{M}$. The projections $e \in A' \cap p\langle \mathcal{M}, e_{B \rtimes \Lambda} \rangle p$ are precisely the orthogonal projections onto the A-$(B \rtimes \Lambda)$-subbimodules $H \subset L^2(p\mathcal{M})$. Moreover under this correspondence, we have $\mathrm{Tr}(e) = \dim_{B \rtimes \Lambda}(H)$. We also have the canonical operator valued weight $\mathcal{T}_\Lambda$ from $\langle \mathcal{M}, e_{B \rtimes \Lambda} \rangle^+$ to the extended positive part of $\mathcal{M}$ such that $\mathrm{Tr} = \mathrm{Tr} \circ \mathcal{T}_\Lambda$. Using the anti-unitary involution

$J : L^2(\mathcal{M}) \to L^2(\mathcal{M}) : J(x) = x^*$, we can therefore alternatively define $z(\Lambda)$ as

$$\begin{aligned} pJz(\Lambda)J &= \bigvee\{e \mid e \in A' \cap p\langle \mathcal{M}, e_{B\rtimes\Lambda}\rangle p \text{ is a projection with } \|\mathcal{T}_\Lambda(e)\| < \infty\} \\ &= \bigvee\{\operatorname{supp}(a) \mid a \in A' \cap p\langle \mathcal{M}, e_{B\rtimes\Lambda}\rangle^+ p \text{ and } \|\mathcal{T}_\Lambda(a)\| < \infty\}. \end{aligned}$$

If now $\Lambda < \Gamma$ and $\Lambda' < \Gamma$ are subgroups, we can repeat the proof of [HPV10, Proposition 6] verbatim to conclude that $z(\Lambda)$ and $z(\Lambda')$ commute, with

$$z(\Lambda \cap \Lambda') = z(\Lambda)z(\Lambda'). \tag{8.1}$$

We are now ready to prove Theorem 8.1.

Proof of Theorem 8.1. Denote by $\omega : \Gamma \times X \to \mathbb{R}$ the logarithm of the Radon–Nikodym cocycle. Put $Y = X \times \mathbb{R}$ and equip Y with the measure m given by $dm = d\mu \times \exp(t)dt$, so that the action $\Gamma \curvearrowright Y$ given by $g \cdot (x,t) = (g \cdot x, \omega(g,x) + t)$ is measure preserving (see [Ma63]). The restricted actions $\Lambda_i \curvearrowright (Y, m)$ are still recurrent.

Put $B = L^\infty(Y)$ and denote by Tr the canonical semifinite trace on $\mathcal{M} = B \rtimes \Gamma$, given by the infinite invariant measure m. Choose a projection $p \in B$ with $0 < \operatorname{Tr}(p) < \infty$. Put $\Sigma_i = \Lambda_i \cap \Sigma$. Since Σ_i is a finite group, the von Neumann algebra $p(B \rtimes \Sigma_i)p$ is of type I. Since the action $\Lambda_i \curvearrowright (Y, m)$ is recurrent, the von Neumann algebra $p(B \rtimes \Lambda_i)p$ is of type II_1. In particular, the inclusion $p(B \rtimes \Sigma_i)p \subset p(B \rtimes \Lambda_i)p$ has no trivial corner in the sense of [HV12, Definition 5.1] and it follows from [HV12, Lemma 5.4] that there exists a unitary $u_i \in p(B \rtimes \Lambda_i)p$ such that $E_{p(B\rtimes\Sigma_i)p}(u_i^n) = 0$ for all $n \in \mathbb{Z} - \{0\}$. Since $\Lambda_i \cap \Sigma = \Sigma_i$, we have $E_{p(B\rtimes\Sigma)p}(x) = E_{p(B\rtimes\Sigma_i)p}(x)$ for all $x \in p(B \rtimes \Lambda_i)p$. So, we deduce that $E_{p(B\rtimes\Sigma)p}(u_i^n) = 0$ for all $n \in \mathbb{Z} - \{0\}$. We put $v_i = u_i^*$ and have thus found unitaries $u_i, v_i \in p(B \rtimes \Gamma_i)p$ satisfying

$$E_{p(B\rtimes\Sigma)p}(u_i) = E_{p(B\rtimes\Sigma)p}(v_i) = E_{p(B\rtimes\Sigma)p}(u_i^* v_i) = 0. \tag{8.2}$$

Define the normal trace preserving $*$-homomorphism

$$\Delta : \mathcal{M} \to \mathcal{M} \mathbin{\overline{\otimes}} L(\Gamma) : \Delta(bu_g) = bu_g \otimes u_g \quad \text{for all } b \in B, g \in \Gamma.$$

We use the unitaries u_i satisfying (8.2) to prove the following two easy statements.

Statement 1. *For $i = 1, 2$, we have $\Delta(p\mathcal{M}p) \not\prec_{p\mathcal{M}p \overline{\otimes} L(\Gamma)} p\mathcal{M}p \mathbin{\overline{\otimes}} L(\Gamma_i)$.*

Statement 2. *The von Neumann subalgebra $\Delta(p\mathcal{M}p) \subset p\mathcal{M}p \mathbin{\overline{\otimes}} L(\Gamma)$ is not amenable relative to $p\mathcal{M}p \mathbin{\overline{\otimes}} L(\Sigma)$.*

Proof of Statement 1. Denote by $|g|$ the length of an element $g \in \Gamma$, i.e. the minimal number of factors that are needed to write g as a product of elements in Γ_1, Γ_2, with the convention that $|g| = 0$ if and only if $g \in \Sigma$. Denote by Q_m the orthogonal projection of $L^2(p\mathcal{M}p)$ onto the closed linear span of $\{pbu_gp \mid b \in B, g \in \Gamma, |g| \leq m\}$. Denote by P_m the orthogonal projection of $\ell^2(\Gamma)$ onto the closed linear span of $\{u_g \mid g \in \Gamma, |g| \leq m\}$. A direct computation yields

$$(1 \otimes P_m)(\Delta(x)) = \Delta(Q_m(x)) \quad \text{for all } x \in p\mathcal{M}p.$$

Define the unitary $w_n = (u_1u_2)^n$. Since $Q_m(w_n) = 0$ whenever $n > m/2$, we have $(1 \otimes P_m)(\Delta(w_n)) = 0$ for all $n > m/2$. It follows in particular that for all $g, h \in \Gamma$,

$$E_{p\mathcal{M}p\overline{\otimes}L(\Gamma_i)}((1 \otimes u_g)\Delta(w_n)(1 \otimes u_h)) = 0 \quad \text{whenever } n > (|g| + |h| + 1)/2.$$

So, for every $x, y \in p\mathcal{M}p\overline{\otimes}L(\Gamma)$, we get $\lim_n \|E_{p\mathcal{M}p\overline{\otimes}L(\Gamma_i)}(x\Delta(w_n)y)\|_2 = 0$. Hence, $\Delta(p\mathcal{M}p) \not\prec p\mathcal{M}p \,\overline{\otimes}\, L(\Gamma_i)$ and Statement 1 is proven.

Proof of Statement 2. Assume that the subalgebra $\Delta(p\mathcal{M}p)$ is amenable relative $p\mathcal{M}p \,\overline{\otimes}\, L(\Sigma)$. So we find a positive $\Delta(p\mathcal{M}p)$-central functional Ω on the basic construction $\langle p\mathcal{M}p \,\overline{\otimes}\, L(\Gamma), e_{p\mathcal{M}p\overline{\otimes}L(\Sigma)}\rangle$ such that $\Omega(x) = (\mathrm{Tr} \otimes \tau)(x)$ for all x in $p\mathcal{M}p \,\overline{\otimes}\, L(\Gamma)$. Note that we can identify

$$\begin{aligned}\langle p\mathcal{M}p \,\overline{\otimes}\, L(\Gamma), e_{p\mathcal{M}p\overline{\otimes}L(\Sigma)}\rangle &= p\mathcal{M}p \,\overline{\otimes}\, \langle L(\Gamma), e_{L(\Sigma)}\rangle \\ &= (p \otimes 1)\langle \mathcal{M} \,\overline{\otimes}\, L(\Gamma), \mathcal{M} \,\overline{\otimes}\, L(\Sigma)\rangle(p \otimes 1).\end{aligned}$$

Since $E_{\mathcal{M}\overline{\otimes}L(\Sigma)} \circ \Delta = \Delta \circ E_{B\rtimes\Sigma}$ and the closed linear span of $\Delta(\mathcal{M})L^2(\mathcal{M}\overline{\otimes}L(\Sigma))$ equals $L^2(\mathcal{M}\overline{\otimes}L(\Gamma))$, there is a unique normal unital $*$-homomorphism satisfying

$$\Psi : \langle \mathcal{M}, e_{B\rtimes\Sigma}\rangle \to \langle \mathcal{M} \,\overline{\otimes}\, L(\Gamma), e_{\mathcal{M}\overline{\otimes}L(\Sigma)}\rangle : \Psi(xe_{B\rtimes\Sigma}y) = \Delta(x)e_{\mathcal{M}\overline{\otimes}L(\Sigma)}\Delta(y)$$

for all $x, y \in \mathcal{M}$. The composition of Ω and Ψ yields a $p\mathcal{M}p$-central positive functional Ω_0 on $p\langle \mathcal{M}, e_{B\rtimes\Sigma}\rangle p$ satisfying $\Omega_0(p) = \mathrm{Tr}(p)$. Note that we can view $p\langle \mathcal{M}, e_{B\rtimes\Sigma}\rangle p$ as the commutant of the right action of $B \rtimes \Sigma$ on $pL^2(\mathcal{M})$.

Denote by $H_i \subset pL^2(\mathcal{M})$ the closed linear span of all pbu_g with $b \in B$ and $g \in \Gamma$ such that a reduced expression of Γ as an alternating product of elements in $\Gamma_1 - \Sigma$ and $\Gamma_2 - \Sigma$ starts with a factor in $\Gamma_i - \Sigma$. Denote $H_0 = pL^2(B)$. So we have the orthogonal decomposition $pL^2(\mathcal{M}) = H_0 \oplus H_1 \oplus H_2$. Denote by $e_i : pL^2(\mathcal{M}) \to H_i$ the orthogonal projection. Note that e_i is a projection in $p\langle \mathcal{M}, e_{B\rtimes\Sigma}\rangle p$. By (8.2), the projections $u_2(e_0 + e_1)u_2^*$ and $v_2(e_0 + e_1)v_2^*$ are orthogonal and lie under e_2. Since Ω_0 is $p\mathcal{M}p$-central, it follows that

$$2\Omega_0(e_0 + e_1) \leq \Omega_0(e_2).$$

It similarly follows that $2\Omega_0(e_2) \leq \Omega_0(e_1)$. Together, it follows that $\Omega_0(e_0 + e_1) = \Omega_0(e_2) = 0$. Since $e_0 + e_1 + e_2 = p$, we obtain the contradiction that $\Omega_0(p) = 0$. So also Statement 2 is proven.

Assume now that $L^\infty(X) \rtimes \Gamma$ admits a Cartan subalgebra that is not unitarily conjugate to $L^\infty(X)$. The first paragraphs of the proof of [HV12, Theorem A] are entirely general and yield an abelian von Neumann subalgebra $A \subset p\mathcal{M}p$ such that $\mathcal{N}_{p\mathcal{M}p}(A)'' = p\mathcal{M}p$ and $A \not\prec Bq$ whenever $q \in B$ is a projection with $\mathrm{Tr}(q) < \infty$. So to prove the theorem, we fix an abelian von Neumann subalgebra $A \subset p\mathcal{M}p$ with $\mathcal{N}_{p\mathcal{M}p}(A)'' = p\mathcal{M}p$. We have to find a projection $q \in B$ with $\mathrm{Tr}(q) < \infty$ and $A \prec Bq$.

Note that $\Delta(A) \subset p\mathcal{M}p \,\overline{\otimes}\, L(\Gamma)$ is an abelian, hence amenable, von Neumann subalgebra whose normalizer contains $\Delta(p\mathcal{M}p)$. We view $p\mathcal{M}p \,\overline{\otimes}\, L(\Gamma)$ as the amalgamated free product of $p\mathcal{M}p \,\overline{\otimes}\, L(\Gamma_1)$ and $p\mathcal{M}p \,\overline{\otimes}\, L(\Gamma_2)$ over their common von Neumann subalgebra $p\mathcal{M}p \,\overline{\otimes}\, L(\Sigma)$. A combination of Theorem A and Statements 1 and 2 above implies that $\Delta(A) \prec p\mathcal{M}p \,\overline{\otimes}\, L(\Sigma)$. So there is no sequence of unitaries (a_n) in $\mathcal{U}(A)$ satisfying $\lim_n \|E_{p\mathcal{M}p\overline{\otimes}L(\Sigma)}(x\Delta(a_n)y)\|_2 = 0$ for all $x, y \in p\mathcal{M}p \,\overline{\otimes}\, L(\Gamma)$. This means that we can find $\varepsilon > 0$ and $h_1, \dots, h_m \in \Gamma$ such that

$$\sum_{i,j=1}^{m} \|E_{p\mathcal{M}p\overline{\otimes}L(\Sigma)}((1 \otimes u_{h_i}^*)\Delta(a)(1 \otimes u_{h_j}))\|_2^2 \geq \varepsilon \quad \text{for all } a \in \mathcal{U}(A). \tag{8.3}$$

Consider the positive element $T = \sum_{i=1}^m p u_{h_i} e_{B\rtimes\Sigma} u_{h_i}^* p$ in $p\langle \mathcal{M}, e_{B\rtimes\Sigma}\rangle p$. The left hand side of (8.3) equals $\mathrm{Tr}(TaTa^*)$. Denote by S the element of smallest $\|\cdot\|_{2,\mathrm{Tr}}$-norm in the weakly closed convex hull of $\{aTa^* \mid a \in \mathcal{U}(A)\}$. Then S is a nonzero element of $A' \cap p\langle \mathcal{M}, e_{B\rtimes\Sigma}\rangle p$ and $\mathrm{Tr}(S) < \infty$. In the notation introduced before this proof, this means that $z(\Sigma) \neq 0$.

Since the action $\Gamma \curvearrowright Y$ is free, we have $\mathcal{M} \cap B' = B$. So the projections $z(\Sigma)$ and $z(\Gamma_i)$ belong to B and are, respectively, Σ- and Γ_i-invariant. We prove below that $z(\Sigma)$ is a Γ-invariant projection in B. We prove this by showing that $z(\Gamma_1) = z(\Sigma) = z(\Gamma_2)$.

Since $\Sigma < \Gamma_i$, we have $z(\Sigma) \leq z(\Gamma_i)$ for every $i = 1, 2$. We claim that equality holds. Assume that $z(\Sigma) < z(\Gamma_1)$. Note that both projections belong to B. Choose a nonzero projection $q \in B$ with $\mathrm{Tr}(q) < \infty$ and $q \leq z(\Gamma_1) - z(\Sigma)$. Choose $H \in \mathcal{E}_{\Gamma_1}$ such that $Hq \neq \{0\}$. By Lemma 8.2, we have

$$\dim_{q(B\rtimes\Gamma_1)q}(Hq) \leq \dim_{B\rtimes\Gamma_1}(H) < \infty.$$

We conclude that $L^2(p\mathcal{M}q)$ admits a nonzero A-$q(B \rtimes \Gamma_1)q$-subbimodule K that is finitely generated as a right Hilbert module. Since $q \perp z(\Sigma)$, we also know that $L^2(p\mathcal{M}q)$ does not admit an A-$q(B \rtimes \Sigma)q$-subbimodule that is finitely generated as a right Hilbert module. We then encode K as an integer n, a projection $q_1 \in M_n(\mathbb{C}) \otimes q(B \rtimes \Gamma_1)q$, a nonzero partial isometry $V \in p(M_{1,n}(\mathbb{C}) \otimes \mathcal{M})q_1$ and a normal unital $*$-homomorphism $\varphi : A \to q_1(M_n(\mathbb{C}) \otimes (B \rtimes \Gamma_1))q_1$ such that

$$aV = V\varphi(a) \text{ for all } a \in A \quad \text{and} \quad \varphi(A) \not\prec_{M_n(\mathbb{C})\otimes q(B\rtimes\Gamma_1)q} q(B \rtimes \Sigma)q. \tag{8.4}$$

Let $u \in \mathcal{N}_{p\mathcal{M}p}(A)$ and write $uau^* = \alpha(a)$ for all $a \in A$. Then V^*uV is an element of $q_1(M_n(\mathbb{C}) \otimes \mathcal{M})q_1$ satisfying

$$V^*uV\varphi(a) = \varphi(\alpha(a))V^*uV \quad \text{for all } a \in A.$$

By (8.4) and [CH08, Theorem 2.4], it follows that $V^*uV \in q_1(M_n(\mathbb{C})\otimes(B\rtimes\Gamma_1))q_1$. This holds for all $u \in \mathcal{N}_{p\mathcal{M}p}(A)$. Since the linear span of $\mathcal{N}_{p\mathcal{M}p}(A)$ is strongly dense in $p\mathcal{M}p$, and writing $q_2 = V^*V$, we have found a nonzero projection $q_2 \in M_n(\mathbb{C}) \otimes (B \rtimes \Gamma_1)$ with the property that

$$q_2(M_n(\mathbb{C}) \otimes \mathcal{M})q_2 = q_2(M_n(\mathbb{C}) \otimes (B \rtimes \Gamma_1))q_2.$$

In the von Neumann algebra $M_n(\mathbb{C}) \otimes (B \rtimes \Gamma_1)$, the projection q_2 is equivalent to a projection in $D_n(\mathbb{C}) \otimes B$, where $D_n(\mathbb{C}) \subset M_n(\mathbb{C})$ is the diagonal subalgebra. So, we find a nonzero projection $q_3 \in B$ satisfying $q_3\mathcal{M}q_3 = q_3(B \rtimes \Gamma_1)q_3$. As in (8.2), there however exists a unitary $v \in q_3(B \rtimes \Gamma_2)q_3$ with the property that $E_{q_3(B\rtimes\Sigma)q_3}(v) = 0$. It follows that v belongs to $q_3\mathcal{M}q_3$, but is orthogonal to $q_3(B \rtimes \Gamma_1)q_3$. We have reached a contradiction and conclude that $z(\Sigma) = z(\Gamma_1)$. By symmetry, we also have $z(\Sigma) = z(\Gamma_2)$.

Since $z(\Gamma_i)$ is a Γ_i-invariant projection in B, we conclude that $z(\Sigma)$ is a nonzero Γ-invariant projection in B. Take now $g_1, \ldots, g_n \in \Gamma$ such that $\Sigma_0 = \bigcap_{k=1}^n g_k\Sigma g_k^{-1}$ is finite. By definition, we have $z(g_k\Sigma g_k^{-1}) = \sigma_{g_k}(z(\Sigma))$. Since $z(\Sigma)$ is Γ-invariant, it follows that $z(g_k\Sigma g_k^{-1}) = z(\Sigma)$ for every k. Using (8.1), we conclude that $z(\Sigma) = z(\Sigma_0)$. In particular, $z(\Sigma_0) \neq 0$. So we find a nonzero A-$(B \rtimes \Sigma_0)$-subbimodule H of $L^2(p\mathcal{M})$ with $\dim_{B\rtimes\Sigma_0}(H) < \infty$. A fortiori, H is an A-B-bimodule. Since Σ_0 is finite, also $\dim_B(H) < \infty$. Taking a projection $q \in B$ with $\mathrm{Tr}(q) < \infty$ and $Hq \neq \{0\}$, it follows from Lemma 8.2 that we have found a nonzero A-Bq-subbimodule of $L^2(p\mathcal{M}q)$ having finite right dimension. This precisely means that $A \prec Bq$, and hence ends the proof of the theorem. □

We can now deduce Proposition D.

Proof of Proposition D. Write $X = \mathbb{R}^5/\mathbb{R}_+ \times [0,1]^\Gamma$ and $Y = \mathbb{R}^5 \times [0,1]^\Gamma$. Put $G = \Gamma \times \mathbb{R}_+$ and consider the action $G \curvearrowright Y$ given by

$$(g, \alpha) \cdot (x, y) = (\alpha\pi(g) \cdot x, g \cdot y) \quad \text{for all } g \in \Gamma,\ \alpha \in \mathbb{R}_+,\ x \in \mathbb{R}^5,\ y \in [0,1]^\Gamma.$$

Note that the restricted action $\Gamma \curvearrowright Y$ is infinite measure preserving and can be identified with the Maharam extension of $\Gamma \curvearrowright X$. Since the Bernoulli action

$\Gamma \curvearrowright [0,1]^\Gamma$ is mixing, we use throughout the proof the fact that the restriction of $\Gamma \curvearrowright Y$ to a subgroup $\Lambda < \Gamma$ is ergodic whenever $\pi(\Lambda)$ acts ergodically on $\mathbb{R}^5$ (see e.g. [Sc82, Proposition 2.3]). Using [PV08, Lemma 5.6], we find in particular that $\Gamma \curvearrowright Y$ is ergodic, meaning that $\Gamma \curvearrowright X$ is of type III_1.

Let $\mathcal{G}$ be a Polish group in Popa's class $\mathcal{U}_{\text{fin}}$, i.e. a closed subgroup of the unitary group of a II_1 factor, e.g. any countable group. We claim that every measurable 1-cocycle $\omega : G \times Y \to \mathcal{G}$ is cohomologous to a continuous group homomorphism $G \to \mathcal{G}$. As explained in detail in [KS12, Step 1 of the proof of Theorem 21], it follows from [PV08, Theorem 5.3] that up to cohomology, we may assume that the restriction of ω to $\text{SL}(5,\mathbb{Z})$ is a group homomorphism. By [PV08, Lemma 5.6], the diagonal action $\text{SL}(3,\mathbb{Z}) \curvearrowright \mathbb{R}^3 \times \mathbb{R}^3$ is ergodic. It follows that the diagonal action $\Sigma \curvearrowright \mathbb{R}^5 \times \mathbb{R}^5$ is ergodic as well. But then also the diagonal action $\Sigma \curvearrowright Y \times Y$ is ergodic. Since the restriction of ω to Σ is a homomorphism and since Σ commutes with the natural copies of $\mathbb{Z}$ and $\mathbb{R}_+$ inside G, it now follows from [PV08, Lemma 5.5] that ω is also a homomorphism on $\mathbb{Z}$ and on $\mathbb{R}_+$. Because $\text{SL}(5,\mathbb{Z})$, $\mathbb{Z}$ and $\mathbb{R}_+$ together generate G, we have proven the claim that ω is cohomologous to a group homomorphism.

We next prove that $\mathbb{R}_+$ is the only open normal subgroup of G that acts properly on Y. Indeed, if G_0 is such a subgroup, we first see that $\mathbb{R}_+ \subset G_0$ because $\mathbb{R}_+$ is connected. So $G_0 = \Gamma_0 \times \mathbb{R}_+$ where Γ_0 is a normal subgroup of Γ that acts properly on Y. Then $\pi(\Gamma_0)$ is a normal subgroup of $\text{SL}(5,\mathbb{Z})$. So either $\pi(\Gamma_0) = \{1\}$ or $\pi(\Gamma_0)$ has finite index in $\text{SL}(5,\mathbb{Z})$. In the latter case, Γ_0 acts ergodically on Y, rather than properly. In the former case, Γ_0 only acts by the Bernoulli shift and the properness forces Γ_0 to be finite. But Γ is an icc group, so that $\Gamma_0 = \{e\}$.

The cocycle superrigidity of $G \curvearrowright Y$, together with the previous paragraph and [PV08, Lemma 5.10], now implies that the only actions that are stably orbit equivalent to $\Gamma \curvearrowright X$ are the induced Γ'-actions, given an embedding of Γ into Γ'.

So to conclude the proof, it remains to show that $L^\infty(X) \rtimes \Gamma$ has a unique Cartan subalgebra up to unitary conjugacy. This follows from Theorem 8.1, using the subgroups $\text{SL}(2,\mathbb{Z}) < \text{SL}(5,\mathbb{Z})$ (embedded in the upper left corner) and $\mathbb{Z} < \Sigma \times \mathbb{Z}$ that act recurrently on X and intersect Σ trivially. □

Remark 8.3. In the formulation of Theorem 8.1, we required the existence of subgroups $\Lambda_i < \Gamma_i$ that intersect Σ finitely and that act in a recurrent way on (X,μ). It is actually sufficient to impose the following more ergodic-theoretic condition. Denote by $\Gamma \curvearrowright (Y,m)$ the (infinite measure preserving) Maharam extension of $\Gamma \curvearrowright (X,\mu)$. Consider the orbit equivalence relations $\mathcal{R}(\Gamma_i \curvearrowright Y)$ and $\mathcal{R}(\Sigma \curvearrowright Y)$, as well as their restrictions to nonnegligible subsets of Y. It is then sufficient to assume that for every Borel set $\mathcal{U} \subset Y$ with $0 < m(\mathcal{U}) < \infty$, almost every $\mathcal{R}(\Gamma_i \curvearrowright Y)_{|\mathcal{U}}$-equivalence class consists of infinitely many $\mathcal{R}(\Sigma \curvearrowright Y)_{|\mathcal{U}}$-equivalence classes. Indeed, writing $B = L^\infty(Y)$, it then follows from [IKT08, Lemma 2.6] that for every projection $p \in B$ with $0 < \mathrm{Tr}(B) < \infty$, there exist unitaries $u_i, v_i \in p(B \rtimes \Gamma_i)p$ satisfying (8.2). So the proof of Theorem 8.1 goes through.

Acknowledgments

The main part of this work was done at the *Institut Henri Poincaré* in Paris and I would like to thank the institute for their hospitality.

This research was supported by ERC Starting Grant VNALG-200749, Research Programme G.0639.11 of the Research Foundation—Flanders (FWO) and KU Leuven BOF research grant OT/08/032.

References

[BHR12] R. Boutonnet, C. Houdayer and S. Raum, Amalgamated free product type III factors with at most one Cartan subalgebra, Compos. Math. **150** (2014), 143–174. Zbl 06333814 MR 3164361

[CH08] I. Chifan and C. Houdayer, Bass–Serre rigidity results in von Neumann algebras, Duke Math. J. **153** (2010), 23–54. Zbl 1201.46057 MR 2641939

[CS11] I. Chifan and T. Sinclair, On the structural theory of II_1 factors of negatively curved groups, Ann. Sci. École Norm. Sup. **46** (2013), 1–33. Zbl 1290.46053 MR 3087388

[CSU11] I. Chifan, T. Sinclair and B. Udrea, On the structural theory of II_1 factors of negatively curved groups, II, Adv. Math. **245** (2013), 208–236. Zbl 1288.46037 MR 3084428

[DI12] I. Dabrowski and A. Ioana, Unbounded derivations, free dilations and indecomposability results for II_1 factors, Trans. Amer. Math. Soc., to appear; arXiv:1212.6425 (2012).

[Ho07] C. Houdayer, Construction of type II_1 factors with prescribed countable fundamental group, J. Reine Angew. Math. **634** (2009), 169–207. Zbl 1209.46038 MR 2560409

[Ho09] ———, Strongly solid group factors which are not interpolated free group factors, Math. Ann. **346** (2010), 969–989. Zbl 1201.46058 MR 2587099

[HPV10] C. Houdayer, S. Popa and S. Vaes, A class of groups for which every action is W*-superrigid, Groups Geom. Dynam. **7** (2013), 577–590. Zbl 06220439 MR 3095710

[HV12] C. Houdayer and S. Vaes, Type III factors with a unique Cartan decomposition, J. Math. Pures Appl. **100** (2013), 564–590. Zbl 1291.46052 MR 3102166

[Io06] A. Ioana, Rigidity results for wreath product II_1 factors, J. Funct. Anal. **252** (2007), 763–791. Zbl 1134.46041 MR 2360936

[Io12] ———, Cartan subalgebras of amalgamated free product II_1 factors, arXiv:1207.0054 (2012).

[IKT08] A. Ioana, A. S. Kechris and T. Tsankov, Subequivalence relations and positive-definite functions, Groups Geom. Dynam. **3** (2009), 579–625. Zbl 1186.37011 MR 2529949

[IPP05] A. Ioana, J. Peterson and S. Popa, Amalgamated free products of weakly rigid factors and calculation of their symmetry groups, Acta Math. **200** (2008), 85–153. Zbl 1149.46047 MR 2386109

[KS12] J. Keersmaekers and A. Speelman, II_1 factors and equivalence relations with distinct fundamental groups, Int. J. Math. **24** (2013), art. 1350016, 24 pp. Zbl 1280.46042 MR 3048005

[Ma63] D. Maharam, Incompressible transformations, Fund. Math. **56** (1964), 35–50. Zbl 0133.00304 MR 0169988

[OP07] N. Ozawa and S. Popa, On a class of II_1 factors with at most one Cartan subalgebra, Ann. of Math. **172** (2010), 713–749. Zbl 1201.46054 MR 2680430

[OP08] ———, On a class of II_1 factors with at most one Cartan subalgebra, II, Amer. J. Math. **132** (2010), 841–866. Zbl 1213.46053 MR 2666909

[Pi84] J.-P. Pier, *Amenable locally compact groups*, Wiley, New York, 1984. Zbl 0621.43001 MR 0767264

[Po01] S. Popa, On a class of type II_1 factors with Betti numbers invariants, Ann. of Math. **163** (2006), 809–899. Zbl 1120.46045 MR 2215135

[Po03] ———, Strong rigidity of II_1 factors arising from malleable actions of w-rigid groups, I, Invent. Math. **165** (2006), 369–408. Zbl 1120.46043 MR 2231961

[Po06] ———, On the superrigidity of malleable actions with spectral gap, J. Amer. Math. Soc. **21** (2008), 981–1000. Zbl 1222.46048 MR 2425177

[PV08] S. Popa and S. Vaes, Cocycle and orbit superrigidity for lattices in $\mathrm{SL}(n,\mathbb{R})$ acting on homogeneous spaces, in *Geometry, rigidity and group actions*, B. Farb and D. Fisher (eds.), Univ. Chicago Press, 2011, 419–451. Zbl 1291.37006 MR 2807839

[PV09] ———, Group measure space decomposition of II_1 factors and W*-superrigidity, Invent. Math. **182** (2010), 371–417. Zbl 1238.46052 MR 2729271

[PV11] ———, Unique Cartan decomposition for II_1 factors arising from arbitrary actions of free groups, Acta Math. **212** (2014), 141–198. Zbl 06305187 MR 3179609

[PV12] ———, Unique Cartan decomposition for II_1 factors arising from arbitrary actions of hyperbolic groups, J. Reine Angew. Math. **694** (2014), 215–239.

[Sc82] K. Schmidt, Asymptotic properties of unitary representations and mixing, Proc. London Math. Soc. **48** (1984), 445–460. Zbl 0539.28010 MR 0735224

[Ue07] Y. Ueda, Remarks on HNN extensions in operator algebras, Illinois J. Math. **52** (2008), 705–725. Zbl 1183.46057 MR 2546003