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Generalized Bousfield Lattices and a Generalized Retract Conjecture

by

Ryo Kato, Katsumi Shimomura and Yutaro Tatehara

Abstract

In [1], Bousfield studied a lattice (Bousfield lattice) on the stable homotopy category of spectra, and in [5], Hovey and Palmieri made the retract conjecture about that lattice. In this paper we generalize the Bousfield lattice and the retract conjecture to ones on a monoid. We also determine the structure of typical examples of generalized Bousfield lattices which satisfy the generalized retract conjecture. In particular we give explicitly the structure of the Bousfield lattice of the stable homotopy category of harmonic spectra.

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§1. Introduction

Let $\mathcal{M}$ be a closed symmetric monoidal category with zero object, and consider an object M of $\mathcal{M}$. We call the full subcategory $\langle M\rangle$ of $\mathcal{M}$ the *Bousfield class* of M if it consists of all objects A of $\mathcal{M}$ such that $MA = 0$ in the monoidal structure. Then we have a partial order on Bousfield classes by $\langle M\rangle \le \langle N\rangle$ if every object of $\langle N\rangle$ is an object of $\langle M\rangle$. The subcategories $\langle S\rangle$ and $\langle O\rangle$ of the unit S and the zero O are respectively the greatest and the least in this order. We call the collection of all Bousfield classes the *Bousfield lattice* of $\mathcal{M}$, and denote it by $\mathbb{B}(\mathcal{M})$. In the case

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R. Kato: Graduate School of Mathematics, Nagoya University, Furo-cho, Chikusa-ku, Nagoya 464-8602, Japan;
e-mail: ryo_kato_1128@yahoo.co.jp
K. Shimomura: Department of Mathematics, Faculty of Science, Kochi University, Kochi 780-8520, Japan;
e-mail: katsumi@kochi-u.ac.jp
Y. Tatehara: Department of Mathematics, Faculty of Science, Kochi University, Kochi 780-8520, Japan;
e-mail: y.tatehara@gmail.com

where a Bousfield lattice is a set, the partial order introduces a lattice structure in it, and we may investigate it algebraically.

In a sense, stable homotopy theory is analyzing stable homotopy categories (cf. [6]). A stable homotopy category is a symmetric monoidal category, and so we may consider its Bousfield lattice. In particular, T. Ohkawa [8] (cf. [2]) showed that the Bousfield lattice $\mathbb{B}$ of the stable homotopy category of spectra is a set, and then Iyengar and Krause [7] generalized this to a stable homotopy category.

In order to investigate a category, we sometimes classify special subcategories of it. From this viewpoint, we study a Bousfield lattice by classifying its localizing subcategories (see [6]). Indeed, every Bousfield class is a localizing subcategory.

In [5], Hovey and Palmieri studied the Bousfield lattice $\mathbb{B}$ deeply. Furthermore, they proposed many conjectures on the structure of $\mathbb{B}$, including the retract conjecture, which is one of our main topics. Dwyer and Palmieri [3] constructed a stable homotopy category where the conjecture does not hold. It seems that so far, no nontrivial category in which the conjecture holds has been known. In this paper, we give some examples of such categories.

As stated above, the Bousfield lattice $\mathbb{B}(\mathcal{M})$ is a set in some cases. It is then a monoid with multiplication compatible with its order. In Section 2 we introduce the notion of monoidal posets and define a functor β from a subcategory of commutative monoids to the category of monoidal posets. Then we define a Bousfield lattice of a monoid to be an object in the image of β, which is an analogy of Bousfield lattices of stable homotopy categories. In particular, $\mathbb{B}$ not only has the structure of a monoidal poset, but also is a Bousfield lattice associated to $\mathbb{B}$ itself. In Section 3, we show analogous properties on a Bousfield lattice to those given by Hovey and Palmieri [5], including the following:

Conjecture 1.1 (Original retract conjecture [5, Conj. 3.12]). *Let h be the Bousfield class of the* mod p *Eilenberg–MacLane spectrum $H\mathbb{Z}/p$ in the Bousfield lattice $\mathbb{B}$. Then there is a lattice isomorphism $r_*\colon \mathbb{B}/J(h) \to \mathbb{DL}$. Here, $J(h)$ is an ideal related to h (see Notation 3.1).*

We generalize it to generalized retract conjectures on a monoidally distributive poset (Conjectures 3.18 and 3.19) and show some related facts. Section 4 is devoted to determining Bousfield lattices obtained from principal ideal domains, and to show the conjecture is true for them. In Section 5, we study the Bousfield lattices of stable homotopy categories of Bousfield localized spectra, and construct isomorphisms between the Bousfield lattice and a Bousfield lattice given in Section 4. In particular, we have the following:

Theorem 1.2. *The generalized retract conjecture holds on the stable homotopy category of harmonic spectra.*

One of our ultimate goals is to determine the lattice structure of $\mathbb{B}$, which seems rather difficult. In the last section, we propose problems on the functor β whose solution may help us to understand the Bousfield lattice $\mathbb{B}$. We expect that these problems will give us hints to reach the above goal.

§2. Monoidal posets and Bousfield lattices

Let M be commutative monoid with unit 1. We call M a *monoid with* 0 if M admits an element $0 \in M$ such that $0 \cdot x = 0 = x \cdot 0$ for any $x \in M$. A typical example is a commutative ring with addition ignored. We denote by $\mathcal{M}_0$ the category of commutative monoids with 0 and monoid homomorphisms preserving zero.

For $M \in \mathcal{M}_0$, $\beta(M)$ denotes the set consisting of all subsets

$$\langle x \rangle = \{y \in M : xy = 0\}$$

of M for $x \in M$.

Lemma 2.1. *$\beta(M)$ for $M \in \mathcal{M}_0$ is also a monoid with* 0 *with inherited multiplication. Therefore, we have the canonical epimorphism $M \to \beta(M)$ in $\mathcal{M}_0$.*

Proof. Define multiplication in $\beta(M)$ by $\langle x \rangle\langle y \rangle = \langle xy \rangle$. We verify that it is well defined: Assume that $\langle x_0 \rangle = \langle x_1 \rangle$ and $\langle y_0 \rangle = \langle y_1 \rangle$. Then

$$\begin{aligned} zx_0y_0 = 0 \;&\Leftrightarrow\; zx_1y_0 = 0 \quad (\text{since } \langle x_0 \rangle = \langle x_1 \rangle) \\ &\Leftrightarrow\; zx_1y_1 = 0 \quad (\text{since } \langle y_0 \rangle = \langle y_1 \rangle), \end{aligned}$$

and $\langle x_0y_0 \rangle = \langle x_1y_1 \rangle$. The elements $\langle 1 \rangle$ and $\langle 0 \rangle$ are the unit and the zero elements of $\beta(M)$. □

Remark 2.2. We notice that $\beta(R) = \mathbb{Z}/2$ if R is a domain.

Lemma 2.3. *Let M be a monoid with* 0*. Then $\beta(M)$ admits a partial order $\leq$ defined by $\langle x \rangle \leq \langle y \rangle$ if $\langle x \rangle \supset \langle y \rangle$. Moreover, $\langle 1 \rangle$ and $\langle 0 \rangle$ are respectively the greatest and the least elements of $\beta(M)$.*

Proof. This is trivial since $\langle 1 \rangle = \{0\}$ and $\langle 0 \rangle = M$. □

By the lemma, the commutative monoid $\beta(M)$ also has a poset structure. We define the following notion by abstracting its crucial properties.

Definition 2.4. A *monoidal poset* $P = (P, \le, \cdot, 1, 0)$ is defined by the following data:

(1) $(P, \cdot, 1, 0)$ is a monoid with 0.
(2) $(P, \le)$ is a poset.
(3) The following are equivalent:

  (a) $x \le y$.
  (b) $cy = 0$ for $c \in P$ implies $cx = 0$.

A *monoidal poset map* $f\colon P \to P'$ is an order preserving monoid homomorphism with $f(0) = 0$.

Lemma 2.3 implies the following.

Corollary 2.5. *$\beta(M)$ for $M \in \mathcal{M}_0$ is a monoidal poset with $1 = \langle 1\rangle$ and $0 = \langle 0\rangle$.*

Lemma 2.6. *Let M be a monoidal poset. Then $\beta(M) = M$ as monoidal posets.*

Remark 2.7. A monoidal poset may seem to be a lattice, but unfortunately this is not always the case. Consider $M = \{1, x_i, y_i, w, 0 : i = 1, 2\}$ with multiplication

1	x_1	x_2	y_1	y_2	w
x_1	w	w	0	w	0
x_2	w	w	w	0	0
y_1	0	w	0	0	0
y_2	w	0	0	0	0
w	0	0	0	0	0

Then the join of y_1 and y_2 does not exist.

Let $\mathcal{MP}$ denote the category of monoidal posets and monoidal poset maps. Then $\mathcal{MP} \subset \mathcal{M}_0$.

Lemma 2.8. *Let M be a monoidal poset. If $x \le y$ and $z \le w$, then $xz \le yw$. In particular, if $x \le y$, then $xz \le yz$ for any z.*

Proposition 2.9. *The category $\mathcal{MP}$ admits direct products.*

Proof. Let $\{M_\lambda\}$ be a family of monoidal posets. Then we have the direct product $\prod_\lambda M_\lambda$ of monoids. Define an order $\le$ on $\prod_\lambda M_\lambda$ by $(x_\lambda) \le (y_\lambda)$ if $(c_\lambda)(y_\lambda) = (0)$ implies $(c_\lambda)(x_\lambda) = (0)$. It is straightforward to verify that this is the desired direct product. □

Lemma 2.10. *Let $\{M_\lambda\}$ be a family of monoidal posets and let $\langle x_\lambda\rangle, \langle y_\lambda\rangle \in \beta(M_\lambda)$ and $\langle (x_\lambda)\rangle, \langle (y_\lambda)\rangle \in \beta(\prod_\lambda M_\lambda)$. Then $\langle x_\lambda\rangle \le \langle y_\lambda\rangle$ for all λ if and only if $\langle (x_\lambda)\rangle \le \langle (y_\lambda)\rangle$ in $\beta(\prod_\lambda M_\lambda)$.*

Proof. Assume that $\langle x_\lambda\rangle \le \langle y_\lambda\rangle$ for any λ. Then

$$\begin{aligned}(c_\lambda)(y_\lambda) = 0 &\Rightarrow c_\lambda y_\lambda = 0 \quad \text{for any } \lambda\\ &\Rightarrow c_\lambda x_\lambda = 0 \quad \text{for any } \lambda \text{ (since } \langle x_\lambda\rangle \le \langle y_\lambda\rangle)\\ &\Rightarrow (c_\lambda)(x_\lambda) = 0,\end{aligned}$$

Conversely, suppose that $\langle (x_\mu)\rangle \le \langle (y_\mu)\rangle$. Then, for any λ,

$$\begin{aligned}y_\lambda c_\lambda = 0 &\Rightarrow (y_\lambda)(c_\lambda)_0 = 0\\ &\Rightarrow (x_\lambda)(c_\lambda)_0 = 0 \quad \text{(since } \langle (x_\mu)\rangle \le \langle (y_\mu)\rangle)\\ &\Rightarrow x_\lambda c_\lambda = 0\end{aligned}$$

in M_λ, where $(c_\lambda)_0$ denotes the element (x_μ) such that $x_\lambda = c_\lambda$ and $x_\mu = 0$ for $\mu \ne \lambda$. □

Corollary 2.11. *Let $\{M_\lambda\}$ be a family of monoidal posets. Define an order $\le'$ on the set $\prod_\lambda M_\lambda$ by $(x_\lambda) \le' (y_\lambda)$ if $x_\lambda \le y_\lambda$ for all λ. Then it is equivalent to the order in the proof of Proposition 2.9.*

Corollary 2.12. *Let $\{M_\lambda\}$ be a family of monoidal posets. Then $\bigvee_\mu (x^\mu_\lambda) = (\bigvee_\mu x^\mu_\lambda)$ for any subset $\{(x^\mu_\lambda)\}_\mu \subset \prod_\lambda M_\lambda$.*

Proof. Since $(x^\mu_\lambda) \le (\bigvee_\mu x^\mu_\lambda)$ for all μ, $\bigvee_\mu (x^\mu_\lambda) \le (\bigvee_\mu x^\mu_\lambda)$. If $(x^\mu_\lambda) \le (z_\lambda)$, then $x^\mu_\lambda \le z_\lambda$, and so $\bigvee_\mu x^\mu_\lambda \le z_\lambda$, that is, $(\bigvee_\mu x^\mu_\lambda) \le (z_\lambda)$. Therefore, $\bigvee_\mu (x^\mu_\lambda) = (\bigvee_\mu x^\mu_\lambda)$ by definition. □

We call an epimorphism $f\colon M \to N$ of $\mathcal{M}_0$ *strong* if $f(x) = 0$ if and only if $x = 0$.

We define a map $\beta(f)\colon \beta(M) \to \beta(N)$ by sending $\langle x\rangle$ to $\langle f(x)\rangle$.

Lemma 2.13. *For a strong epimorphism $f\colon M \to N$, the map $\beta(f)$ is not only a monoidal poset map but also a strong epimorphism.*

Proof. As f is a strong epimorphism, $c\cdot f(x) = 0 \Leftrightarrow f(c')\cdot f(x) = 0 \Leftrightarrow f(c'\cdot x) = 0 \Leftrightarrow c'\cdot x = 0$ for an element c' such that $f(c') = c$. This shows that $\langle x\rangle = \langle y\rangle$ implies $\langle f(x)\rangle = \langle f(y)\rangle$. It is easy to see that $\beta(f)$ is a strong epimorphism. □

We also consider the subcategories $\mathcal{M}_0^{\mathrm{epi}}$ and $\mathcal{MP}^{\mathrm{epi}}$ of $\mathcal{M}_0$ and $\mathcal{MP}$, respectively, obtained by taking as morphisms strong epimorphisms only.

Corollary 2.14. *The operation* β *above defines a functor* $\beta\colon \mathcal{M}_0^{\mathrm{epi}} \to \mathcal{MP}^{\mathrm{epi}} \subset \mathcal{M}_0^{\mathrm{epi}}$.

By the above argument, we redefine Bousfield lattices as follows. The definition is one of our main topics in this paper.

Definition 2.15. For a monoid $M \in \mathcal{M}_0^{\mathrm{epi}}$ we call the monoidal poset $\beta(M)$ the *Bousfield lattice associated to* M.

In earlier papers, a Bousfield lattice comes from a closed symmetric monoidal category with a zero object. However, its set-theoretic confusion complicates our argument too much. Our new definition settles this problem, and the following proposition says that this argument is consistent.

Proposition 2.16. *The Bousfield lattice* $\mathbb{B}$ *of the stable homotopy category of spectra is a Bousfield lattice in the sense of our definition.*

Proof. By forgetting the ordering on $\mathbb{B}$, we regard $\mathbb{B}$ as a monoid with $1 = \langle S\rangle$ and $0 = \langle *\rangle$. Then it is clear that $\beta(\mathbb{B}) = \mathbb{B}$. □

Proposition 2.17. *The functor* β *satisfies the following:*

(1) $\beta(\prod_\lambda M_\lambda) = \prod_\lambda \beta(M_\lambda)$.
(2) $\beta\beta(M) = \beta(M)$.

Proof. (1) Let $\{p_\lambda\colon \beta(\prod_\lambda M_\lambda) \to \beta(M_\lambda)\}$ be a family of epimorphisms defined by $\langle(x_\lambda)\rangle \mapsto \langle x_\lambda\rangle$, and $\{f_\lambda\colon W \to \beta(M_\lambda)\}$ a family of poset maps. We notice that p_λ is well defined by Lemma 2.10. For an element $w \in W$, we take an element $w_\lambda \in W_\lambda$ so that $f_\lambda(w) = \langle w_\lambda\rangle$, and define $g\colon W \to \beta(\prod_\lambda M_\lambda)$ by $g(w) = \langle(w_\lambda)\rangle$. Then g is also a well defined poset map by Lemma 2.10 and

$$p_\lambda g(w) = p_\lambda(\langle(w_\lambda)\rangle) = \langle w_\lambda\rangle = f_\lambda(w).$$

Suppose that there is another poset map $g'\colon W \to \beta(\prod_\lambda M_\lambda)$ satisfying $p_\lambda g'(w) = f_\lambda(w)$ for $w \in W$, and g' maps w to $\langle(w'_\lambda)\rangle$. Then

$$\begin{aligned} p_\lambda g'(w) = f_\lambda(w) \text{ for any } \lambda &\Leftrightarrow \langle w'_\lambda\rangle = \langle w_\lambda\rangle \text{ for any } \lambda \\ &\Leftrightarrow \langle(w'_\lambda)\rangle = \langle(w_\lambda)\rangle \quad \text{(by Lemma 2.10)} \\ &\Leftrightarrow g'(w) = g(w). \end{aligned}$$

Therefore, $\beta(\prod_\lambda M_\lambda)$ is the product $\prod_\lambda \beta(M_\lambda)$.

(2) follows from Lemma 2.6. □

§3. Retract conjecture

From now on, we assume that every monoidal poset considered is a complete lattice.

Since a monoidal poset M is a sup-lattice with least element $0 = \langle 0 \rangle$, M is a bounded lattice.

Notation 3.1. For a monoidal poset M, we introduce the following notations:

$$\begin{aligned} a_M(x) &:= \bigvee \{y \in M \colon xy = 0\} \quad \text{ for } x \in M, \\ DL(M) &:= \{x \in M : x^2 = x\}, \\ r_M(x) &:= \bigvee \{w \in DL(M) \colon w \leq x\} \quad \text{ for } x \in M, \\ J_M(x) &:= \{y \in M \colon y \leq x \wedge a_M(x)\} \quad \text{ for } x \in M, \\ N(M) &:= \{x \in M \colon x^n = 0 \text{ for some } n \geq 1\}, \\ A(M) &:= \{x \in M \colon r_M(x) = 0\}. \end{aligned}$$

We will omit M from notations if M is clear from the context.

It is well known that the subposet $DL(M)$ is also a complete lattice. Indeed, the following holds.

Proposition 3.2. *$DL(M)$ is closed under arbitrary joins.*

Proof. By Lemma 2.8, $(\bigvee_{\lambda \in \Lambda} x_\lambda)^2 \leq \bigvee_{\lambda \in \Lambda} x_\lambda$. Suppose that x_λ is in DL for $\lambda \in \Lambda$. Then $x_\lambda = x_\lambda^2 \leq (\bigvee_{\lambda \in \Lambda} x_\lambda)^2$, and so $\bigvee_{\lambda \in \Lambda} x_\lambda \leq (\bigvee_{\lambda \in \Lambda} x_\lambda)^2$. □

Lemma 3.3. *In $DL(M)$, the meet of x and y is xy.*

Proof. Since $x \wedge y \leq x$ and $x \wedge y \leq y$, if $x \wedge y \in DL(M)$ then $x \wedge y \leq xy$. □

Remark 3.4. $DL(M)$ is not always sublattice of M by Lemma 3.3.

In investigating the original Bousfield lattice $\mathbb{B}$, the operations r and a play important roles (see [5]). Here we give their properties on monoidal posets.

Proposition 3.5. *Let M be a monoidal poset, and $r = r_M \colon M \to M$ be the map defined in Notation 3.1.*

(1) *r is order-preserving, i.e. $x \leq y$ implies $r(x) \leq r(y)$.*
(2) *$r(x)^2 = r(x)$ and $r^2(x) = r(x)$ for $x \in M$.*
(3) *$r(x) \leq x^n$ for any $n \geq 1$.*
(4) *$r(xy) = r(x)r(y) = r(x \wedge y)$ for $x, y \in M$.*

Proof. (1) is trivial, and (2) follows from Proposition 3.2. For (3), $r(x) \leq x$ by definition, and we have $r(x) = r(x)^n \leq x^n$.

Since $r(x)r(y) \leq xy$ and $r(x)r(y) \in DL(M)$, we have $r(x)r(y) \leq r(xy)$. We also see $r(x \wedge y) \leq r(x)r(y)$, since $r(x \wedge y) \leq r(x)$ and $r(x \wedge y) \leq r(y)$. Therefore, $r(xy) \leq r(x \wedge y) \leq r(x)r(y) \leq r(xy)$, and we obtain (4). □

The behavior of the map r is the same as on $\mathbb{B}$, but that of the operation a not. Indeed, for any $x \in M$ and $\{y_\lambda\}_\lambda \subset M$, the relation $x(\bigvee_\lambda y_\lambda) \geq \bigvee_\lambda (xy_\lambda)$ is not always an equality. To make the operator a have good properties, we introduce the following notion.

Definition 3.6. A monoidal poset M is *monoidally distributive* if $x(\bigvee_\lambda y_\lambda) = \bigvee_\lambda (xy_\lambda)$ for any $x \in M$ and $\{y_\lambda\}_\lambda \subset M$.

Remark 3.7. If M is a monoidally distributive poset, then $DL(M)$ is a distributive lattice by Lemma 3.3.

In the same way as in [5], we have

Proposition 3.8. *Let M be a monoidally distributive poset. Then*

(1) *$a(-)$ is order-reversing.*

(2) *$xy = 0$ if and only if $x \leq a(y)$.*

(3) *$aa(x) = x$.*

Lemma 3.9. *Let M be a monoidally distributive poset. Let $c \in M$ be such that $c^n = 0$ for a positive integer n. Then, for any $x \in M$, $(x \vee c)^n \leq x$ and $r(x \vee c) = r(x)$.*

Proof. Under the assumption, we compute

$$(x \vee c)^n = x^n \vee x^{n-1}c \vee \cdots \vee xc^{n-1} = x(x^{n-1} \vee x^{n-2}c \vee \cdots \vee c^{n-1}) \leq x$$

for any $x \in M$. So, if $z \leq x \vee c$ for $z \in DL(M)$, then $z \leq x$. Thus, $r(x \vee c) = r(x)$ by definition of r. □

Proposition 3.10. *Let M be a monoidally distributive poset. Then $J_M(x) \subset N(M) \subset A(M)$ for any $x \in M$.*

Proof. Since $(x \wedge a_M(x))(x \wedge a_M(x)) \leq xa_M(x) = 0$ by Proposition 3.8(2), we have $J_M(x) \subset N(M)$. Suppose that $x^n = 0$; then $r(x) = r(x)^n = r(x^n) = r(0) = 0$ by Proposition 3.5(4). So we have $N(M) \subset A(M)$. □

Proposition 3.11. *Let M_λ be a monoidal poset for any $\lambda \in \Lambda$. Then*

(1) *$r((x_\lambda)) = (r(x_\lambda))$ for any $(x_\lambda) \in \prod_\lambda M_\lambda$.*

(2) *r preserves arbitrary joins on M_λ for any $\lambda \in \Lambda$ if and only if r preserves arbitrary joins on $\prod_\lambda M_\lambda$.*

Proof. (1) is stated in Corollary 2.12.

(2) Suppose that r preserves arbitrary joins on M_λ for any $\lambda \in \Lambda$. Then, for $\{(x_\lambda^\mu)\}_\mu \subset \prod_\lambda M_\lambda$,

$$\begin{aligned} r\Big(\bigvee_\mu (x_\lambda^\mu)\Big) &= r\Big(\Big(\bigvee_\mu x_\lambda^\mu\Big)\Big) && \text{(by Corollary 2.12)} \\ &= \Big(r\Big(\bigvee_\mu x_\lambda^\mu\Big)\Big) && \text{(by (1))} \\ &= \Big(\bigvee_\mu r(x_\lambda^\mu)\Big) \\ &= \bigvee_\mu (r(x_\lambda^\mu)) && \text{(by Corollary 2.12).} \end{aligned}$$

Therefore, r preserves arbitrary joins on $\prod_\lambda M_\lambda$.

Conversely, if r preserves arbitrary joins on $\prod_\lambda M_\lambda$, then

$$\begin{aligned} \Big(r\Big(\bigvee_\mu x_\lambda^\mu\Big)\Big) &= r\Big(\Big(\bigvee_\mu x_\lambda^\mu\Big)\Big) && \text{(by (1))} \\ &= r\Big(\bigvee_\mu (x_\lambda^\mu)\Big) && \text{(by Corollary 2.12)} \\ &= \bigvee_\mu (r(x_\lambda^\mu)) \\ &= \Big(\bigvee_\mu r(x_\lambda^\mu)\Big) && \text{(by Corollary 2.12).} \end{aligned}$$

It follows that r preserves arbitrary joins on M_λ for any $\lambda \in \Lambda$ as desired. □

Remark 3.12. We notice that M_λ is a monoidally distributive poset for any $\lambda \in \Lambda$ if and only if $\prod_{\lambda\in\Lambda} M_\lambda$ is a monoidally distributive poset. Indeed, if M_λ is monoidally distributive for any $\lambda \in \Lambda$, then $(c_\lambda)(\bigvee_\mu (x_\lambda^\mu)) = (c_\lambda)(\bigvee_\mu x_\lambda^\mu) = (c_\lambda(\bigvee_\mu x_\lambda^\mu)) = (\bigvee_\mu c_\lambda x_\lambda^\mu) = \bigvee_\mu (c_\lambda x_\lambda^\mu)$ for $(c_\lambda) \in \prod_\lambda M_\lambda$ and $\{(x_\lambda^\mu)\}_\mu \subset \prod_\lambda M_\lambda$ by Corollary 2.12. Thus, $\prod_\lambda M_\lambda$ is a monoidally distributive poset. Conversely, if $\prod_\lambda M_\lambda$ is a monoidally distributive poset, then $(c_\lambda(\bigvee_\mu x_\lambda^\mu)) = (c_\lambda)(\bigvee_\mu x_\lambda^\mu) = (c_\lambda)(\bigvee_\mu (x_\lambda^\mu)) = \bigvee_\mu (c_\lambda x_\lambda^\mu) = (\bigvee_\mu c_\lambda x_\lambda^\mu)$ by Corollary 2.12. Therefore, M_λ is a monoidally distributive poset for any $\lambda \in \Lambda$ by Lemma 2.10.

Recall that an *ideal* I of a poset is any subset of M such that:

(1) if $x \in I$, and $y \le x$, then $y \in I$, and
(2) for $x, y \in I$, there is an element $z \in I$ such that $x \le z$ and $y \le z$.

Suppose that a monoidal poset M is an ordinary lattice. Then any ideal of M is also an ideal as a lattice, and for an ideal I, M/I is the lattice of equivalencr classes under the equivalence relation defined by

$$x \sim y \quad \text{if and only if} \quad x \vee c = y \vee c \text{ for some } c \in I, \tag{3.13}$$

with order given by $[x] \leq [y] \Leftrightarrow x \vee c \leq y \vee c$ for some $c \in I$. We notice that M/I is complete if M and I are complete. If M is monoidally distributive, then M/I has multiplication $[x][y] := [xy]$. Indeed, if $x \vee i = x' \vee i$ and $y \vee j = y' \vee j$ for $x, x', y, y' \in M$ and $i, j \in I$, then $(x \vee i)(y \vee j) = (x' \vee i)(y' \vee j)$ turns into

$$\begin{aligned} xy \vee (x \vee i)j \vee (y \vee j)i &= x'y' \vee (x' \vee i)j \vee (y' \vee j)i \\ &= x'y' \vee (x \vee i)j \vee (y \vee j)i. \end{aligned}$$

Since $(x \vee i)j \vee (y \vee j)i \in I$, the multiplication is well defined.

Remark 3.14. M/I is not always a monoidal poset. Indeed, let $M = \{1, x, y, 0\}$ be a monoidal poset with multiplication $x^2 = x, xy = 0, y^2 = 0$. Then, for the ideal $I = \{y, 0\}$, $M/I = \{1, x, 0\}$ and $\beta(M/I) = \{1, 0\}$. Since $M/I \neq \beta(M/I)$, M/I is not a monoidal poset by Lemma 2.6.

Lemma 3.15. *Let M be a monoidally distributive poset. Then $N(M)$ is an ideal of M, and $J_M(x)$ is a principal ideal of M for any $x \in M$.*

Proof. Suppose that $x^n = 0$ and $y^m = 0$. Then $(x \vee y)^{n+m} = \bigvee_{a+b=n+m} x^a y^b$. Since here $a < n$ implies $b \geq m$, it follows that $(x \vee y)^{n+m} = 0$. So $N(M)$ is an ideal of M. By definition, $J_M(x)$ is a principal ideal of M. □

Now, consider the following correspondence:

$$r_* \colon M/I \to DL(M), \qquad [x] \mapsto \{r(y) \colon y \in [x]\}.$$

We notice that if r_* is a mapping (i.e. a single-valued mapping), then it is a surjection.

Theorem 3.16. *Let M be a monoidally distributive poset and I an ideal in M.*

(1) *If I is contained in N, then r_* is a mapping.*

(2) *If r_* is a mapping, then $I \subset A$.*

(3) *If r_* is an injection, then $I = A$.*

(4) *If r_* is an injection and $I \subset N$, then:*

   (a) *For any x and y in M, we have $r(x \vee y) = r(x) \vee r(y)$. In particular, if I is a principal ideal, then r preserves arbitrary joins.*

   (b) *For any $x \in M$, there exists an integer n such that $x^n = r(x)$.*

Proof. (1) If $x \vee c = y \vee c$ for $x, y \in M$ and $c \in I \subset N$, then $r(x) = r(y)$ by Lemma 3.9.

(2) For $x \in I$, $[x] = 0 = [0]$ in M/I, and so $r(x) = r_*([x]) = r_*([0]) = r(0) = 0$. Thus, $x \in A$.

(3) For $x \in A$, $r_*([x]) = r(x) = 0 = r_*([0])$. It follows that $[x] = [0]$, since r_* is an injection, which implies $x \in I$. So we obtain $A = I$ by (2).

(4) For $x \in M$, $r_*([x]) = r(x) = r^2(x) = r_*([r(x)])$ and $[x] = [r(x)]$, since r_* is an injection. So we have an element $c_x \in N$ such that $x \vee c_x = r(x) \vee c_x$, and then:

(a) Since $x \vee y \vee c_x \vee c_y = r(x) \vee r(y) \vee c_x \vee c_y$, $r(x \vee y) = r(x) \vee r(y)$ by Lemma 3.9. Suppose that I is a principal ideal and take a generator m of I. Then $(\bigvee_\lambda x_\lambda) \vee m = (\bigvee_\lambda r(x_\lambda)) \vee m$ for any subset $\{x_\lambda\}_\lambda \subset M$. Therefore $r(\bigvee_{\lambda\in\Lambda} x_\lambda) = \bigvee_{\lambda\in\Lambda} r(x_\lambda)$ by Lemma 3.9.

(b) Since there exists an integer n such that $c_x^n = 0$, we have

$$x^n \leq (x \vee c_x)^n = (r(x) \vee c_x)^n \leq r(x)$$

by Lemma 3.9. □

Hovey and Palmieri introduced a map $r_*\colon M/J(h) \to DL$, and proposed Conjecture 1.1 of the introduction. Here, we generalize this map to our setting.

Lemma 3.17. *For a monoidal poset M the map $r_M\colon M \to M$ factors through $DL(M)$. Furthermore, it induces the map $r_*\colon M/J_M(y) \to DL(M)$ for any $y \in M$, sending the class $[x]$ to $r_M(x)$.*

Proof. The former statement follows from Proposition 3.5(2), and the latter from Proposition 3.10 and Theorem 3.16(1). □

By Theorem 3.16, we see that $J(h) = A$ if Conjecture 1.1 holds. This makes us conjecture the following:

Conjecture 3.18 (Generalized retract conjecture 1 (GRC1)). *Let M be a monoidal poset. If M is a complete lattice and is monoidally distributive, and if $A = A(M)$ is an ideal of M, then $r_*\colon M/A \to DL$ is a lattice isomorphism.*

Conjecture 3.19 (Generalized retract conjecture 2 (GRC2)). *Let M be a monoidal poset. If M is a complete lattice and is monoidally distributive, then $r_*\colon M/N \to DL(M)$ is a lattice isomorphism.*

By Theorem 3.16(3), we see the following:

Corollary 3.20. GRC2 *implies* GRC1*.*

Example 3.21. Consider the monoidal poset $M = \beta(\mathbb{Z}/2^m\mathbb{Z})$. Then

$$\begin{aligned} M &= \{1, 2, 2^2, \ldots, 2^{m-1}, 2^m = 0\}, \\ DL(M) &= \{1, 0\}, \\ N(M) &= \{2, 2^2, \ldots, 2^{m-1}, 0\}. \end{aligned}$$

Thus $M/N(M) \cong DL(M)$. That is, GRC2 holds on $\beta(\mathbb{Z}/2^m\mathbb{Z})$.

Theorem 3.22. *For a monoidally distributive poset M, the following are equivalent:*

(1) $r_*\colon M/N \to DL$ *is an isomorphism.*
(2) *Any class* $[x] \in M/N$ *satisfies* $[x^2] = [x]$.

Proof. The statement (1) implies (2), since $r_*([x]) = r_*([x^2])$.

For the converse, it suffices to show that r_* is injective. If $[x^2] = [x]$, then $[x] = [x^n]$ for any $n > 0$ by induction. So, we have an element $c_x \in N$ for each $x \in M$ such that

$$x \vee c_x = x^n \vee c_x \quad \text{for any } n > 0. \tag{3.23}$$

Since $c_x \in N$, we have an integer $L = L(x) > 0$ such that $c_x^L = 0$. Then

$$x^L \leq (x \vee c_x)^L = (x^n \vee c_x)^L \leq x^n$$

for any $n > 0$ by Lemma 3.9. In particular, $x^L = (x^L)^2$ and so

$$x^{L(x)} = r(x). \tag{3.24}$$

by Proposition 3.5.

Now suppose that $r_*([x]) = r_*([y])$. Then $r(x) = r(y)$, and $x^{L(x)} = y^{L(y)}$ by (3.24). By (3.23),

$$x \vee c_x \vee c_y = x^{L(x)} \vee c_x \vee c_y = y^{L(y)} \vee c_y \vee c_x = y \vee c_x \vee c_y$$

and $[x] = [y]$ by the definition (3.13). □

Furthermore, Proposition 3.11 leads us to the following.

Proposition 3.25. *Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of monoidally distributive posets. Then the following are equivalent:*

(1) GRC *holds on* M_λ *for any* $\lambda \in \Lambda$.
(2) GRC *holds on* $\prod M_\lambda$.

Here, GRC *is* GRC1 *or* GRC2.

As an application, we extend a result of Dwyer and Palmieri:

Theorem 3.26 (Dwyer–Palmieri [3]). *There is a ring Λ such that the original retract conjecture does not hold on the derived category $D(\Lambda)$ of Λ.*

In the proof of it, Dwyer and Palmieri define Λ to be a truncated polynomial ring over a field k, and take $\langle k\rangle$ instead of $h = \langle H\mathbb{Z}/p\rangle$. Here $\langle k\rangle$ denotes the

Bousfield class of a complex $\{X_i\}$ with $X_0 = k$, and $X_i = 0$ if $i \neq 0$. By a similar argument to that of Hovey and Palmieri in [5], if r_* is an isomorphism from $\mathbb{B}(D(\Lambda))/J(\langle k\rangle)$ to DL, then any Bousfield class $x \in \mathbb{B}(D(\Lambda))$ satisfies $x^2 = x^3$. They show the theorem by constructing a Bousfield class $y \in \mathbb{B}(D(\Lambda))$ such that $y > y^2 > \cdots > y^n > \cdots$. By Theorem 3.16, the existence of y implies the following:

Theorem 3.27. *The map $r_*\colon \mathbb{B}(D(\Lambda))/N \to DL$ is not isomorphic.*

§4. A Bousfield lattice associated to a quotient of PID

We abbreviate 'principal ideal domain' to 'PID'. Furthermore, we write x for $\langle x\rangle \in \beta(M)$, where no confusion arises.

Theorem 4.1. *Let P be a PID and put $q = p_0^{e_0} \cdots p_{m-1}^{e_{m-1}} \in P$ for prime elements p_i and integers $e_i > 0$. Let B denote the Bousfield lattice $\beta(P/qP)$. Then:*

(1) *$B = \{x \in P\colon x \mid q\}$ as sets. In particular q is the zero element 0.*

(2) *$x \geq y$ if and only if $x \mid y$.*

(3) *$DL = \{p_0^{s_0} \cdots p_{m-1}^{s_{m-1}}\colon s_i = 0$ or $e_i\}$.*

(4) *$N = \{x \in B\colon p_0 \cdots p_{m-1} \mid x$ in $P\}$.*

(5) *$B = \prod_{i=0}^{n-1} \beta(P/p_i^{e_i} P)$.*

Proof. For an element $x \in P$, we consider an integer $e_i(x)$ and an element $x_{(q)}$ defined by

$$e_i(x) := \max\{e\colon e \leq e_i \text{ and } p_i^e \mid x\} \quad \text{and} \quad x_{(q)} := \textstyle\prod_{0\leq i<m} p_i^{e_i(x)}.$$

We see that

$$x = x_{(q)} \in \beta(P/qP) \quad \text{for any } x \in P. \tag{4.2}$$

Indeed, $x_{(q)}$ divides x, and so $x \leq x_{(q)}$. If $xy = 0$ in P/qP, then xy is divisible by q in P. Therefore, $q \mid x_{(q)}y_{(q)}$ and so $q \mid x_{(q)}y$. Hence $x_{(q)}y = 0$ in P/qP and so $x_{(q)} \leq x$.

The statements (1)–(4) follow immediately from (4.2), and (5) from (1). □

Corollary 4.3. *We have isomorphisms of monoidal posets*

$$\beta(P/p_0^{e_0} \cdots p_{n-1}^{e_{n-1}} P) = \prod_{i=0}^{n-1} \beta(\mathbb{Z}/2^{e_i}\mathbb{Z}) \quad \textit{and}$$

$$DL(\beta(P/p_0^{e_0} \cdots p_{n-1}^{e_{n-1}} P)) = \prod_{i=0}^{n-1} \mathbb{Z}/2.$$

Corollary 4.4. *For any PID P and a non-zero element $q \in P$, the Bousfield lattice $\beta(P/qP)$ is monoidally distributive.*

Proof. Noticing the relation

$$(p_0^{s_0} \cdots p_{n-1}^{s_{n-1}}) \vee (p_0^{t_0} \cdots p_{n-1}^{t_{n-1}}) = p_0^{\ell_0} \cdots p_{n-1}^{\ell_{n-1}} \quad \text{with} \quad \ell_i = \min\{s_i, t_i\},$$

the proof is straightforward. □

Theorem 4.5. *If P is a PID and $q \in P \setminus \{0\}$, then* GRC2 *holds on $\beta(P/qP)$, and hence so does* GRC1.

Proof. The ideal $N(\beta(P/qP))$ has the greatest element $g = p_0 \cdots p_{n-1}$. We compute

$$\begin{aligned}(p_0^{s_0} \cdots p_{n-1}^{s_{n-1}}) \vee g &= p_0^{\min\{s_0,1\}} \cdots p_{n-1}^{\min\{s_{n-1},1\}} = p_0^{\min\{2s_0,1\}} \cdots p_{n-1}^{\min\{2s_{n-1},1\}} \\ &= (p_0^{2s_0} \cdots p_{n-1}^{2s_{n-1}}) \vee g = (p_0^{s_0} \cdots p_{n-1}^{s_{n-1}})^2 \vee g.\end{aligned}$$

So the theorem follows from Theorem 3.22. □

Remark 4.6. Here is another proof: since $\beta(P/qP) = \prod_{i=0}^{n-1} \beta(\mathbb{Z}/2^{e_i}\mathbb{Z})$ and GRC2 holds on $\beta(\mathbb{Z}/2^{e_i}\mathbb{Z})$, GRC2 holds on $\beta(P/qP)$ by Proposition 3.25.

§5. Bousfield lattices of stable homotopy categories

Let $\mathcal{L}_E$ for a spectrum E denote the stable homotopy category of E-local spectra, and $\mathbb{B}(\mathcal{L}_E)$ the Bousfield lattice in the sense of Bousfield. Then we have the Bousfield localization functor $L_E \colon \mathcal{S} \to \mathcal{L}_E$. The monoidal structure of $\mathcal{L}_E$ is given by $XY = L_E(X \wedge Y)$. We consider the Johnson–Wilson spectra $E(n)$ and the Morava K-theories $K(n)$ for $n \geq 0$. From the chromatic viewpoint, investigating the categories $\mathcal{L}_n$ $(= \mathcal{L}_{E(n)})$ and $\mathcal{L}_{K(n)}$ is one of main targets of stable homotopy theory. We determine the Bousfield lattices of these categories.

We begin with a simple category. A spectrum F is called a *field* if it is a ring spectrum and $F \wedge X = \bigvee \Sigma^a F$ for all spectra X.

Proposition 5.1. *Let F be a field. Then $\mathbb{B}(\mathcal{L}_F) = \mathbb{Z}/2$.*

Proof. Since F is a ring spectrum, we have $FX = F \wedge X$. We easily see that $\langle X \rangle \geq \langle FX \rangle$. Suppose that $(FX)C = 0$. Then XC is F-acyclic and so $XC = 0$. It follows that $\langle X \rangle = \langle FX \rangle = \langle \bigvee \Sigma^i F \rangle = 0$ or $\langle F \rangle$, which shows the lemma. □

By [4], the Eilenberg–MacLane spectrum $H\mathbb{Z}/p$ and the Morava K-theories $K(n)$ are fields.

Corollary 5.2. $\mathbb{B}(\mathcal{L}_{H\mathbb{Z}/p}) = \mathbb{Z}/2 = \mathbb{B}(\mathcal{L}_{K(n)})$.

Theorem 5.3. *Let $p_0, \ldots, p_n$ be $n+1$ distinguished prime numbers. Then $\mathbb{B}(\mathcal{L}_n)$ is isomorphic to $\beta(\mathbb{Z}/p_0 \cdots p_n) = \prod_{i=0}^{n} \mathbb{Z}/2$ in $\mathcal{MP}$.*

Proof. The Bousfield lattice $\mathbb{B}(\mathcal{L}_n)$ consists of $\langle L_n X\rangle$ for all spectra X, which equals, by Ravenel [9],

$$\begin{aligned}\langle L_n X\rangle = \langle L_n S^0\rangle \cdot \langle X\rangle \ &= \ \langle E(n)\rangle \cdot \langle X\rangle \\ = \Big(\bigvee_{0\le i\le n} \langle K(i)\rangle\Big)\cdot\langle X\rangle \ &= \ \bigvee_{0\le i\le n \text{ and } K(i)\wedge X\neq 0} \langle K(i)\rangle\end{aligned}$$

since L_n is smashing and $K(n)$ is a field. Here $\langle X\rangle \cdot \langle Y\rangle$ is the Bousfield class of the smash product $X \wedge Y$. We define a map $f\colon \mathbb{B}(\mathcal{L}_n) \to \beta(\mathbb{Z}/p_0\cdots p_n)$ by $f(\bigvee_{i\in S}\langle K(i)\rangle) = \prod_{i\notin S} p_i$ for $S \subset \{0,1,\ldots,n\}$. Then f preserves multiplication, since

$$\Big(\bigvee_{i\in S}\langle K(i)\rangle\Big)\Big(\bigvee_{j\in T}\langle K(j)\rangle\Big) = \bigvee_{i\in S\cap T}\langle K(i)\rangle,$$

$$\Big(\prod_{i\notin S} p_i\Big)\Big(\prod_{j\notin T} p_j\Big) = \prod_{i\notin S \text{ or } i\notin T} p_i \ = \ \prod_{i\notin S\cap T} p_i.$$

Moreover, for the order, we have

$$\begin{aligned}\bigvee_{i\in S}\langle K(i)\rangle \le \bigvee_{i\in T}\langle K(i)\rangle \ &\Leftrightarrow \ S\subset T \ \Leftrightarrow \ I(n)-S \supset I(n)-T \\ &\Leftrightarrow \prod_{i\notin S} p_i \le \prod_{i\notin T} p_i,\end{aligned}$$

and f is a monoidal poset map. □

A similar argument shows the following

Theorem 5.4. *Let $E = \bigvee_{i\in F} K(i)$ be a spectrum for a finite subset F of $\mathbb{Z}_{\ge 0}$. Then $\mathbb{B}(\mathcal{L}_E)$ is isomorphic to $\prod_{i\in F}\mathbb{Z}/2$.*

This together with Theorem 4.5 implies

Corollary 5.5. GRC2 *holds on $\mathbb{B}(\mathcal{L}_E)$ for a spectrum $E = \bigvee_{i\in F} K(i)$ on a finite subset F of $\mathbb{Z}_{\ge 0}$.*

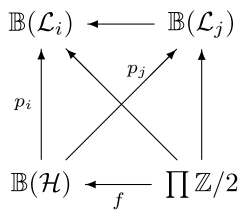
The chromatic tower $\mathcal{L}_0 \leftarrow \mathcal{L}_1 \leftarrow \mathcal{L}_2 \leftarrow \cdots$ induces the inverse system

$$\mathbb{B}(\mathcal{L}_0) \leftarrow \mathbb{B}(\mathcal{L}_1) \leftarrow \mathbb{B}(\mathcal{L}_2) \leftarrow \cdots . \tag{5.6}$$

Moreover, we notice that $B_\infty := \lim_n \mathbb{B}(\mathcal{L}_n) = \prod_n \mathbb{Z}/2$ in $\mathcal{MP}$. We call a spectrum *harmonic* if it is $(\bigvee_{i\ge 0} K(i))$-local.

Theorem 5.7. *Let $\mathcal{H}$ be the stable homotopy category of harmonic spectra. Then $\mathbb{B}(\mathcal{H})$ is isomorphic to B_∞ in $\mathcal{MP}$.*

Proof. Let $f\colon \prod \mathbb{Z}/2 \to \mathbb{B}(\mathcal{H})$ be the poset map defined by $(x_n) \mapsto \bigvee_{x_n=1} \langle K(n)\rangle$ and let $p_n\colon \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{L}_n)$ be the poset map defined by $\langle X\rangle \mapsto \langle X\rangle \cdot \langle E(n)\rangle$. Then we have the following commutative diagram:



for any i and j with $i \le j$, since

$$p_i f((x_n)) = p_i\Big(\bigvee_{x_n=1} \langle K(n)\rangle\Big) = \bigvee_{x_n=1} \langle K(n)\rangle \cdot \langle E(i)\rangle$$
$$= \bigvee_{i\ge n,\, x_n=1} \langle K(n)\rangle.$$

Therefore, $\mathbb{B}(\mathcal{H})$ is the inverse limit of the above system (5.6) by definition. □

Proof of Theorem 1.2. This follows from Theorem 5.7 and Proposition 3.25. □

In the same way, we obtain

Theorem 5.8. *Let T be a set of field spectra, and put $\bigvee T = \bigvee_{F\in T} F$. Then $\mathbb{B}(\mathcal{L}_{\bigvee T}) = \prod \mathbb{Z}/2$.*

§6. Problems

In this section we state some problems.

Problem 6.1. *What condition on $X \xrightarrow{f} Y$ in $\mathcal{M}_0^{\mathrm{epi}}$ makes $\beta(f)$ an isomorphism?*

Suppose that the problem is settled and we find a map from $\mathbb{B}$ to a commutative monoid Y such that $\beta(f)$ is an isomorphism. Then we may study $\mathbb{B} = \beta(\mathbb{B})$ by observing $\beta(Y)$ in virtue of Proposition 2.16, which may let us consider the lattice from a different viewpoint.

Problem 6.2. *Let M be a monoid with 0. Is there a ring R such that $\beta(M)$ is isomorphic to R as a monoid?*

Example 6.3. Let $p_0, \ldots, p_n$ be $n+1$ distinguished primes. Then $\beta(\mathbb{Z}/p_0 \ldots p_n) = \prod_{i=0}^{n} \mathbb{Z}/2$ as monoids by Theorem 5.3.

If this is possible, we may approach these from the ring-theoretic viewpoint.

Problem 6.4. *Are $\mathbb{B}/J(h)$ and $\mathbb{DL}$ monoidal posets?*

References

[1] A. K. Bousfield, The Boolean algebra of spectra, Comment. Math. Helv. **54** (1979), 368–377. Zbl 0421.55002 MR 0543337

[2] W. G. Dwyer and J. H. Palmieri, Ohkawa's theorem: there is a set of Bousfield classes. Proc. Amer. Math. Soc. **129** (2001), 881–886. Zbl 0963.55005 MR 1712921

[3] ———, The Bousfield lattice for truncated polynomial algebras, Homology Homotopy Appl. **10** (2008), 413–436. Zbl 1161.18005 MR 2426110

[4] M. J. Hopkins and J. H. Smith, Nilpotence and stable homotopy theory. II, Ann. of Math. **148** (1998), 1–49. Zbl 0924.55010 MR 1652975

[5] M. Hovey and J. H. Palmieri, The structure of the Bousfield lattice, in: *Homotopy invariant algebraic structures* (Baltimore, MD, 1998), Contemp. Math. 239, Amer. Math. Soc., 1999, 175–196. Zbl 0947.55014 MR 1718080

[6] M. Hovey, J. H. Palmieri and N. P. Strickland, Axiomatic stable homotopy theory, Mem. Amer. Math. Soc. **128** (1997), no. 610. Zbl 0881.55001 MR 1388895

[7] S. B. Iyengar and H. Krause, The Bousfield lattice of a triangulated category and stratification, Math. Z. **273** (2013), 1215–1241. Zbl 06149061 MR 3030697

[8] T. Ohkawa, The injective hull of homotopy types with respect to generalized homology functors, Hiroshima Math. J. **19** (1989), 631–639. Zbl 0697.55008 MR 1035147

[9] D. C. Ravenel, Localization with respect to certain periodic homology theories, Amer. J. Math. **106** (1984), 351–414. Zbl 0586.55003 MR 0737778