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Uniform Resolvent Estimates for Magnetic Schrödinger Operators and Smoothing Effects for Related Evolution Equations

by

Kiyoshi Mochizuki

Abstract

We prove uniform resolvent estimates for the magnetic Schrödinger operator in an exterior domain under smallness conditions on the magnetic fields and the scalar potential. The results are then used to obtain smoothing effects for the corresponding Schrödinger and Klein–Gordon evolution equations.

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§1. Introduction and results

Let Ω be an exterior domain in $\mathbb{R}^n$ $(n \geq 3)$ with star-shaped complement with respect to the origin 0 and smooth boundary $\partial\Omega$ (the case $\Omega = \mathbb{R}^n$ is not excluded). In this paper we consider in Ω the magnetic Schrödinger equation

$$(1) \qquad -\sum_{j=1}^{n}\{\partial_j + ib_j(x)\}^2 u + c(x)u - \kappa^2 u = f(x), \quad x \in \Omega,$$

with Dirichlet boundary condition

$$(2) \qquad u(x, \kappa) = 0, \quad x \in \partial\Omega.$$

Here $\partial_j = \partial/\partial x_j$ $(j = 1, \dots, n)$, $i = \sqrt{-1}$, $\kappa \in \Pi_\pm = \{\kappa = \sigma + i\epsilon \in \mathbb{C};\ \pm\sigma > 0, \epsilon > 0\}$ and $f \in L^2 = L^2(\Omega)$; $b_j(x)$ are real-valued C^1-functions of $x \in \mathbb{R}^n$ and $c(x)$ is a real-valued continuous function of $x \in \mathbb{R}^n \setminus \{0\}$; $b(x) = (b_1(x), \dots, b_n(x))$ represents a magnetic potential. Thus the magnetic field is defined by its rotation

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K. Mochizuki: Department of Mathematics, Chuo University, Kasuga, Bunnkyou, Tokyo 112-8551, Japan;
e-mail: mochizuk@math.chuo-u.ac.jp

$\nabla\times b(x)$. The external potential $c(x)$ may have a singularity like $O(|x|^{-2})$ at $x=0$. In this case we assume $c(x)>-\beta/4|x|^2$, $\beta<(n-2)^2$. Throughout this paper, we further require that $\max\{|\nabla\times b(x)|,|c(x)|\}$ decays sufficiently fast as $|x|\to\infty$.

Notation. Let $a\cdot b$ and $a\times b$ respectively denote the inner product and the exterior product of $a,b\in\mathbb{R}^n$. More generally, we put

$$\nabla\cdot v(x)=\partial_1v_1(x)+\cdots+\partial_nv_n(x),\qquad \nabla\times v(x)=(\partial_jv_k(x)-\partial_kv_j(x))_{1\le j<k\le n}$$

for $\nabla=(\partial_1,\dots,\partial_n)$ and $v(x)=(v_1(x),\dots,v_n(x))$. We also put $r=|x|$, $\tilde{x}=x/r$ and $\partial_r=\partial/\partial r=\tilde{x}\cdot\nabla$. The inner product and norm of L^2 are denoted by

$$(f,g)=\int f(x)\overline{g(x)}\,dx\quad\text{and}\quad \|f\|=\sqrt{(f,f)}.$$

Here $\int dx$ indicates integration over the domain Ω. Moreover, for $\rho>0$ we put $\Omega_\rho=\{x\in\Omega;\,|x|<\rho\}$ and $S_\rho=\{x\in\overline{\Omega};\,|x|=\rho\}$ ($\overline{\Omega}=\Omega\cup\partial\Omega$). We are thus able to consider $\int_{\Omega_\rho}dx=\int_0^\rho\int_{S_\sigma}dS\,d\sigma$.

Now, putting $\nabla_b=\nabla+ib(x)$ and $\Delta_b=\nabla_b\cdot\nabla_b$, we define in the Hilbert space L^2 the operator L as follows:

$$\text{(3)}\qquad \begin{cases} Lu=-\Delta_bu+c(x)u \quad \text{for } u\in\mathcal{D}(L),\\ \mathcal{D}(L)=\{u\in L^2\cap H^2_{\mathrm{loc}}(\overline{\Omega}\setminus\{0\});\,-\Delta_bu+cu\in L^2,\,\nabla_bu\in[L^2]^n,\,u|_{\partial\Omega}=0\}.\end{cases}$$

Here $H^j=H^j(\Omega)$ ($j=1,2,\dots$) is the usual Sobolev space on Ω and $H^2_{\mathrm{loc}}(\overline{\Omega}\setminus\{0\})$ is the space of H^2 functions on each compact subset of $\overline{\Omega}\setminus\{0\}$.

Note that the Hardy inequality is easily modified as

$$\text{(4)}\qquad \int\frac{(n-2)^2}{4r^2}|u|^2\,dx\le\int|\tilde{x}\cdot\nabla_bu|^2\,dx.$$

Then, as the Friedrichs extension of a lower semi-bounded symmetric operator $-\Delta_b+c$ initially defined on $C_0^\infty(\Omega\setminus\{0\})$, L forms a selfadjoint operator in L^2 with essential spectrum contained in the half line $[0,\infty)$ (see, e.g., Mochizuki [12] and Kalf et al. [7]). Hence, the resolvent $R(\kappa^2)=(L-\kappa^2)^{-1}$ of L can be defined for each $\kappa\in\Pi_\pm$.

The main purpose of this paper is to show the following theorem.

Theorem 1. *Let u be the solution of the problem* (1), (2).

(i) *Assume that*

$$\text{(A1)}\qquad \max\{|\nabla\times b(x)|,|c(x)|\}\le\mu(r)\quad \textit{in }\Omega,$$

where $\mu=\mu(r)$ is a smooth function of $r>0$ satisfying

$$\text{(5)}\qquad \mu>0,\qquad \mu'\le 0,\qquad \mu\in L^1(\mathbb{R}_+).$$

Then

$$\int \left\{ \mu(|\nabla_b u|^2 + |\kappa u|^2) - \mu' \frac{n-1}{2r}|u|^2 \right\} dx$$
$$\leq 4\|\mu\|_{L^1}^2 \int \mu^{-1}(5|f(x)|^2 + 4|\max\{|\nabla \times b|, |c|\}u|^2)\, dx$$

for each $\kappa \in \Pi_\pm$ *and* f *satisfying* $(1 + \mu(r)^{-1/2})f \in L^2$. *Here* $\|\mu\|_{L^1} = \int_0^\infty \mu(\sigma)\, d\sigma$.

(ii) *Assume that*

$$\text{(A2)} \qquad \max\{|\nabla \times b(x)|, |c(x)|\} \leq \epsilon_0 r^{-2} \qquad \text{in } \Omega,$$

where $0 < \epsilon_0 < 1/4\sqrt{2}$. *Then*

$$\int \frac{1}{r^2}|u|^2\, dx \leq C_1 \int r^2 |f|^2\, dx, \qquad C_1 = \frac{32}{(n-2)^2 - 32\epsilon_0^2},$$

for each $\kappa \in \Pi_\pm$ *and* f *satisfying* $(1 + r)f \in L^2$.

(iii) *Assume that*

$$\text{(A3)} \qquad \max\{|\nabla \times b(x)|, |c(x)|\} \leq \epsilon_0 \min\{\mu(r), r^{-2}\} \qquad \text{in } \Omega.$$

Then

$$\int \left\{ \mu(|\nabla_b u|^2 + |\kappa u|^2) - \mu' \frac{n-1}{2r}|u|^2 \right\} dx \leq C_2 \int \max\{\mu^{-1}, r^2\}|f(x)|^2\, dx,$$
$$C_2 = 4(5 + 4\epsilon_0^2 C_1)\|\mu\|_{L^1}^2,$$

for each $\kappa \in \Pi_\pm$ *and* f *satisfying* $\max\{\mu(r)^{-1/2}, 1 + r\}f \in L^2$.

(i) shows that the multiplication operator $\sqrt{\mu(r)}$ is locally L-smooth near $\kappa = \infty$. The notion of the smooth perturbation introduced by Kato [8] and the local smoothness condition were used in Mochizuki [11] for non-small complex potentials. (ii) shows the L-smoothness of r^{-1}. Note that (ii) and (iii) generalize the corresponding results of Kato–Yajima [9] (see also Kuroda [10] and Watanabe [15]), where the operator in question is the Laplace operator in $\mathbb{R}^n$ ($n \geq 3$). The Fourier transformation method employed there is not applicable in our case. In this paper our arguments are based on the partial integration method widely used to show the principle of limiting absorption. The weight functions introduced in Mochizuki [12] will play an important role. (i) is a result of precise examination of previous results (see, e.g., [12]–[14]). On the other hand, to show (ii) and (iii) it is necessary to introduce yet another identity (see Lemma 3 of §3). The weight function of [12] is especially effective in this argument.

Remark 1. Assertion (ii) can be generalized to the potential $c(x) = c_1(x)+c_2(x)$, where $c_1(x)$ is a small function satisfying (A2), and $c_2(x)$ is a bounded nonnegative function satisfying

$$c_2(x) \le Cr^{-2} \quad \text{for some } C > 0.$$

In this case, to ensure the solvability of an integral equation corresponding to (1), we further require that 0 is not a resonance of L.

As a corollary of Theorem 1, we are able to obtain space-time weighted estimates (smoothing effects) for the Schrödinger evolution equation

$$i\frac{\partial u}{\partial t} - Lu = 0, \quad u(0) = f \in L^2, \tag{6}$$

and the relativistic Schrödinger evolution equation

$$i\frac{\partial u}{\partial t} - \sqrt{L+m^2}\, u = 0, \quad u(0) = f \in L^2, \tag{7}$$

with $m \ge 0$. Note that the smoothing effects for (7) give those for the Klein–Gordon ($m > 0$) or the wave equation ($m = 0$) in the energy space.

Theorem 2. (i) *Assume* (A2). *Then for* $f \in L^2$ *we have*

$$\left| \int_0^{\pm\infty} \|r^{-1} e^{-itL} f\|^2 \, dt \right| \le 2\sqrt{C_1}\|f\|^2.$$

(ii) *Assume (A3). Then for* $f \in L^2$ *we have*

$$\left| \int_0^{\pm\infty} \|\min\{\sqrt{\mu(r)}, r^{-1}\} e^{-it\sqrt{L+m^2}} f\|^2 \, dt \right| \le 4\sqrt{m^2C_1 + C_2}\|f\|^2.$$

(iii) *Assume that* $b(x) \equiv 0$ *and* $c(x) \equiv 0$. *Then* L *reduces to the usual Laplacian*

$$L_0 = -\Delta, \quad \mathcal{D}(L_0) = H^2 \cap H_0^1.$$

In this case, we have

$$\left| \int_0^{\pm\infty} \|\sqrt{\mu(r)} e^{-it\sqrt{L_0}} f\|^2 \, dt \right| \le 8\sqrt{5}\|\mu\|_{L^1}\|f\|^2.$$

Similar results have been obtained by many authors in connection with local smoothing properties (see, e.g., Ben Artzi–Klainerman [1], Yajima [16], Cuccagna–Schirmer [2], D'Ancona–Fanelli [3], Erdogan–Goldberg–Schlag [4] and Georgiev–Stefanov–Tarulli [5]). Note that these works are restricted to the initial value problem and the vector potential $b(x)$ itself is required to be small and to decay sufficiently fast. No such requirement is imposed in our case. The decay conditions similar to (A1) on the magnetic field $\nabla \times b(x)$ have been used in Ikebe–Uchiyama [6] to show growth estimates of generalized eigenfunctions corresponding to each positive spectrum.

As for the Schrödinger equation (6), we can have the following more general results.

Theorem 3. (i) *Assume* (A2). *Then for* $h(t)$ *satisfying* $r^{-1}h(t) \in L^2(\mathbb{R}\times\Omega)$ *we have*

$$\left|\int_0^{\pm\infty}\left\|r^{-1}\int_0^t e^{-i(t-\tau)L}h(\tau)\,d\tau\right\|^2 dt\right| \le C_1\left|\int_0^{\pm\infty}\|rh(t)\|^2\,dt\right|.$$

(ii) *Assume* (A3). *Then for* $h(t)$ *satisfying* $\max\{\sqrt{\mu(r)}^{-1}, r\}h(t) \in L^2(\mathbb{R}\times\Omega)$ *we have*

$$\begin{aligned}\left|\int_0^{\pm\infty}\left\|\min\{\sqrt{\mu(r)}, r^{-1}\}\int_0^t \nabla_b e^{-i(t-\tau)L}h(\tau)\,d\tau\right\|^2 dt\right| \\ \le \max\{C_1, C_2\}\left|\int_0^{\pm\infty}\|\max\{\sqrt{\mu(r)}^{-1}, r\}h(t)\|^2\,dt\right|.\end{aligned}$$

The rest of the paper will be organized as follows. In the next section we prepare two identities related to the solution of (1), (2). They are used in §3 to prove Theorem 1. Finally, Theorems 2 and 3 are proved in §4 based on Theorem 1.

§2. Functional identities for solutions

In this section we prepare two functional identities related to the solution of (1), (2).

We first multiply both sides of (1) by $-\overline{i\kappa u}$ to obtain

$$\nabla\cdot\{(\nabla_b u)\overline{i\kappa u}\} - \overline{i\kappa}\{|\nabla_b u|^2 + c(x)|u|^2 - \kappa^2|u|^2\} = -f\,\overline{i\kappa u}. \tag{8}$$

Integrating the real part of this equation over Ω_ρ ($\rho > 0$), by using the boundary condition (2) we obtain

$$\begin{aligned}\mathrm{Re}\int_{\Omega_\rho}\nabla\cdot\{(\nabla_b u)\overline{i\kappa u}\}\,dx &= \mathrm{Re}\int_{S_\rho}(\tilde{x}\cdot\nabla_b u)\overline{i\kappa u}\,dS \\ &= \frac{1}{2}\int_{S_\rho}\{-|\nabla_b u - i\kappa\tilde{x}u|^2 + |\nabla_b u|^2 + |\kappa u|^2\}\,dS,\end{aligned}$$

and it follows that

$$\begin{aligned}\frac{1}{2}\int_{S_\rho}\{-|\nabla_b u - i\kappa u|^2 + |\nabla_b u|^2 + |\kappa u|^2\}\,dS \\ + \mathrm{Im}\,\kappa\int_{\Omega_\rho}(|\nabla_b u|^2 + c|u|^2 + |\kappa u|^2)\,dx = -\,\mathrm{Re}\int_{\Omega_\rho} f\,\overline{i\kappa u}\,dx.\end{aligned}$$

The following proposition is a direct consequence of this identity multiplied by $\mu(\rho)$ and integrated over $(0,\infty)$.

Proposition 1. *Let $u \in \mathcal{D}(L)$ satisfy* (1), (2) *with $\kappa \in \Pi_\pm$ and $f \in L^2$. Then for μ satisfying* (5) *we have*

$$\frac{1}{2}\int \mu\left\{-|\nabla_b u - i\kappa\tilde{x}u|^2 + |\nabla_b u|^2 + |\kappa u|^2\right\} dx + \operatorname{Im}\kappa\int_0^\infty \mu(\rho)\,d\rho\int_{\Omega_\rho}(|\nabla_b u|^2 + c|u|^2 + |\kappa u|^2)\,dx = -\operatorname{Re}\int_0^\infty \mu(\rho)\,d\rho\int_{\Omega_\rho} f\,\overline{i\kappa u}\,dx.$$

Next, we put $v = e^{-i\kappa r}r^{(n-1)/2}u$, $g = e^{-i\kappa r}r^{(n-1)/2}f$ and rewrite (1) as follows:

$$(9)\qquad -\nabla_b\cdot\nabla_b v + \left(-2i\kappa + \frac{n-1}{r}\right)\tilde{x}\cdot\nabla_b v + \left(\frac{(n-1)(n-3)}{4r^2} + c\right)v = g.$$

Let $\varphi = \varphi(r)$ be a positive increasing function of $r > 0$ such that

$$(10)\qquad \varphi(r) = O(r) \quad\text{and}\quad \frac{\varphi'(r)}{\varphi(r)} \le \frac{1}{r},$$

and let $\phi = \phi(r) = e^{-2\operatorname{Im}\kappa r}r^{-n+1}\varphi(r)$. We multiply both sides of (9) by $\phi\,\overline{\tilde{x}\cdot\nabla_b v}$ to obtain

$$\begin{aligned}
&-\operatorname{Re}\nabla\cdot\{(\phi\nabla_b v)\overline{\tilde{x}\cdot\nabla_b v}\} + \phi'|\tilde{x}\cdot\nabla_b v|^2 + \frac{\phi}{r}(|\nabla_b v|^2 - |\tilde{x}\cdot\nabla_b v|^2)\\
&\quad + \frac{1}{2}\nabla\cdot(\phi\tilde{x}|\nabla_b v|^2) - \left(\frac{\phi'}{2} + \phi\frac{n-1}{2r}\right)|\nabla_b v|^2\\
&\quad - \operatorname{Re}\phi\{(\tilde{x}\times\nabla_b v)\cdot\overline{(\nabla\times ib)v}\} + \phi\left(2\operatorname{Im}\kappa + \frac{n-1}{r}\right)|\tilde{x}\cdot\nabla_b v|^2\\
&\quad + \operatorname{Re}\phi\left(\frac{(n-1)(n-3)}{4r^2} + c\right)v\overline{\tilde{x}\cdot\nabla_b v} = \operatorname{Re}\{\phi g\overline{\tilde{x}\cdot\nabla_b v}\}.
\end{aligned}$$

We integrate this over Ω_ρ. Then noting

$$\phi'(r) = \phi(r)\left(-2\operatorname{Im}\kappa - \frac{n-1}{r} + \frac{\varphi'}{\varphi}\right)$$

and

$$\begin{aligned}
\operatorname{Re}\phi\frac{(n-1)(n-3)}{4r^2}v\,\overline{\tilde{x}\cdot\nabla_b v} &= \frac{1}{2}\nabla\cdot\left\{\phi\tilde{x}\frac{(n-1)(n-3)}{4r^2}|v|^2\right\}\\
&\quad + \phi\left(\operatorname{Im}\kappa - \frac{\varphi'}{2\varphi}\right)\frac{(n-1)(n-3)}{4r^2}|v|^2 + \phi\frac{(n-1)(n-3)}{4r^3}|v|^2,
\end{aligned}$$

we obtain the following proposition.

Proposition 2. *Let v satisfy* (9) *with boundary condition $v|_{\partial\Omega} = 0$. Then*

$$\int_{S_\rho} \phi\left\{-|\tilde{x}\cdot\nabla_b v|^2 + \frac{1}{2}|\nabla_b v|^2 + \frac{1}{2}\frac{(n-1)(n-3)}{4r^2}|v|^2\right\} dS$$
$$+ \int_{\partial\Omega\cap\{|x|<\rho\}} \phi\left\{-(\nu\cdot\nabla_b v)(\overline{\tilde{x}\cdot\nabla_b v}) + \frac{1}{2}(\nu\cdot\tilde{x})|\nabla_b v|^2\right\} dS$$
$$+ \int_{\Omega_\rho} \phi\left\{\left(\frac{1}{r} - \frac{\varphi'}{\varphi}\right)\left(|\nabla_b v|^2 - |\tilde{x}\cdot\nabla_b v|^2 + \frac{(n-1)(n-3)}{4r^2}|v|^2\right)\right.$$
$$+ \left(\operatorname{Im}\kappa + \frac{\varphi'}{2\varphi}\right)\left(|\nabla_b v|^2 + \frac{(n-1)(n-3)}{4r^2}|v|^2\right)$$
$$\left. + \operatorname{Re}[-(\tilde{x}\times\nabla_b v)\cdot\overline{(\nabla\times ib)v} + cv\overline{\tilde{x}\cdot\nabla_b v}]\right\} dx = \operatorname{Re}\int_{\Omega_\rho} \phi g\overline{\tilde{x}\cdot\nabla_b v}\, dx.$$

where $\nu = \nu(x)$ is the outer unit normal to the boundary $\partial\Omega$.

§3. Proof of Theorem 1

We shall show Theorem 1 by a series of lemmas.

Lemma 1. *Assume $c(x) \geq -(n-2)^2/4r^2$. Then for μ satisfying* (5) *we have*

$$\frac{1}{2}\int\left\{\mu\operatorname{Im}\kappa\,\frac{1}{r}|u|^2 - \mu'\frac{n-1}{2r}|u|^2 + \mu(|\nabla_b u|^2 + |\kappa u|^2)\right\} dx$$
$$\leq \frac{1}{2}\int \mu\left(|\theta|^2 + \frac{(n-1)(n-3)}{4r^2}|u|^2\right) dx + \|\mu\|_{L^1}\int |f(x)|\,|i\kappa u|\, dx,$$

where

$$\theta = \nabla_b u + \tilde{x}\left(\frac{n-1}{2r} - i\kappa\right)u.$$

Proof. Note that

$$\mu|\nabla_b u - i\kappa\tilde{x}u|^2 = -\nabla\cdot\left\{\tilde{x}\mu\frac{n-1}{2r}|u|^2\right\} + \mu\left(|\theta|^2 + \frac{(n-1)(n-3)}{4r^2}|u|^2\right)$$
$$+ \mu'\frac{n-1}{2r}|u|^2 - \mu\operatorname{Im}\kappa\,\frac{n-1}{r}|u|^2$$

and

$$|\nabla_b u|^2 + c|u|^2 \geq |\tilde{x}\cdot\nabla_b u|^2 - \frac{(n-2)^2}{4r^2}|u|^2$$
$$= \left|\tilde{x}\cdot\nabla_b u + \frac{n-2}{2r}u\right|^2 - \nabla\cdot\left(\tilde{x}\frac{n-2}{2r}|u|^2\right).$$

in Proposition 1. Then since

$$\liminf_{\rho\to\infty}\int_{S_\rho}\mu\frac{n-1}{2r}|u|^2\,dS=0,$$

we have

$$\begin{aligned}
&-\frac{1}{2}\int\mu|\nabla_b u-i\kappa\tilde{x}u|^2\,dx+\operatorname{Im}\kappa\int_0^\infty\mu(\rho)\,d\rho\int_{\Omega_\rho}(|\nabla_b u|^2+c|u|^2)\,dx\\
&\quad\geq\;-\frac{1}{2}\int\left\{\mu\left(|\theta|^2+\frac{(n-1)(n-3)}{4r^2}|u|^2\right)+\mu'\frac{n-1}{2r}|u|^2-\mu\operatorname{Im}\kappa\,\frac{n-1}{r}|u|^2\right\}dx\\
&\qquad+\operatorname{Im}\kappa\int_0^\infty\mu(\rho)\,d\rho\int_{\Omega_\rho}-\nabla\cdot\left(\tilde{x}\frac{n-2}{2r}|u|^2\right)dx\\
&\quad=\;-\frac{1}{2}\int\left\{\mu\left(|\theta|^2+\frac{(n-1)(n-3)}{4r^2}|u|^2\right)+\mu'\frac{n-1}{2r}|u|^2-\mu\operatorname{Im}\kappa\,\frac{1}{r}|u|^2\right\}dx,
\end{aligned}$$

and the desired inequality. □

Lemma 2. *For $\varphi(r)$ satisfying* (10) *we have*

$$\begin{aligned}
\frac{1}{4}\int\varphi\left(\operatorname{Im}\kappa+\frac{\varphi'}{2\varphi}\right)&\left(|\theta|^2+\frac{(n-1)(n-3)}{4r^2}|u|^2\right)dx\\
&\leq\int\frac{\varphi^2}{\varphi'}(|f|^2+|\max\{|\nabla\times b|,|c|\}u|^2)\,dx.
\end{aligned}$$

Proof. In the identity of Proposition 2, the Dirichlet condition (2) implies that $\tau\cdot\nabla v=0$ for any tangential vector τ to the boundary. On the other hand, since the starshapedness of the boundary implies $\nu\cdot\tilde{x}\leq 0$, it follows that

$$\int_{\partial\Omega}\phi\left\{-(\nu\cdot\nabla_b v)(\tilde{x}\cdot\overline{\nabla_b v})+\frac{1}{2}(\nu\cdot\tilde{x})|\nabla_b v|^2\right\}dS=-\frac{1}{2}\int_{\partial\Omega}\phi(\nu\cdot\tilde{x})|\nu\cdot\nabla_b v|^2\,dS\geq 0.$$

Moreover, since $1/r\geq\varphi'/\varphi$, $|\nabla_b v|\geq|\tilde{x}\cdot\nabla_b v|$, $n\geq 3$ and

$$\left|-(\tilde{x}\times\nabla_b v)\cdot\overline{(\nabla\times ib)v}+cv\overline{\tilde{x}\cdot\nabla_b v}\right|\leq|\max\{|\nabla\times b|,|c|\}v|\,|\nabla_b v|,$$

it follows that

$$\begin{aligned}
\int_{\Omega_\rho}\phi\left(\operatorname{Im}\kappa+\frac{\varphi'}{2\varphi}\right)&\left(|\nabla_b v|^2+\frac{(n-1)(n-3)}{4r^2}|v|^2\right)dx\\
&\leq\int_{\Omega_\rho}\phi(|g|+|\max\{|\nabla\times b|,|c|\}v|)|\nabla_b v|\,dx+\frac{1}{2}\int_{S_\rho}\phi|\tilde{x}\cdot\nabla_b v|^2\,dS.
\end{aligned}$$

If we let $\rho\to\infty$, then $\varphi(r)=O(r)$ leads to

$$\liminf_{\rho\to\infty}\int_{S_\rho}\phi|\tilde{x}\cdot\nabla_b v|^2\,dS=\liminf_{\rho\to\infty}\int_{S_\rho}\varphi|\tilde{x}\cdot\theta|^2\,dS=0,$$

and we obtain the inequality of the lemma. □

Proof of Theorem 1(i). We combine Lemmas 1 and 2 with $\varphi(r) = \int_0^r \mu(\sigma)\,d\sigma$. It is obvious that this φ satisfies (10). Then since $\varphi(r) \le \|\mu\|_{L^1}$, it follows that

$$\frac{1}{2}\int\left\{-\mu'\frac{n-1}{2r}|u|^2+\mu\left(|\nabla_b u|^2+|\kappa u|^2\right)\right\}dx$$
$$\le 4\|\mu\|_{L^1}^2\int\mu^{-1}(|f|^2+|\max\{|\nabla\times b|,|c|\}u|^2)\,dx+\|\mu\|_{L^1}\int|f|\,|i\kappa u|\,dx.$$

Thus, noting

$$\|\mu\|_{L^1}\int|f|\,|i\kappa u|\,dx\le\|\mu\|_{L^1}^2\int\mu^{-1}|f|^2\,dx+\frac{1}{4}\int\mu|\kappa u|^2\,dx,$$

we deduce the desired inequality. □

To complete the proof of (ii) and (iii), we need one more lemma.

Lemma 3. *We have*

$$\int(2\operatorname{Im}\kappa\,r+1)\frac{1}{4r^2}|u|^2\,dx\le\int(2\operatorname{Im}\kappa\,r+1)|\tilde{x}\cdot\theta|^2\,dx.$$

Proof. We put $\psi = e^{-\operatorname{Im}\kappa\,r}(2\operatorname{Im}\kappa\,r+1)^{1/2}$. Then we have

$$(2\operatorname{Im}\kappa\,r+1)|\tilde{x}\cdot\theta|^2=r^{-n+1}\psi^2|\tilde{x}\cdot\nabla_b v|^2.$$

Note that

$$\begin{aligned}
\psi^2|\tilde{x}\cdot\nabla_b v|^2&=|\tilde{x}\cdot\nabla_b(\psi v)-\psi' v|^2=\left|\left\{\tilde{x}\cdot\nabla_b(\psi v)-\frac{\psi'}{2}v-\xi v\right\}-\frac{\psi'}{2}v+\xi v\right|^2\\
&=\left|\tilde{x}\cdot\nabla_b(\psi v)-\frac{\psi'}{2}v-\xi v\right|^2-\partial_r\left\{\left(\frac{\psi'}{2\psi}-\frac{\xi}{\psi}\right)|\psi v|^2\right\}\\
&\quad+\left(\frac{\psi''\psi-\psi'^2}{2}-\xi'\psi+\xi\psi'\right)|v|^2+2\left(\frac{\psi'^2}{4}-\xi^2\right)|v|^2+\left|\frac{\psi'}{2}v-\xi v\right|^2\\
&=\left|\tilde{x}\cdot\nabla_b(\psi v)-\frac{\psi'}{2}v-\xi v\right|^2-\partial_r\left\{\left(\frac{\psi'}{2\psi}-\frac{\xi}{\psi}\right)|\psi v|^2\right\}\\
&\quad+\frac{1}{2}\left(\frac{1}{2}\psi'^2+\psi''\psi-2\xi'\psi-2\xi^2\right)|v|^2,
\end{aligned}$$

where $\xi=\xi(r)$ is another weight function given later. By definition we have

$$\psi'=\psi\frac{-2(\operatorname{Im}\kappa)^2 r}{2\operatorname{Im}\kappa\,r+1}<0,\qquad \psi''\psi=\psi'^2+\frac{1}{r(2\operatorname{Im}\kappa\,r+1)}\psi'\psi.$$

Then since

$$\frac{1}{2}\psi'^2+\psi''\psi=\frac{3}{2}\psi'^2+\left\{\frac{1}{r(2\operatorname{Im}\kappa\,r+1)}-\frac{1}{r}\right\}\psi'\psi+\frac{1}{r}\psi'\psi\ge\frac{1}{r}\psi'\psi,$$

we can now choose $\xi = \psi/2r$ to obtain

$$\frac{1}{2}\psi'^2 + \psi''\psi - 2\xi'\psi - 2\xi^2 \geq \frac{1}{r}\psi'\psi - 2\xi'\psi - 2\xi^2 = \frac{1}{2r^2}\psi^2,$$

and it follows that

$$r^{-n+1}\psi^2|\tilde{x}\cdot\nabla_b v|^2 \geq \nabla\cdot\left\{\tilde{x}\left(-\frac{\psi'}{2\psi}+\frac{1}{2r}\right)r^{-n+1}|\psi v|^2\right\} + \frac{\psi^2}{4r^2}r^{-n+1}|v|^2.$$

Integrate both sides over Ω_ρ. Then since

$$\int_{S_\rho}\left(-\frac{\psi'}{2\psi}+\frac{1}{2r}\right)r^{-n+1}|\psi v|^2\,dS = O(\rho)\int_{S_\rho}|u|^2\,dS,$$

letting $\rho\to\infty$, we deduce the inequality of the lemma. □

Proof of Theorem 1(ii). We choose $\varphi = r$ in Lemma 2. Then noting (A2), we have

$$\int (2\,\mathrm{Im}\,\kappa\, r + 1)|\theta|^2\,dx \leq 8\int (r^2|f|^2 + \epsilon_0^2 r^{-2}|u|^2)\,dx.$$

Combining this and Lemma 3, we obtain the inequality of (ii). □

Proof of Theorem 1(iii). (A3) and the inequality of (ii) imply that

$$\int \mu^{-1}|\max\{|\nabla\times b|, |c|\}u|^2\,dx \leq \epsilon_0^2\int \mu^{-1}|\min\{\mu, r^{-2}\}u|^2\,dx \leq \epsilon_0^2 C_1\int r^2|f|^2\,dx.$$

Substituting this in the inequality of (i), we obtain the desired inequality. □

§4. Proof of Theorems 2 and 3

First we shall summarize abstract results which allow us to employ the resolvent estimate for a selfadjoint operator to show a space-time weighted estimate for the associated evolution equation.

Let Λ be a selfadjoint operator in the Hilbert space $\mathcal{H}$, and for $z\in\mathbb{C}\setminus\mathbb{R}$ let $\mathcal{R}(z)$ be the resolvent of Λ. Suppose that A is a densely defined, closed operator from $\mathcal{H}$ to another Hilbert space $\mathcal{H}_1$.

Proposition 3. *Assume that there exists $C>0$ such that*

$$\sup_{z\notin\mathbb{R}}\|A\mathcal{R}(z)A^*f\|_{\mathcal{H}_1} < \sqrt{C}\|f\|_{\mathcal{H}_1} \tag{11}$$

for $f \in \mathcal{D}(A^*)$*. Then*

$$\left|\int_0^{\pm\infty}\left\|\int_0^t Ae^{-i(t-\tau)\Lambda}A^*h(\tau)\,d\tau\right\|_{\mathcal{H}_1}^2 dt\right| \le C\left|\int_0^{\pm\infty}\|h(t)\|_{\mathcal{H}_1}^2\,dt\right|, \tag{12}$$

$$\sup_{t\in\mathbb{R}_\pm}\left\|\int_0^t e^{i\tau\Lambda}A^*h(\tau)\,d\tau\right\|_{\mathcal{H}}^2 \le 2\sqrt{C}\left|\int_0^{\pm\infty}\|h(t)\|_{\mathcal{H}_1}^2\,dt\right| \tag{13}$$

for each $h \in L^2(\mathbb{R};\mathcal{D}(A^*))$*, and*

$$\left|\int_0^{\pm\infty}\|Ae^{-it\Lambda}f\|_{\mathcal{H}_1}^2\,dt\right| \le 2\sqrt{C}\|f\|_{\mathcal{H}}^2 \tag{14}$$

for each $f \in \mathcal{H}$*.*

Proof. To show (12) and (13), by the standard approximation procedure, we can assume $h \in C_0^\infty(\mathbb{R};\mathcal{D}(A^*))$.

We put $v(t) = \int_0^t e^{-i(t-\tau)\Lambda}A^*h(\tau)\,d\tau$, and consider its Laplace transform

$$\tilde{v}(z) = \pm\int_0^{\pm\infty} e^{izt}v(t)\,dt, \qquad \pm\operatorname{Im} z > 0.$$

Then since $\tilde{v}(z) = -i\mathcal{R}(z)A^*\tilde{h}(z)$, it follows from the Plancherel theorem and the assumption (11) that

$$\begin{aligned}\left|\int_0^{\pm\infty} e^{\mp 2\epsilon t}(Av(t),g(t))_{\mathcal{H}_1}\,dt\right| &= \left|(2\pi)^{-1}\int_{-\infty}^{\infty}(A\tilde{v}(\lambda\pm i\epsilon),\tilde{g}(\lambda\pm i\epsilon))_{\mathcal{H}_1}\,d\lambda\right| \\ &\le (2\pi)^{-1}\int_{-\infty}^{\infty}\|A\mathcal{R}(\lambda\pm i\epsilon)A^*\tilde{h}(\lambda\pm i\epsilon)\|_{\mathcal{H}_1}\|\tilde{g}(\lambda\pm i\epsilon)\|_{\mathcal{H}_1}\,d\lambda \\ &\le \left|C\int_0^{\pm\infty}e^{\mp 2\epsilon t}\|h(t)\|_{\mathcal{H}_1}^2\,dt\int_0^{\pm\infty}e^{\mp 2\epsilon t}\|g(t)\|_{\mathcal{H}_1}^2\,dt\right|^{1/2}\end{aligned}$$

for any $g \in C_0^\infty(\mathbb{R};\mathcal{D}(A^*))$. Letting $\epsilon \downarrow 0$, we obtain inequality (12).

Next, note that the Fubini theorem implies

$$\begin{aligned}\left\|\int_0^t e^{i\tau\Lambda}A^*h(\tau)\,d\tau\right\|_{\mathcal{H}_1}^2 = &\int_0^t\left(\int_0^s Ae^{-i(s-\tau)\Lambda}A^*h(\tau)\,d\tau, h(s)\right)_{\mathcal{H}_1} ds \\ &+\int_0^t\left(h(\tau),\int_0^\tau Ae^{-i(\tau-s)\Lambda}A^*h(s)\,ds\right)_{\mathcal{H}_1} d\tau.\end{aligned}$$

This and (12) show that (13) holds.

(14) is the dual assertion of (13). □

Proof of Theorem 2(i) and Theorem 3(i). Set $\Lambda = L$, $\mathcal{H} = \mathcal{H}_1 = L^2$ and $A = r^{-1}$ (multiplication operator). Then $A^* = A$ and $\mathcal{R}(z) = R(z)$, and if we let $z = \kappa^2$, then it follows from Theorem 1(ii) that

$$\|AR(z)A^*f\| = \|r^{-1}R(z)A^*f\| \le \sqrt{C_1}\|rA^*f\| = \sqrt{C_1}\|f\|.$$

Thus, the estimates (12) and (14) can be written as

$$\left|\int_0^{\pm\infty}\left\|r^{-1}\int_0^t e^{-i(t-\tau)L}h(\tau)\,d\tau\right\|^2 dt\right| \le C_1\left|\int_0^{\pm\infty}\|rh(t)\|^2\,dt\right|,$$
$$\left|\int_0^{\pm\infty}\|r^{-1}e^{-itL}f\|^2\,dt\right| \le 2\sqrt{C_1}\|f\|^2,$$

as desired. □

Proof of Theorem 3(ii). Put $A = \min\{\sqrt{\mu(r)}, r^{-1}\}$. Then by Theorem 1(iii) we have

$$\|A\nabla_b R(\lambda \pm i\epsilon)A^*\tilde{h}(\lambda \pm i\epsilon)\| \le \sqrt{C_2}\|\tilde{h}(\lambda \pm i\epsilon)\|.$$

Thus, we can use the argument proving (12) to obtain the desired conclusion. □

To show Theorem 2(ii) we consider the Klein–Gordon equation

$$i\partial_t u = \Lambda u, \qquad u(t) = \{w(t), \partial_t w(t)\}, \qquad \Lambda = \begin{pmatrix} 0 & i \\ -i(L+m^2) & 0 \end{pmatrix},$$

in the energy space $\mathcal{H} = H_b^1 \times L^2$, where H_b^1 is the completion of $C_0^\infty(\Omega)$ in the norm

$$\|f_1\|_{H_b^1}^2 = \int \{|\nabla_b f_1|^2 + (c(x)+m^2)|f_1|^2\}\,dx.$$

Then Λ with domain

$$\mathcal{D}(\Lambda) = \{f_1 \in H_b^1;\ \Delta_b f_1 \in L^2\} \times \{f_2 \in H_b^1 \cap L^2\}$$

forms a selfadjoint operator in $\mathcal{H}$, and its resolvent is given by

$$\mathcal{R}(z) = (L+m^2-z^2)^{-1}\begin{pmatrix} z & i \\ -i(L+m^2) & z \end{pmatrix}.$$

Let $A:\ \mathcal{H} \to \mathcal{H}_1 = L^2$ be defined by

$$Af = \min\{\sqrt{\mu(r)}, r^{-1}\}\sqrt{L+m^2}\, f_1 \qquad \text{for } f = \{f_1, f_2\} \in \mathcal{H}.$$

Then the adjoint operator A^* is given by

$$A^*g = \{\sqrt{L+m^2}^{\,-1}\min\{\sqrt{\mu(r)}, r^{-1}\}g, 0\} \qquad \text{for } g \in L^2.$$

Proof of Theorem 2(ii) and (iii). By definition

(15) $$A\mathcal{R}(z)A^*g = \min\{\sqrt{\mu(r)}, r^{-1}\}z(L+m^2-z^2)^{-1}\min\{\sqrt{\mu(r)}, r^{-1}\}g$$

for $g \in \mathcal{D}(A^*)$. Then since

$$\begin{aligned}&\int |\min\{\sqrt{\mu(r)}, r^{-1}\} z(L+m^2-z^2)^{-1} f|^2\, dx \\ &\quad \le m^2 \int r^{-2} |(L+m^2-z^2)^{-1} f|^2\, dx + \int \mu| - m^2 + z^2|\, |(L+m^2-z^2)^{-1} f|^2\, dx,\end{aligned}$$

using Theorem 1(ii) and (iii), we obtain

$$\|A\mathcal{R}(z)A^* g\| \le \sqrt{m^2 C_1 + C_2}\|g\|.$$

We now return to Proposition 3. Then (14) shows that

$$\begin{aligned}\left|\int_0^{\pm\infty} \|Ae^{-it\Lambda} f\|^2\, dt\right| &= \left|\int_0^{\pm\infty} \|\min\{\sqrt{\mu(r)}, r^{-1}\}\sqrt{L+m^2}\, w(t)\|^2\, dt\right| \\ &\le 2\sqrt{m^2 C_1 + C_2}\|f\|_{\mathcal{H}}^2.\end{aligned}$$

Since

$$w(t) = \cos(t\sqrt{L+m^2}) f_1 + \sqrt{L+m^2}^{-1} \sin(t\sqrt{L+m^2}) f_2,$$

this inequality implies (ii).

To show (iii) we have only to use Theorem 1(i) with b and c identically equal to 0. In fact, let Λ_0 represent the operator Λ with $L = L_0$ and $m = 0$, and let $\mathcal{R}_0(z)$ be its resolvent. Then choosing

$$A_0 f = \sqrt{\mu(r)}\sqrt{L_0} f_1,$$

we have corresponding to (15)

$$A_0 \mathcal{R}_0(z) A_0^* g = \sqrt{\mu(r)} z (L_0 - z^2)^{-1} \sqrt{\mu(r)} g,$$

and hence

$$\|A_0 \mathcal{R}_0(z) A_0^* g\| \le 2\sqrt{5}\|\mu\|_{L^1}\|g\|.$$

Thus, following the above argument leads us to the conclusion. □

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