Publ. RIMS, Kyoto Univ.
44 (2008), 495–506

Mordell-Weil Groups of a Hyperkähler Manifold—A Question of F. Campana

Dedicated to Professor Heisuke Hironaka on his 77-th birthday

By

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Abstract

Among other things, we show that Mordell-Weil groups of finitely many different abelian fibrations of a hyperkähler manifold have essentially no relation, as its birational transformation. Precise definition of the terms "essentially no relation" will be given in Introduction.

§1. Introduction—Background and the Statement of Main Result

Let M be a hyperkähler manifold M, i.e., a simply-connected compact Kähler manifold admitting an everywhere non-degenerate global holomorphic 2-form σ_M s.t. $H^0(M, \Omega_M^2) = \mathbf{C}\sigma_M$. In this note, we assume that M is projective. We denote by $\mathrm{Bir}\, M$ the group of birational transformations of M. A morphism $\varphi : M \longrightarrow B$ onto a normal projective variety B is called an *abelian fibration* if its generic fiber $M_\eta (\ni O)$ is a positive dimensional abelian variety defined over $\mathbf{C}(B)$. The group $M_\eta(\mathbf{C}(B))$, called the Mordell-Weil group of φ and denoted from now by $\mathrm{MW}(\varphi)$, can be naturally regarded as an abelian subgroup of $\mathrm{Bir}\, M$. In [Og3] (see also [Og4]), we have shown the following:

Communicated by K. Saito. Received March 12, 2007. Revised September 17, 2007.
2000 Mathematics Subject Classification(s): 14J50, 14J28.
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Theorem 1.1.

(1) *Let* $G < \mathrm{Bir}\, M$. *Then either* $G > \mathbf{Z} * \mathbf{Z}$ *or* G *is an almost abelian group of rank at most* $\mathrm{Max}\,(1, \rho(M) - 2)$, *the latter of which is finitely generated.*

(2) *Assume that* M *admits* 2 *different abelian fibrations* $\varphi_i : M \longrightarrow B_i$ *s.t.* $\mathrm{rank}\, \mathrm{MW}(\varphi_i) > 0$ $(i = 1, 2)$. *Let* $f_i \in \mathrm{MW}(\varphi_i)$ *s.t.* $\mathrm{ord}\, f_i = \infty$. *Then the subgroup* $\langle f_1, f_2 \rangle$ *of* $\mathrm{Bir}\, M$ *contains* $\mathbf{Z} * \mathbf{Z}$ *as its subgroup.*

Unfortunately, the proof in [Og3], which is based on Tits' alternative [Ti] together with properties of Salem polynomial (cf. [Mc]), tells us no explicit way to find $\mathbf{Z} * \mathbf{Z}$ in $\langle f_1, f_2 \rangle$. Then F. Campana asked the following:

Question 1.2. Under the same condition as in Theorem (1.1)(2),

$$\langle f_1, f_2 \rangle = \langle f_1 \rangle * \langle f_2 \rangle \simeq \mathbf{Z} * \mathbf{Z}$$

for *suitably chosen* $f_i \in \mathrm{MW}(\varphi_i)$ $(i = 1, 2)$?

In the view of an observation of Cantat [Ca], one might ask an even stronger:

Question 1.3. Under the same condition as in Theorem (1.1)(2),

$$\langle f_1, f_2 \rangle = \langle f_1 \rangle * \langle f_2 \rangle \simeq \mathbf{Z} * \mathbf{Z}$$

for *any* $f_i \in \mathrm{MW}(\varphi_i)$ s.t. $\mathrm{ord}\,(f_i) = \infty$ $(i = 1, 2)$ at least when $\dim M = 2$?

The aim of this note is to give an affirmative answer to the first question in a slightly more general form and a negative answer to the second questions, via an elementary consideration in hyperbolic geometry. An applicability of hyperbolic geometry is suggested by S. Cantat in connection with [Og3].

Definition 1.4. Let $\mathbf{Z}_+$ be the set of positive integers and let Λ be a (possibly infinite) subset of $\mathbf{Z}_+$. Let G be a group. We say that subgroups G_i $(i \in \Lambda)$ of G have *essentially no relation* if there are finite index subgroups $H_i < G_i$ s.t. the group $\langle H_i | i \in \Lambda \rangle (\subset G)$ generated by H_i $(i \in \Lambda)$ is the free product $*_{i \in \Lambda} H_i$.

Our main result is the following:

Theorem 1.5.

(1) *Let* $\varphi_i : M \longrightarrow B_i$ $(i = 1, 2, \ldots, s)$ *be mutually different abelian fibrations on a hyperkähler manifold* M *s.t.* $\mathrm{rank}\, MW(\varphi_i) =: r_i > 0$. *Then, in*

$\operatorname{Bir} M$, *the Mordell-Weil groups* $\mathrm{MW}(\varphi_i)$ $(i = 1, 2, \ldots, s)$ *have essentially no relation, i.e., there are finite index subgroups* $H_i < \mathrm{MW}(\varphi_i)$ *s.t.*

$$\langle H_1, H_2, \ldots, H_s \rangle = H_1 * H_2 * \cdots * H_s \simeq \mathbf{Z}^{r_1} * \mathbf{Z}^{r_2} * \cdots * \mathbf{Z}^{r_s} \ .$$

In particular, for given $g_i \in \mathrm{MW}(\varphi_i)$ *with* $\operatorname{ord} g_i = \infty$, *there are positive integers* m_i $(i = 1, 2, \ldots, s)$ *s.t.*

$$\langle g_1^{m_1}, g_2^{m_2}, \ldots, g_s^{m_s} \rangle = \langle g_1^{m_1} \rangle * \langle g_2^{m_2} \rangle * \cdots * \langle g_s^{m_s} \rangle \simeq \mathbf{Z} * \mathbf{Z} * \cdots * \mathbf{Z} \ (s\text{−factors}) \ .$$

(2) *Let* $\varphi_i : M \longrightarrow B_i$ $(i \in \mathbf{Z}_+)$ *be mutually different abelian fibrations on a hyperkähler manifold* M *s.t.* $\operatorname{rank} MW(\varphi_i) =: r_i > 0$. *Then, there is an infinite subset* Λ *of* $\mathbf{Z}_+$ *s.t. the Mordell-Weil groups* $\mathrm{MW}(\varphi_i)$ $(i \in \Lambda)$ *have essentially no relation in* $\operatorname{Bir} M$, *i.e., there are finite index subgroups* $H_i < \mathrm{MW}(\varphi_i)$ *s.t.*

$$\langle H_i | i \in \Lambda \rangle = *_{i \in \Lambda} H_i \simeq *_{i \in \Lambda} \mathbf{Z}^{r_i} \ .$$

(3) *There are a K3 surface* S *admitting* 2 *different abelian* (*i.e. Jacobian*) *fibrations* $\varphi_i : S \longrightarrow \mathbf{P}^1$ *and elements* $f_i \in \mathrm{MW}(\varphi_i)$ *with* $\operatorname{ord} f_i = \infty$ $(i = 1, 2)$ *s.t. in* $\operatorname{Aut} S$,

$$\langle f_1 \, , \, f_2 \rangle \simeq \mathbf{Z}/2 * \mathbf{Z}/3 \ , \ \text{whence,} \ \langle f_1 \, , \, f_2 \rangle \neq \langle f_1 \rangle * \langle f_2 \rangle (\simeq \mathbf{Z} * \mathbf{Z}) \ ,$$

but, for each integer $n \geq 2$,

$$\langle f_1^n \, , \, f_2^n \rangle = \langle f_1^n \rangle * \langle f_2^n \rangle \simeq \mathbf{Z} * \mathbf{Z} \ .$$

As we shall show in Example (4.1), there are K3 surfaces and hyperkähler manifolds admitting infinitely many different abelian fibrations of positive Mordell-Weil rank.

§2. From Hyperkähler Manifold to Hyperbolic Geometry

In this section, we recall some important theorems which make a clear bridge between groups of birational autmorphisms of a projective hyperkähler manifold and elementary hyperbolic geometry.

Throughout this section, M is a (not necessarily projective) hyperkähler manifold. By [Be] (see also [GHJ, Part III]), there is a natural symmetric bilinear form called Beauville form, $(*, **) : H^2(M, \mathbf{Z}) \times H^2(M, \mathbf{Z}) \longrightarrow \mathbf{Z}$. Let $NS(M)$ be the Néron-Severi group of M, i.e., the subgroup of $H^2(M, \mathbf{Z})$ generated by the first Chern classes of holomorphic line bundles. We regard $NS(M)$ as a (possibly degenerate) sublattice of $H^2(M, \mathbf{Z})$ by $(*, **)$. We denote by $\rho(M)$ the Picard number of M, i.e., the rank of $NS(M)$.

The following theorem due to Huybrechts [Hu] (see also [GHJ, Part III] for (1) and [Og2] for (3)) is quite essential:

Theorem 2.1.

(1) *M is projective iff $NS(M)$ is hyperbolic, i.e., non-degenerate, of signature* $(1, \rho(M) - 1)$.

(2) *Let g be a bimeromorphic automorphism of M. Then the natural action g^* on $H^2(M, \mathbf{Z})$ is a Hodge isometry of M. Moreover, the induced representaion* $\mathrm{Bir}\, M \longrightarrow O(H^2(M, \mathbf{Z}))$ *has finite kernel.*

(3) *When M is projective, the induced representaion* $\mathrm{Bir}\, M \longrightarrow O(NS(M))$ *has finite kernel.*

The statements (1) and (3) make first bridge between $\mathrm{Bir}\, M$ and geometry of the hyperbolic lattice $NS(M)$, when M is projective.

Let M be a projective hyperkähler manifold admitting an abelian fibration $\varphi : M \longrightarrow B$ (See Introduction for the precise definition). The morphism φ is given by the complete linear system $|H|$ of some nef divisor H on M. This H is of the form $\varphi^* H_B$, where h is a very ample divisor on B. For a general manifold, there are lot of choices of such H (even up to multiple). However, for a hyperkähler manifold, a result due to Matsushita [Ma] says:

Theorem 2.2. *The base space B is $\mathbf{Q}$-factorial and of Picard number* 1 (*and of dimension* $\dim M/2$). *In particular, the choice of H is unique up to multiple. Moreover,* $(H^2) = 0$.

As an immediate consequence of Theorems (2.1), (2.2) with Theorem (1.1)(1), we obtain the following bridge between the Mordell-Weil group $\mathrm{MW}(\varphi)$ and geometry of the hyperbolic lattice $NS(M)$:

Corollary 2.3. *Under the notations above, let $O_H(NS(M))(\subset O(NS(M)))$ be the stabilizer subgroup of the class $H \in NS(M)$. Then, we have a natural representaion* $\mathrm{MW}(\varphi) \longrightarrow O_H(NS(M))$ *with finite kernel.*

§3. From Group Theory and Elementary Hyperbolic Geometry

(3.1) The next Theorem and its proof are taken from Tits' article [Ti]. This simple theorem is very useful in our proof of Theorems (1.5) and (3.2).

Theorem 3.1. *Let Λ be a* (*possibly infinite*) *subset of $\mathbf{Z}_+$ of cardinality $|\Lambda| \geq 2$. Let G be a group which acts faithfully on a non-empty set S. Let G_i $(i \in \Lambda)$ be subgroups of G, let S_i $(i \in \Lambda)$ be subsets of S s.t. $S \setminus \cup_{i\in\Lambda} S_i \neq \emptyset$, and let p be an element of $S \setminus \cup_{i\in\Lambda} S_i$. Put $G_i^0 := G_i \setminus \{id\}$ and $\tilde{S}_i := S_i \cup \{p\}$. Suppose that $G_i^0(\tilde{S}_j) \subset S_i$ for all pairs $i \neq j$. Then $\langle G_i | i \in \Lambda\rangle = *_{i\Lambda} G_i$.*

Proof. Consider $g_1 g_2 \cdots g_m$, where $g_k \in G^0_{i_k}$ and $i_1 \neq i_2 \neq \cdots \neq i_m$. Using $|\Lambda| \geq 2$, one can choose an integer $i \in \Lambda$ s.t. $i \neq i_m$. Then $G^0_{i_m}(\tilde{S}_i) \subset S_{i_m}$, $G^0_{i_{m-1}} G^0_{i_m}(\tilde{S}_i) \subset S_{i_{m-1}}$, and finally $G^0_{i_1} G^0_{i_2} \cdots G^0_{i_{m-1}} G^0_{i_m}(\tilde{S}_i) \subset S_{i_1}$. Thus $g_1 g_2 \cdots g_m(p) \subset S_{i_1}$. Hence $g_1 g_2 \cdots g_m(p) \neq p$ by $p \notin S_{i_1}$, and therefore $g_1 g_2 \cdots g_m \neq id$. □

(3.2) By a hyperbolic lattice $N = (N, (*, **))$ of rank ρ, we mean a pair of a free **Z**-module N of finite rank ρ and a **Z**-valued non-degenerate symmetric bilinear form $(*, **)$ on N with signature $(1, \rho - 1)$. We denote by $O(N)$ the group of isometries of N. What we need from elementary hyperbolic geometry is the following:

Theorem 3.2. *Let $N = (N, (*, **))$ be a hyperbolic lattice of rank ρ. Let $\tilde{\Lambda}$ be a (possibly infinite) subset of $\mathbf{Z}_+$ of cardinality $|\tilde{\Lambda}| \geq 2$. Suppose that v_i $(i \in \tilde{\Lambda})$ are primitive elements of $N \setminus \{0\}$ s.t. $(v_i^2) = 0$ and $v_i \neq \pm v_j$ if $i \neq j$, and that $G_i \simeq \mathbf{Z}^{r_i}$ $(i \in \tilde{\Lambda},\ r_i > 0)$ are subgroups of $O(N)$ s.t. $G_i(v_i) = v_i$. Then:*

(1) *Assume that $|\tilde{\Lambda}| < \infty$, say, $\tilde{\Lambda} = \{1, 2, \ldots, s\}$. Then G_i $(i = 1, 2, \ldots, s)$ have essentially no relation.*

(2) *Assume that $|\tilde{\Lambda}| = \infty$. Then there is an infinite subset Λ of $\tilde{\Lambda}$ s.t. G_i $(i \in \Lambda)$ have essentially no relation.*

Remark 3.3. The statement similar to (1) should be known for experts. In the rest of this section, we shall give a unform, self-contained proof of (1) and (2).

Proof. We denote the scalar extension $N \otimes K$ of N by a field K by N_K. Whenever we discuss topology, we regard $N_{\mathbf{R}} \simeq \mathbf{R}^\rho$ as the Euclidean space by a suitable Euclidean norm $|| * ||$.

1. Let $\mathcal{C}$ be the *positive cone* of N, that is, one of the two connected components of $\{x \in N_{\mathbf{R}} | (x, x) > 0\}$. We denote by $\overline{\mathcal{C}}$ the closure of $\mathcal{C}$ in $N_{\mathbf{R}}$, and by $\partial\overline{\mathcal{C}}$, the boundary of $\mathcal{C}$. We define $S := \{x \in \partial\mathcal{C} \,|\, ||x|| = 1\}$. This space S can be identified with the set of the rays $\{\mathbf{R}_+ x \,|\, x \in \partial\mathcal{C} \setminus \{0\}\}$, by the natural bijection $\mathbf{R}_+ x \leftrightarrow x/||x||$, and is also homeomorphic to the sphere $S^{\rho-2}$. We denote $x/||x||$ by $\overline{x}$ and call it the *unit vector corresponding to* x.

2. Let v be a primitive element of $N \setminus \{0\}$ s.t. $(v^2) = 0$. Following [Th, Problem 2.5.24], we consider the spaces:

$$\tilde{L}^{\mathbf{Q}} := \{x \in N_{\mathbf{Q}} | (x, v) = 1\}\ ,\ \tilde{L} := \{x \in N_{\mathbf{R}} | (x, v) = 1\}\ ;$$

$$L^{\mathbf{Q}} := \tilde{L}^{\mathbf{Q}}/\mathbf{Q}v\ ,\ L := \tilde{L}/\mathbf{R}v\ ;$$

$$P := \{x \in \partial\mathcal{C} \mid (x, v) = 1\} \subset \tilde{L}\ .$$

Here, since $v \in N$ and $(v, v) = 0$, the mapping $av : x \mapsto x + av$, $a \in \mathbf{Q}$ (resp. $a \in \mathbf{R}$) defines a faithful action of $\mathbf{Q}v$ (resp. of $\mathbf{R}v$) on $\tilde{L}^{\mathbf{Q}}$ (resp. on $\tilde{L}$). □

Remark 3.4. Though the rational spaces $\tilde{L}^{\mathbf{Q}}$ and $L^{\mathbf{Q}}$ are not explicitly considered in [Th], it is important to consider these spaces for our proof.

By definition, the space $L^{\mathbf{Q}}$ (resp. L) is a $(\rho - 2)$-dimensional rational (resp. real) affine space. Or more explicitly, by taking a point in $u \in L^{\mathbf{Q}}$ as 0 and by taking a rational basis of the tangent space at u, we can identify $L^{\mathbf{Q}}$ (resp. L) with the vector space $\mathbf{Q}^{\rho-2}$ (resp. $\mathbf{R}^{\rho-2}$). Under these identifications, we have also $L = (L^{\mathbf{Q}})_{\mathbf{R}}$. By $p : \tilde{L}^{\mathbf{Q}} \longrightarrow L^{\mathbf{Q}}$ and $p : \tilde{L} \longrightarrow L$, we denote the natural quotient maps. We define the map π by $\pi : P \longrightarrow S; x \mapsto x/||x||$.

Let $G \simeq \mathbf{Z}^r$ be an abelian subgroup of $O(N)$ s.t. $Gv = v$. The group G acts on $\tilde{L}^{\mathbf{Q}}$, $\tilde{L}$, $L^{\mathbf{Q}}$, L, P and S. Here the action of G on S is defined by $x \mapsto g(x)/||g(x)||$. By definiton, the actions of G on these spaces are equivariant under p and π.

3. In this paragraph, again closely following [Th, Problem 2.5.14], we shall prove three lemmas, which are crucial in our proof of Theorem (3.2). We use the same notations as in the previous paragraphs.

Lemma 3.5. *The map $p|P : P \longrightarrow L$ is a homeomorphism and the map*

$$\iota := \pi \circ (p|P)^{-1} : L \longrightarrow S$$

is a homeomorphism onto $S \setminus \{\overline{v}\}$, i.e., S is a one point compactification of the real affine space L through ι. Moreover, the actions of G on L and S are equivariant under ι.

Proof. The last assertion follows from the last statement in the paragraph **2**. Let us show the first two assertions. For a given $x \in \tilde{L}$, there is a unique α s.t. $(x + \alpha v, x + \alpha v) = 0$, namely $\alpha = -(x, x)/2$. Since $(x + \alpha v, v) = 1 > 0$, we have $x + \alpha v \in \partial\mathcal{C}$, and therefore $x + \alpha v \in P$. By the uniqueness of α, the map $x + \mathbf{R}v \mapsto x + \alpha v$ is well-defined and gives the inverse $q : L \longrightarrow P$ of $p|P$.

By $(v, v) = 0$, we have $\pi(P) \subset S \setminus \{\overline{v}\}$. Let $x \in \partial\mathcal{C} \setminus \{0\}$. By the Schwartz inequality, we have $(x, v) > 0$ unless $x \in \mathbf{R}v$. Thus, for $x \in S \setminus \{\overline{v}\}$, we have $x/(x, v) \in P$ and $\pi(x/(x, v)) = x$. Thus, $x \mapsto x/(x, v)$ gives the inverse of $\pi : P \longrightarrow S \setminus \{\overline{v}\}$. Hence $\iota = \pi \circ q$ satisfies the requirement. □

Lemma 3.6. *There is a finite index subgroup G' of G s.t. G' acts on the real affine space L as parallel transformations. Moreover, the action of G' on L is faithful and discrete.*

Remark 3.7. For both statements, it seems quite essential that the action G is defined over the *lattice* N, or in other words, over $\mathbf{Z}$. Being parallel transformations is false even for a cyclic orthogonal subgroup of $N_{\mathbf{Q}}$ fixing some non-zero rational vector v with $(v, v) = 0$, as there is an infinite-order 3×3 rotation matrix of rational entries. An account of [Ha, Page 135] seems to miss this point.

Proof. Since v is a primitive vector with $(v, v) = 0$, one can choose an integral basis of $v_N^{\perp}$ as $\langle v\,,\, w_1\,,\, \dots\,,\, w_n\,\rangle$. We also choose $u \in N_{\mathbf{Q}}$ s.t. $(v, u) = 1$. Such u exists, as N is non-degenerate. Then $\langle v, w_1, \dots, w_n, u\rangle$ forms $\mathbf{Q}$-basis of $N_{\mathbf{Q}}$. In this notation, $\rho = n + 2$.

Let $g \in G$. Since $g(v) = v$ and $g(v_N^{\perp}) = v_N^{\perp}$, the matrix representation $M(g)$ of g w.r.t. the basis above is of the form:

$$M(g) := \begin{pmatrix} 1 & {}^t\mathbf{a}(g) & c(g) \\ \mathbf{o} & A(g) & \mathbf{b}(g) \\ 0 & {}^t\mathbf{o} & d(g) \end{pmatrix} .$$

Here $A(g)$ is the matrix representation of the action of g on the *lattice* $N := v_{NS(M)}^{\perp}/\mathbf{Z}v$ w.r.t. its *integral* basis $\langle [w_i]\rangle_{i=1}^n$. Thus $A(g)$ is an *integral* matrix, while $\mathbf{b}(g) \in \mathbf{Q}^n$ is, in general, a rational vector. Since N is of negative definite, its orthogonal transformation $A(g)$ is diagonalizable, and the eigenvalues of $A(g)$ are of absolute value 1. On the other hand, since $A(g)$ is an integral matrix, its eigenvalues are algebraic integers. Thus, by Kronecker's theorem, the eigenvalues of $A(g)$ are roots of 1. Since $A(g)$ is diagonalizable, $A(g)$ is then of finite order. Let us denote by $m(g)$ the order of $A(g)$.

Let $\langle g_i\rangle_{i=1}^r$ be a generator of the free abelian group G. Put $g_i' := g_i^{2m(\tilde{g}_i)}$ and define the subgroup G' of G by $G' := \langle g_1', g_2', \dots, g_r'\rangle$. Since G is a finitely generated abelian group, G' is a finite index subgroup of G. Moreover, for each $g \in G'$, we have $A(g) = I_n$ (the identity matrix) and $d(g) = 1$, i.e.,

$$M(g) := \begin{pmatrix} 1 & {}^t\mathbf{a}(g) & c(g) \\ \mathbf{o} & I_n & \mathbf{b}(g) \\ 0 & {}^t\mathbf{o} & 1 \end{pmatrix} .$$

For $d(g) = 1$, observe that $\det M(\tilde{g}_i) = \pm 1$, whence $\det M(\tilde{g}_i^{2m(\tilde{g}_i)}) = 1$, and therefore $\det M(g) = 1$ for $g \in G'$.

Let $Q \in L$. Then Q is uniquely expressed in the form:

$$Q = [u] + \sum_{i=1}^{n} x_i(Q)[w_i] \ ,$$

and the vector valued function $\mathbf{x} := (x_i)_{i=1}^n$ gives an affine coordinate of both L and $L^{\mathbf{Q}}$. Under this coordinate, one can identify $L^{\mathbf{Q}} = \mathbf{Q}^n$, $L = (\mathbf{Q}^n) \otimes \mathbf{R} = \mathbf{R}^n$. Then, by the shape of $M(g)$, the action of $g \in G'$ on $L = \mathbf{R}^n$ is the parallel transformation by $\mathbf{b}(g) \in \mathbf{Q}^n$. This shows the first statement. For the discreteness and faithfulness of the action of G' on $L = \mathbf{R}^n$, it suffices to show the claim below. Indeed, $\langle r(g_i')\rangle_{i=1}^r$ then forms a part of real basis of $L = (\mathbf{Q}^n) \otimes \mathbf{R}$. □

Claim 3.8. The map $r : G' \simeq \mathbf{Z}^r \longrightarrow \mathbf{Q}^n; g \mapsto \mathbf{b}(g)$ is an injective group homomorphism.

Remark 3.9. Note that the map $(a, b) \mapsto a + b\sqrt{2}$ defines an injective group homomorphism from $\mathbf{Z}^2$ to the group $\mathbf{R}$ of the parallel transformations of $\mathbf{R}$, but the image $\{a + b\sqrt{2} | a, b \in \mathbf{Z}\}$ is not discrete in $\mathbf{R}$. So, the fact that G acts on the underlying rational space $L^{\mathbf{Q}}$ seems crucial for the discreteness.

Proof. Assume that $\mathbf{b}(g) = \mathbf{o}$. Then, by the shape of $M(g)$, we have $g(v) = v$, $g(w_i) = w_i + a_i(g)$, $g(u) = u + c(g)v$. From

$$(u, u) = (g(u), g(u)) = (u, u) + 2c(g)(u, v) = (u, u) + 2c(g) \ ,$$

we have $c(g) = 1$. Then, from

$$(u, w_i) = (g(u), g(w_i)) = (u, w_i) + a_i(g)(u, v) = (u, w_i) + a_i(g) \ ,$$

we have $a_i(g) = 0$. Hence $M(g) = I_{n+2}$, i.e., $g = id$. □

Lemma 3.10. *Let $U \subset S$ be a compact neighborhood of the unit vector $\overline{v}$ and let V be a non-empty compact subset of $S \setminus \{\overline{v}\}$. Then, there is a finite index subgroup H of G s.t. $(H \setminus \{id\})(V) \subset U$.*

Proof. Identifying L with $S \setminus \{\overline{v}\}$ by ι (see Lemma (3.5)), we may assume that

$$V \subset L \text{ and } B := \{x \in L| \, ||x|| > r\} \subset U \setminus \{\overline{v}\} \subset L$$

for some $r > 0$. Here $|| * ||$ is some Euclidean norm of L (w.r.t. some fixed origin).

Let $G' \simeq \mathbf{Z}^r$ be a finite index subgroup of G in Lemma (3.6). Then, G' acts on L by parallel transformations, say by $\{t(g) \in \mathbf{R}^n | g \in G\}$. Since the action of G' is discrete by Lemma (3.6), we have $\inf \{||t(g)||; g \in G' \setminus \{id\}\} > 0$.

Thus, there is some positive integer c s.t. $g^c(V) \subset B$ for all $g \in G' \setminus \{id\}$. Set $H := \{g^c | g \in G'\}$. Since G' is a finitely generated abelian group, this set H is a finite index subgroup of G'. This H satisfies the requirement. □

4. Now we shall complete the proof of Theorem (3.2).

Let us first assume that $|\tilde{\Lambda}| = \infty$. Then the sequence $\{\overline{v}_i\}_{i\in\tilde{\Lambda}} (\subset S)$ has an infinite subsequence which converges to some point $\overline{v}_\infty \in S$. Getting rid of the terms $\overline{v}_i = \overline{v}_\infty$ from this subsequence, we obtain a subsequence $\{\overline{v}_i\}_{i\in\Lambda}$ s.t. $\lim_{i\longrightarrow\infty} \overline{v}_i = \overline{v}_\infty$ and $\overline{v}_i \neq \overline{v}_\infty$ for all $i \in \Lambda$. Then $\min_{j\in(\Lambda\setminus\{i\})\cup\{\infty\}} ||\overline{v}_i - \overline{v}_j|| > 0$ for each $i \in \Lambda$. Thus, we can choose *open* neighborhoods $V_i (\subset S)$ of $\overline{v}_i$ $(i \in \Lambda)$ s.t. the closure $\overline{V}_i$ $(i \in \Lambda)$ are compact, $\overline{v}_\infty \notin \overline{V}_i$ for each $i \in \Lambda$ and $\overline{V}_i \cap \overline{V}_j = \emptyset$ whenever $i \neq j$. Then take *open* neighborhoods U_i^0 of $\overline{v}_i$ s.t. $U_i := \overline{U_i^0} \subset V_i$. We put $\tilde{U}_i = U_i \cup \{\overline{v}_\infty\}$ for each $i \in \Lambda$. Then, using compactness of S and Lemma (3.10) applied for $\overline{v} = \overline{v}_i$ $(i \in \Lambda)$ inductively, we find finite index subgroups H_i of G_i $(i \in \Lambda)$ s.t. $(H_i \setminus \{id\})(\cup_{j\neq i}\tilde{U}_j) \subset U_i$. In particular, $(H_i \setminus \{id\})(\tilde{U}_j) \subset U_i$ whenever $i \neq j$.

Next consider the case where $|\tilde{\Lambda}| < \infty$. In this case, we put $\Lambda = \tilde{\Lambda}$. Since Λ is a finite set, we can choose a point $p \in S$ and compact neighborhoods $U_i (\subset S)$ of $\overline{v}_i$ $(i \in \Lambda)$ s.t. $p \notin U_i$ for each $i \in \Lambda$ and $U_i \cap U_j = \emptyset$ whenever $i \neq j$. We also put $\tilde{U}_i = U_i \cup \{p\}$ for each $i \in \Lambda$. Then, there are finite index subgroups H_i of G_i $(i \in \Lambda)$ s.t. $(H_i \setminus \{id\})(\tilde{U}_j) \subset U_i$ whenever $i \neq j$.

In both cases $\langle H_i | i \in \Lambda\rangle = *_{i\in\Lambda} H_i$ by Theorem (2.1). This completes the proof. □

§4. Synthesis

In this section, we shall show Theorem (1.5).

Proof of Theorem (1.5)(1), (2).

Let $v_i \in NS(M)$ be the primitive nef element which is proportional to the divisor H_i defining φ_i. Then $(v_i^2) = 0$ by Theorem (2.2). Let $\overline{v}_i \in S$ be the unit element corresponding to v_i. We note that $\overline{v}_i \neq \overline{v}_j$ if $i \neq j$.

Let $G_i := \mathrm{Im}\,(\mathrm{MW}(\varphi_i) \longrightarrow O(NS(M)))$. Then G_i preserves $\overline{v}_i$ and the kernel of $\mathrm{MW}(\varphi_i) \longrightarrow G_i$ is finite by Corollary (1.3). Hence, the result follows from Theorem (3.2).

Proof of Theorem (1.5)(2).

Let E be an elliptic curve. Let us consider the product abelian surface $A := E \times E$, and its associated Kummer K3 surface $S := \mathrm{Km}\, A$. By definition, S is the minimal resolution of the quotient surface A/ι (ι is defined below).

Let $\tilde{p}_i : A \longrightarrow E$ be the projection to the i-th factor, and let $p_i : S \longrightarrow \mathbf{P}^1$ be the Jacobian fibration on S, induced by $\tilde{p}_i$.

Let

$$\tilde{f}_2 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \ , \ \tilde{f}_1 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \ , \ \iota := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and put $\tilde{G} := \langle \tilde{f}_1, \tilde{f}_2 \rangle$ in $\mathrm{SL}(2, \mathbf{Z})$. Then $\tilde{G} = \mathrm{SL}(2, \mathbf{Z})$ and $\mathrm{SL}(2, \mathbf{Z})$ acts faithfully on A by $\tilde{f}_2(x, y) = (x + y, y)$, $\tilde{f}_1(x, y) = (x, x + y)$. By the shape of $\tilde{f}_i$, we see that $\tilde{f}_i \in \mathrm{MW}(\tilde{p}_i)$ and $\mathrm{ord}\, \tilde{f}_i = \infty$. Since $\tilde{f}_i \iota = \iota \tilde{f}_i$, we also see that $\tilde{f}_i$ descends to $f_i \in \mathrm{MW}(p_i)$ and $\mathrm{ord}\, f_i = \infty$.

Set $G := \langle f_1, f_2 \rangle$ in $\mathrm{Aut}\, S$. Then, $G \simeq \mathrm{SL}(2, \mathbf{Z})/\langle \iota \rangle = \mathrm{PSL}(2, \mathbf{Z})$.

We shall show that (S, f_1, f_2) satisfies the requirement.

It is well-known that $\mathrm{PSL}(2, \mathbf{Z}) \simeq \mathbf{Z}/2 * \mathbf{Z}/3$ (see eg. [Kn, Page 147]). In particular, $\langle f_1, f_2 \rangle \neq \langle f_1 \rangle * \langle f_2 \rangle$; if otherwise, $\mathbf{Z}/2 * \mathbf{Z}/3 \simeq \mathbf{Z} * \mathbf{Z}$, a contradiction.

It remains to show that $\langle f_1^n, f_2^n \rangle = \langle f_1^n \rangle * \langle f_2^n \rangle$ for each $n \geq 2$. Our proof of this fact closely follows [Ha, Example 1]. Note that $\mathrm{PSL}(2, \mathbf{Z}) < \mathrm{PGL}(2, \mathbf{Z})$.

In $\mathrm{PGL}(2, \mathbf{Z})$, we put

$$g_2 := f_2^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \; g_1 := f_1^n := \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}, \; j := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $j^2 = id$ and $g_1 = j g_2 j$ in $\mathrm{PGL}(2, \mathbf{Z})$. So, if there would be a non-trivial relation among g_1 and g_2, say, $h(g_1, g_2) = id$, then substituting $g_1^l = j g_2^l j$ into the relation $h(g_1, g_2) = id$, we would have a non-trivial relation among g_2 and j. Thus, we suffice to show that $\langle g_2, j \rangle = \langle g_2 \rangle * \langle j \rangle$ in $\mathrm{PGL}(2, \mathbf{Z})$.

To prove this claim, we consider the natural fractional linear action of $\mathrm{PGL}(2, \mathbf{Z})$ on $\mathbf{P}^1 = \mathbf{C} \cup \{\infty\}$, and the following subsets and a point of $\mathbf{P}^1$:

$$U_1 := \{z \in \mathbf{C} \,|\, |z| < 1\},\; U_2 := \{z \in \mathbf{C} \,|\, |\mathrm{Re}\, z| > 1\} \cup \{\infty\},\; P := 2\sqrt{-1} \notin U_1 \cup U_2.$$

Then $j(U_2 \cup \{P\}) \subset U_1$ and $g_2^k(U_1 \cup \{P\}) \subset U_2$ for each $k \neq 0$ (by $n \geq 2$). Thus, by Theorem (2.1), we have $\langle g_2\, , \, j \rangle = \langle g_2 \rangle * \langle j \rangle$, and we are done.

Example 4.1. Let $\varphi : S \longrightarrow \mathbf{P}^1$ be a (necessarily projective) Jacobian K3 surface s.t. $\mathrm{rank}\, \mathrm{MW}(\varphi) = 18$ and $\rho(S) = 20$, which are maximum. Such a Jacobian K3 surface exists by [Co] and [Ni]. It is also known that, given a Jacobian K3 surface S', one can find such a Jacobian K3 surface S arbitrarily close to S' in the period domain ([Og1]). In $\mathrm{Aut}\, S$ and $\mathrm{Aut}\, S^{[n]}$, we can find an infinite sequence of subgroups

$$\mathbf{Z}^{18} \subset \mathbf{Z}^{18} * \mathbf{Z}^{18} \subset \mathbf{Z}^{18} * \mathbf{Z}^{18} * \mathbf{Z}^{18} \cdots \subset *_s \mathbf{Z}^{18} \subset \cdots .$$

Here $S^{[n]}$ is the Hilbert scheme of n-points on S, which is a hyperkähler manifold of dimension $2n$ ([Be]).

Proof. Let $e \in \mathrm{Pic}\, S \simeq NS(S)$ be the fiber class of φ. Since φ has at least three singular fibers (see eg., [Ca], [VZ]), the subgroup $\mathrm{Aut}\, \varphi := \{f \in$

$\mathrm{Aut}\, S \mid f^*(e) = e\}$ of $\mathrm{Aut}\, S$ is an almost abelian group (of rank 18). On the other hand, since $\rho(S) = 20$, we have $\mathbf{Z} * \mathbf{Z} \subset \mathrm{Aut}\, S$ by [Og3]. In particular, $\mathrm{Aut}\, S$ is not an almost abelian group.

Hence the coset $\mathrm{Aut}\, S/\mathrm{Aut}\, \varphi$ is an infinite set. Thus, the $\mathrm{Aut}\, S$-orbit of e is an infinite set as well. Therefore, S admits infinitely many different Jacobian fibrations with Mordell-Weil rank 18. Now the result for S follows from Theorem (1.5)(2). The Mordell-Weil group of a Jacobian fibration $S \longrightarrow \mathbf{P}^1$ can be naturally embedded into the Mordell-Weil group of the induced abelian fibration $S^{[n]} \longrightarrow \mathbf{P}^n$. This implies the result for $S^{[n]}$. □

Acknowledgements

I would like to express my thank to Professors F. Campana, S. Cantat, J.M. Hwang, M. Paun, and J. Winkelman for their valuable discussions, mostly in Johns Hopkins University, Luminy and Oberwolfach. Finally, I would like to express my deep thank to the referee for valuable comments, especially the comment which indicates a way to divide an algebro-geometric part and a purely elementary hyperbolic geometrical part cleanly. This makes our explanation much cleaner.

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