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On the Group of S^1-equivariant Homeomorphisms of the 3-Sphere

By

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Abstract

Let G be a locally compact abelian group and let ξ be a principal G-bundle:

$$G \longrightarrow E \xrightarrow{p} B \ ,$$

where E is path connected and B is a locally finite CW complex. We recall that G acts freely on the right of E. Then we denote by $\mathrm{Top}(B)$ the topological group of homeomorphisms of B and by $\mathrm{Top}_G(E)$ the group of G-equivariant homeomorphisms of E. Furthermore, let α be a map of B into the classifying space BG whose homotopy class $[\alpha]$ classifies the principal bundle ξ. Then we have

Corollary 4. *Let G be a locally compact abelian group, then we have the following Serre fibration* :

$$\mathrm{map}(B, G) \longrightarrow \mathrm{Top}_G(E) \xrightarrow{\Phi} \mathrm{Top}^{[\alpha]}(B) \ ,$$

where $\mathrm{Top}^{[\alpha]}(B)$ *is the subspace of* $\mathrm{Top}(B)$ *consisting of homeomorphisms* $f : B \to B$ *such that* $f^*([\alpha]) = [\alpha]$.

As a special case, let $S^1 \to S^3 \to S^2$ be the Hopf principal bundle. By using Corollary 4 we have

Theorem 5. *There exists a weak homotopy equivalence*

$$\mathrm{Top}_{S^1}(S^3) \underset{w}{\simeq} \mathrm{Spin}^c(3) \ .$$

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§1. The G-equivariant Homeomorphisms

In this paper, all function spaces are supposed to have compact open topology.

Let G be a locally compact topological group and let ξ be a principal G-bundle denoted by

$$\xi : G \longrightarrow E \xrightarrow{p} B \ ,$$

where E is path connected, B is a locally finite CW complex and G acts freely on the right of E. If we denote by $\mathcal{G}(\xi)$ the space of self bundle maps of ξ and denote by map(B, B) the space of self maps of B, by the bundle map theory (see [2] and [3]) we have the following Serre fibration :

$$\xi_G \longrightarrow \mathcal{G}(\xi) \xrightarrow{\Phi} \mathrm{map}(B, B) \ .$$

Here ξ_G is weakly homotopy equivalent to the loop space $\Omega(\mathrm{map}(B, BG; \alpha))$ of map$(B, BG; \alpha)$ ([1], [2]), which is the path connected component of map(B, BG) containing the classifying map $\alpha : B \longrightarrow BG$ for the principal bundle ξ. We know that ξ_G can be identified with the space map(B, G) if G is abelian ([3]).

Let Top(B) denote the group of homeomorphisms of B and $\mathrm{Top}_G(E)$ denote the group of G-equivariant homeomorphisms of E. Then Φ may not be surjective but we have the following

Lemma 1. $\Phi^{-1}(\mathrm{Top}(B)) = \mathrm{Top}_G(E)$.

Proof. First we shall show that each map $\tilde{f}$ of $\Phi^{-1}(\mathrm{Top}(B))$ is injective. Put $\Phi(\tilde{f}) = f$. For any distinct points x_1, x_2 of E with $p(x_1) \neq p(x_2)$, we obviously see $\tilde{f}(x_1) \neq \tilde{f}(x_2)$. For distinct points x_1, x_2 with $p(x_1) = p(x_2)$ there exists an element a of G such that

$$x_2 = x_1 \cdot a \quad (a \neq e) \ ,$$

where e is the identity element of G. This implies $\tilde{f}(x_2) = \tilde{f}(x_1) \cdot a$. Since the action of G is free we have $\tilde{f}(x_1) \neq \tilde{f}(x_2)$.

Surjectivity of $\tilde{f}$ also can be proved easily, and continuity of $\tilde{f}^{-1}$ follows from the fact that a bijective bundle map $\tilde{f}$ is a homeomophism if its induced map f is a homeomorphism.

Let α be a map of B into the classifying space BG whose homotopy class $[\alpha]$ classifies the principal bundle ξ. And let $\mathrm{Top}^{[\alpha]}(B)$ denote the subspace of $\mathrm{Top}(B)$ consisting of homeomorphisms $f: B \to B$ which satisfy

$$f^*([\alpha]) = [\alpha] \ .$$

Immediately we have the following

Lemma 2. *The Φ image of* $\mathrm{Top}_G(E)$ *is just* $\mathrm{Top}^{[\alpha]}(B)$.

Consequently we have

Theorem 3. *With notation above, there exists the following Serre fibration* :

$$\xi_G \longrightarrow \mathrm{Top}_G(E) \xrightarrow{\Phi} \mathrm{Top}^{[\alpha]}(B) \ .$$

Corollary 4. *Let G be a locally compact abelian group, then we have the following Serre fibration* :

$$\mathrm{map}(B, G) \longrightarrow \mathrm{Top}_G(E) \xrightarrow{\Phi} \mathrm{Top}^{[\alpha]}(B) \ .$$

Now, let $\mathbf{R}P^n$ be the n-dimensional real projective space then we have the principal bundle :

$$Z_2 \longrightarrow S^n \longrightarrow \mathbf{R}P^n \ .$$

Thus we have

Example 1. *There is a Serre fibration* :

$$Z_2 \longrightarrow \mathrm{Top}_{Z_2}(S^n) \xrightarrow{\Phi} \mathrm{Top}(\mathbf{R}P^n) \ .$$

Similarly, let $\mathbf{C}P^n$ be the n-dimensional complex projective space then we have the principal bundle :

$$S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbf{C}P^n \ .$$

Since $\mathrm{map}(\mathbf{C}P^n, S^1)$ is homotopy equivalent to S^1, we have

Example 2. There is a Serre fibration :

$$S^1 \longrightarrow \mathrm{Top}_{S^1}(S^{2n+1}) \xrightarrow{\Phi} \mathrm{Top}^+(\mathbf{C}P^n) \ ,$$

where $\mathrm{Top}^+(\mathbf{C}P^n)$ denotes the component of identity mapping in $\mathrm{Top}(\mathbf{C}P^n)$.

As a special case of Example 2, we have a Serre fibration :

$$S^1 \longrightarrow \mathrm{Top}_{S^1}(S^3) \xrightarrow{\Phi} \mathrm{Top}^+(S^2) \ .$$

§2. $\mathrm{Top}_{S^1}(S^3)$ and $\mathrm{Top}_{SO(2)}(SO(3))$

Let us define a $(S^3 \times S^1)$-action on S^3 as follows :

$$\rho : (S^3 \times S^1) \times S^3 \longrightarrow S^3$$

is given by

$$\rho((q, z), q') = qq'z \quad (q, q' \in \mathbf{H}, z \in \mathbf{C}, |q| = |q'| = |z| = 1) \ .$$

Then we can easily verify that this action is not effective and the kernel of the action ρ is the central subgroup of the group $S^3 \times S^1$ which consists of two elements $(1, 1)$ and $(-1, -1)$. Therefore $\mathrm{Spin}^c(3) \cong (S^3 \times S^1) / \{(1, 1) \cup (-1, -1)\}$† acts effectively on S^3. Also we can easily prove that each element of $\mathrm{Spin}^c(3)$ induces an S^1-equivariant homeomorphism of S^3. Thus we have the inclusion map $i : \mathrm{Spin}^c(3) \to \mathrm{Top}_{S^1}(S^3)$. With this notation, we have

Theorem 5. *The inclusion* map $i : \mathrm{Spin}^c(3) \to \mathrm{Top}_{S^1}(S^3)$ *gives a following weak homotopy equivalence*

$$\mathrm{Spin}^c(3) \underset{w}{\simeq} \mathrm{Top}_{S^1}(S^3) \ .$$

Proof. We have the following principal bundle :

$$S^1 \longrightarrow \mathrm{Spin}^c(3) \xrightarrow{\pi} S^3 / \{1, -1\}$$

and the map i defines the following commutative diagram

†The fact that $(S^3 \times S^1) / \{(1, 1) \cup (-1, -1)\}$ is actually $\mathrm{Spin}^c(3)$ was pointed out by many participants in the Kinosaki Symposium held in autumn of 1994. The author expresses his thanks here.

$$\begin{array}{ccccc} S^1 & \longrightarrow & \mathrm{Spin}^c(3) & \xrightarrow{\pi} & S^3/\{1,-1\} \\ \downarrow & & \downarrow i & & \downarrow j \\ S^1 & \longrightarrow & \mathrm{Top}_{S^1}(S^3) & \xrightarrow{\Phi} & \mathrm{Top}^+(S^2) \end{array}$$

where $\{1, -1\}$ is the center of S^3 and π is the map induced by the projection of $S^3 \times S^1$ onto S^3. By Kneser's theorem ([4]), we know that j is a homotopy equivalence. Considering the exactness of homotopy sequences of our fibrations, we see that i is a weak homotopy equivalence.

Next, let $SO(2)$ act on the right of $SO(3)$ as usual. We proceed to study the group of $SO(2)$-equivariant homeomorphisms of $SO(3)$.

Let us define a $(SO(3) \times SO(2))$-action on $SO(3)$ similar to the case of S^3 as follows :

$$\rho' : (SO(3) \times SO(2)) \times SO(3) \longrightarrow SO(3)$$

is given by

$$\rho'(\sigma, g, \tau) = \sigma\tau g \quad (\sigma, \tau \in SO(3), g \in SO(2)) .$$

Then we can easily prove that this action is effective and each element of $SO(3) \times SO(2)$ is an $SO(2)$-equivariant homeomorphism of $SO(3)$. So, we have the inclusion map $i' : SO(3) \times SO(2) \longrightarrow \mathrm{Top}_{SO(2)}(SO(3))$.

On the other hand, we have the $SO(2)$-principal bundle :

$$SO(2) \longrightarrow SO(3) \longrightarrow S^2 .$$

For this prinicpal bundle Corollary 4 provides the following Serre fibration :

$$SO(2) \longrightarrow \mathrm{Top}_{SO(2)}(SO(3)) \xrightarrow{\Phi} \mathrm{Top}^+(S^2) .$$

By the same manner in the proof of Theorem 5, we get the following

Theorem 6. *The inclusion map*

$$i' : SO(3) \times SO(2) \longrightarrow \mathrm{Top}_{SO(2)}(SO(3))$$

gives a weak homotopy equivalence

$$SO(3) \times SO(2) \underset{w}{\simeq} \mathrm{Top}_{SO(2)}(SO(3)) .$$

References

[1] Gottlieb, D. H., On fibre spaces and the evaluation map, *Ann. of Math.*, **87** (1968), 42—55; ibid. **91** (1970), 640—642.
[2] ——, Applications of bundle map theory, *Trans. Amer. Math. Soc.*, **171** (1972), 23—50.
[3] James, I. M., The space of bundle maps, *Topology*, **2** (1963), 45—59.
[4] Kneser, H., Die Deformationssätze der einfach zusammenhängenden Flächen, *Math. Zeit.*, **25** (1926), 362—372.