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Some Remarks on Hypoelliptic Operators which are not Micro-hypoelliptic

Dedicated to Professor Shigetake Matsuura on his sixtieth birthday

By

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§1. Introduction

In this note we give an example of hypoelliptic operators which are not micro-hypoelliptic. Non-micro-hypoellipticiy of the example arises from the oscillation of the coefficient with a zero of infinite order.

Let us consider the following semi-elliptic operator with infinite degeneracy:

$$L = a(x, y, D_x) + g(x)b(x, y, D_y) \quad \text{in} \quad \mathbf{R}^n = \mathbf{R}_x^{n_1} \times \mathbf{R}_y^{n_2}. \tag{1.1}$$

Here $g(x) \in C^\infty$ and satisfies

$$g(x) > 0 \quad \text{for} \quad x \neq 0 \quad \text{and} \quad \partial_x^\beta g(0) = 0 \quad \text{for any } \beta. \tag{A.1}$$

Here $a(x, y, D_x)$ and $b(x, y, D_y)$ are differential operators with C^∞ coefficients of order 2ℓ and $2m$. We assume that $a(x, y, D_x)$ and $b(x, y, D_y)$ are strongly elliptic with respect to x and y, respectively, that is, for $C_1, C_2 > 0$

$$\operatorname{Re} a(x, y, \xi) \geq C_1 |\xi|^{2\ell} \quad \text{and} \tag{A.2}$$

$$\operatorname{Re} b(x, y, \eta) \geq C_2 |\eta|^{2m} \tag{A.3}$$

hold if $|\xi|$ and $|\eta|$ are sufficiently large. In [3] the one of authors (T.M.) proved that the operator L is hypoelliptic, i.e.

$$\text{sing supp } u = \text{sing supp } Lu \quad \text{for} \quad u \in \mathcal{D}'. \tag{1.2}$$

This ameliorates the old work [2] (c.f. Fediĭ [1]) of another author (Y.M.). Actually, in [2] the following condition was required to show (1.2) in case of $m \geq 2$:

$$\text{(G)} \quad \begin{cases} \textit{There exist constants } C \textit{ and } \sigma \ (0 < \sigma < 1/\{2(m - \ell + m\ell)\}) \\ \textit{such that } |\partial_x^\beta g(x)| \leq C\, g(x)^{1-\sigma|\beta|} \quad \textit{for} \quad |\beta| \leq 2(m - \ell + m\ell). \end{cases}$$

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In the recent paper [4] the one of authors (T.M.) also has studied the micro-hypoellipticity of L and has given the following theorem:

Theorem A. *Let $z=(x_0, y_0; \xi_0, \eta_0)$ be a point in $T^*(\mathbf{R}^n)$ ($x_0, \xi_0 \in \mathbf{R}^{n_1}$ and $y_0, \eta_0 \in \mathbf{R}^{n_2}$) with $|\eta_0| \neq 0$. Let L be the operator* (1.1) *satisfying* (A.1)−(A.3).
(i) *In the case where $\ell \geq m$, L is micro-hypoelliptic at z, that is, $z \notin$ WF (Lu) implies $z \notin$ WF u*
(ii) *In the case where $\ell < m$, L is still micro-hypoelliptic at z if $g(x)$ satisfies the following condition*:

$$\text{(A.4)} \quad \begin{cases} \textit{There exist constants } C \quad \textit{and} \quad \tau \ (0<\tau<1/\{2(m-\ell)\}) \\ \textit{such that } |\partial_x^\beta g(x)| \leq C\, g(x)^{1-\tau|\beta|} \quad \textit{for} \quad |\beta| \leq 2(m-\ell). \end{cases}$$

We remark that Theorem A is valid in the case where $g(x)$ vanishes finitely at $x=0$. In this case, (A.4) implies $g(x)=o(|x|^{2(m-\ell)})$. If $x \in \mathbf{R}^1$ and $g(x)=x^{2k}$ for an integer $k>0$ then it follows from (A.4) that $k>m-\ell$. By Parenti-Rodino [5], it is known that if $0<k\leq m-\ell$, hypoelliptic operator $D_x^{2\ell}+x^{2k}D_y^{2m}$ in $\mathbf{R}^2$ is not micro-hypoelliptic at $(0, 0; 0, 1) \in T^*$ $(\mathbf{R}^2)$. The condition (A.4) is satisfied when $g(x)=\psi(x)^k$ for some integer $k>m-\ell$ and some C^∞ function $\psi(x)$ with $\psi(x)>0$ for $x \neq 0$. This fact can be seen by noticing that $\psi'(x)^2 \leq$ *Const.*$\psi(x)$ near the origin. On the other hand, we see that for integer $k>0$

$$\text{(1.3)} \qquad g_k(x)=\exp(-1/|x|)\sin^{2k}\pi/|x|+\exp(-1/|x|^2)$$

(cf. Remark 2 in [2, Section 1])

does not satisfies (A.4) if $k \leq m-\ell$. In fact, for $|\beta|=2k$ and integer $j>0$ we have

$$\partial_x^\beta g_k(x)=O(e^{-j}),\ g_k(x)=O(e^{-j^2}),\ |x|=1/j, \text{ as } j\to\infty.$$

In order to consider the necessity of (A.4), we set

$$\text{(1.4)} \qquad L_k=a(x, D_x)+g_k(x)b(x, D_y) \quad \text{in} \quad \mathbf{R}_x^1\times\mathbf{R}_y^{n_2},$$

where a and b satisfies (A.2) and (A.3), respectively (but they are independent of y variable).

Theorem B. *Let ℓ, m and k be positive integers such that $m \geq \ell+2$ and $k \leq m-\ell-1$. If $g_k(x)$ is the function* (1.3) *with $x \in \mathbf{R}^1$ and if L_k is the above operator then L_k is not micro-hypoelliptic at $z=(0, y_0;\ 0, \eta_0) \in T^*(\mathbf{R}_x^1 \times \mathbf{R}_y^{n_2})$ with $\eta_0 \neq 0$.*

If (A.4)$'$ denotes the condition (A.4) with τ replaced by τ' $(0<\tau'<1/\{2(m-\ell-1)\}$ then Theorem B shows that (A.4)$'$ is necessary in general for L to be micro-hypoelliptic. Unfortunately, in case of $m=\ell+1$ the theorem says nothing concerning the necessity of conditions like (A.4). In the next section we shall give the proof of Theorem B influenced by [5] though our method is a little different from the one there. To end Introduction authors wish to express their hearty gratitude to Professor N. Iwasaki for useful discussions.

§2. Proof of Theorem B

For the sake of simplicity we shall prove Theorem B in case of $y \in \mathbf{R}^1$ ($n_2 = 1$), $y_0 = 0$ and $\eta_0 = 1$ since the proof in general case is similar. Throughout this section we assume that $m \geq \ell + 2$. We construct a singular solution $u(x, y)$ in the form

$$u(x, y) = \sum_{j=1}^{\infty} \eta_j^{-4} \exp(i\eta_j y) u_j(x), \tag{2.1}$$

where $\eta_j = \exp\{j^2/4m\}$. We require that $u_j(x) \in C_0^\infty$ satisfies

$$\operatorname{supp} u_j \subset \{|x - j^{-1}| \leq j^{-2}/3\} \equiv \Omega_j, \tag{2.2}$$

$$\hat{u}_j(0) > 1/(2\eta_j^{1/2}) \tag{2.3}$$

and

$$|\hat{u}_j(\xi)| \leq C_1 \eta_j, \tag{2.4}$$

where $C_1 > 0$ is a constant independent of j. Hereafter we denote constants by $C_k (k = 1, 2, \dots)$ and c. Note that

$$L_k u = \sum_{j=1}^{\infty} \eta_j^{-4} \exp(i\eta_j y)\{a(x, D_x) + g_k(x) b(x, \eta_j)\} u_j(x).$$

Setting $f_j(x) = \{a(x, D_x) + g_k(x) b(x, \eta_j)\} u_j(x)$ we require that the Fourier transform of $f_j(x)$ satisfies with $C_2 > 0$ independent of j

$$|\hat{f}_j(\xi)| \leq C_2 \eta_j^{-N_j} \quad \text{on} \quad \{|\xi| \leq \eta_j/2\}, \tag{2.5}$$

where $N_j \to \infty$ ($j \to \infty$). Furthermore, with $C_3 > 0$ and $c > 0$ independent of j we require

$$|\hat{f}_j(\xi)| \leq C_3 \langle \eta_j \rangle^c \quad \text{for all} \quad \xi \in \mathbf{R}^1. \tag{2.6}$$

Once we could obtain $u_j(x)$ (and $f_j(x)$) satisfying (2.2)–(2.6) $u(x, y)$ of the form (2.1) would be the desired singular solution. In fact, let $\varphi(x)$ be arbitrary C_0^∞ function such that $\varphi = 1$ in a neighborhood of the origin and $\hat{\varphi}(0) = 1$. The support of u_j shrinks to $x = 0$ when j tend to ∞ and the sum of finite terms of the right hand side of (2.1) belongs to C^∞. In considering the wave front set of $u(x, y)$ near the origin we may regard the Fourier transform of $\varphi(x)\varphi(y)u(x, y)$ as follows:

$$\mathcal{F}_{\substack{y\to\eta\\x\to\xi}}[\varphi(x)\varphi(y)u(x, y)](\xi, \eta) = \sum \eta_j^{-4}\widehat{\varphi}(\eta-\eta_j)\widehat{u}_j(\xi)$$
$$\equiv U(\xi, \eta).$$

If $j \neq j'$ we have

$$|\eta_j - \eta_{j'}| = |\eta_j^{1/2} - \eta_{j'}^{1/2}||\eta_j^{1/2} + \eta_{j'}^{1/2}|$$
$$\geq \max(\eta_j^{1/2}, \eta_{j'}^{1/2})$$

because $\eta_j^{1/2} - \eta_{j-1}^{1/2} > 1$. Write

$$U(0, \eta_{j'}) = \eta_{j'}^{-4}\widehat{\varphi}(0)\widehat{u}_{j'}(0) + \sum_{j\neq j'} \eta_j^{-4}\widehat{\varphi}(\eta_{j'} - \eta_j)\widehat{u}_j(0).$$

Since $\widehat{\varphi}(\eta) \in \mathcal{S}$ we have $|\widehat{\varphi}(\eta)| \leq C_N < \eta >^{-N}$ for any integer N and some constant C_N the second term of the right hand side is majorated by $\eta_{j'}^{-10}$ with a constant factor. By means of (2.3) we have $U(0, \eta_{j'}) \geq \eta_{j'}^{-9/2}/3$ ($j' \to \infty$) and hence we see $(0, 0; 0, 1) \in \mathrm{WF}\, u$. On the other hand, we have

$$V(\xi, \eta) \equiv \mathcal{F}_{\substack{y\to\eta\\x\to\xi}}[\varphi(x)\varphi(y)L_k u](\xi, \eta) = \sum \eta_j^{-4}\widehat{\varphi}(\eta-\eta_j)\widehat{f}_j(\xi)$$

For any fixed $\eta > 0$, the terms with j satisfying $|\eta^{1/2} - \eta_j^{1/2}| \geq 1$ are negligible because of (2.6). If there exists j satisfying $|\eta^{1/2} - \eta_j^{1/2}| < 1$ then it follows from (2.5) that $V(\xi, \eta) = 0(\eta^{-N_j})$ on $\{(\xi, \eta); |\xi| \leq \eta/3\}$. Consequently, we see $(0, 0; 0, 1) \notin \mathrm{WF}\, L_k u$.

Let us look for $u_j(x)$ and $f_j(x)$ satisfying (2.2)−(2.6). We shall consider the function $g_k(x)$ near $x = 1/j$. If $\alpha_j(x) = \int_0^1 \rho(1/j + (x - 1/j)\theta)d\theta$ with $\rho(t) = (-1/t^2)\cos \pi/t$ then $\sin \pi/x = \alpha_j(x)(x - 1/j)$ near $x = 1/j$ and hence

$$g_k(x) = \beta_j(x)\{(x - j^{-1})^{2k} + \gamma_j(x)\} \quad \text{near} \quad x = j^{-1},$$

where $\beta_j(x) = \alpha_j(x)^{2k} \exp(-1/x)$ and $\gamma_j(x) = \alpha_j(x)^{-2k} \exp\{1/x - 1/x^2\}$. Note that

(2.7) $$|\beta_j(x)| \geq \exp(-2j) \quad \text{in} \quad \Omega_j' \equiv \{|x - j^{-1}| \leq j^{-2}/2\}$$

and for any integer $q > 0$

(2.8) $$|\beta_j^{(q)}(x)| \leq C_q j^{4k+2q} \exp(-1/x)$$
$$\leq C_q'(\log \eta_j)^{2k+q} \exp(-1/x) \quad \text{in} \quad \Omega_j'$$

hold with constants C_q and C_q' independent of j. Here we used $\eta_j = \exp\{j^2/4m\}$. Similarly, we have

(2.9) $$|\gamma_j^{(q)}(x)| \leq C_q''(\log \eta_j)^{3q/2}\, \eta_j^{-2m} \quad \text{in} \quad \Omega_j'.$$

Note that

$$\begin{aligned} f_j(x) &= \{a(x, D_x) + g_k(x) b(x, \eta_j)\} u_j(x) \\ &= \beta_j(x) b(x, \eta_j) \eta_j^{-k} [\{(x - j^{-1}) \eta_j^{1/2}\}^{2k} + \gamma_j(x) \eta_j^k \\ &\quad + \beta_j(x)^{-1} b(x, \eta_j)^{-1} a(x, D_x) \eta_j^k] u_j(x). \\ &\equiv \beta_j(x) b(x, \eta_j) \eta_j^{-k} \tilde{f}_j(x). \end{aligned}$$

Since (2.2) is required we may assume that $\beta_j(x)$ and $\gamma_j(x)$ belong to C_0^∞ and satisfy (2.8)−(2.9) in $(-\infty, \infty)$, by multiplying the cut function in Ω_j' (equal to 1 on Ω_j). If $\hat{\beta}_j(\xi, \eta_j)$ denotes the Fourier transform of $\beta_j(x) b(x, \eta_j)$ then we have $|\hat{\beta}_j(\xi, \eta_j)| \leq C_q (\log \eta_j)^{q+2k} \eta_j^{2m} <\xi>^{-q}$ for any $q > 0$. Hence it suffices to require $\tilde{f}_j(x)$ satisfies (2.5) in $\{|\xi| \leq \eta_j\}$ and (2.6) instead of $f_j(x)$. In fact,

$$\eta_j^k \hat{f}_j(\xi) = \int \hat{\beta}_j(\zeta - \xi, \eta_j) \hat{\tilde{f}}_j(\zeta) \, d\zeta = \int_{|\zeta| \geq \eta_j} \cdot \, d\zeta + \int_{|\zeta| \leq \eta_j} \cdot \, d\zeta$$

If $|\xi| \leq \eta_j/2$ then the first term is estimated above from $C_{N_j} (\log \eta_j)^{N_j + 2k} \eta_j^{-N_j + 2m}$. The similar bound holds also for the second term because of (2.5) for $\tilde{f}_j(x)$.

Now we shall consider the equation

(2.10) $[\{(x - j^{-1}) \eta_j^{1/2}\}^{2k} + \gamma_j(x) \eta_j^k + \beta_j(x)^{-1} b(x, \eta_j)^{-1} a(x, D_x) \eta_j^k] u_j(x) = \tilde{f}_j(x).$

We shall omit the suffix j for a while (by fixing j). If we write

$$\gamma(x) + \beta(x)^{-1} b(x, \eta)^{-1} a(x, D_x) = \sum_{s=0}^{2\ell} D_x^s c_s(x, \eta)$$

by means of (2.7)−(2.9) and (A.3) we see that for any $\varepsilon > 0$

$$|D_x^q c_s(x, \eta)| \leq C_{q,\varepsilon} \eta^{-2m+\varepsilon} (\log \eta)^{3q/2}. \tag{2.11}$$

Note that the left hand side of (2.10) equals

$$\begin{aligned} &\iint \exp -i\{(x - j^{-1}) \eta^{1/2} t - t\tau\} \times \{\tau^{2k} + \alpha(t, \tau)\} \hat{v}(\tau) \, d\tau dt / 2\pi \\ &= \bar{\mathcal{F}}_{t \to (x - j^{-1}) \eta^{1/2}} [\{D_t^{2k} + \alpha(t, D_t)\} v(t)], \end{aligned}$$

where

$$\alpha(t, \tau) = \sum_{s=0}^{2\ell} (-t\eta^{-1/2})^s c_s(\eta^{-1/2} \tau + j^{-1}, \eta) \eta^{s+k}$$

and $\hat{v}(\tau) = u_j(\eta^{-1/2} \tau + j^{-1})$. It follows from (2.2) that

$$\operatorname{supp} \hat{v}(\tau) \subset \{|\tau| \leq \eta^{1/2} j^{-2}/3\} \equiv \omega_0. \tag{2.12}$$

We choose a positive $\delta_0 < 1/2$ such that $\eta^{\delta_0} < \eta^{1/2} j^{-2}/3$ with $\eta = \eta_j = \exp(j^2/4m)$. Since $1 \leq k \leq m - \ell - 1$ it follows from (2.11) that we have for any $\varepsilon > 0$ and any $0 < \delta < \delta_0$

$$|\partial_t^q \partial_\tau^{q'} \alpha(t, \tau)| \leq C_{\varepsilon, q, q'} \eta^{\varepsilon - (k+2) - \delta(q+q')} \quad \text{if} \quad |t| \leq 10\eta^{1/2}. \tag{2.13}$$

If $h_j(t)=h(t)$ is defined by

$$\{D_t^{2k}+\alpha(t,\ D_t)\}\,v(t)=h(t) \tag{2.14}$$

then the proof of Theorem B is reduced to find some $v(t),\ h(t)\in\mathcal{S}$ satisfying (2.12) and the following:

$$|v(0)|>1/2 \quad\text{and}\quad |v(t)|\leq Const.\eta^{2\delta+1/4} \tag{2.15}$$

$$|h(t)|\leq Const.\eta^{-N} \quad\text{on}\quad |t|\leq\eta^{1/2},$$

$$\|h\|_{L^\infty}\leq Const.\eta^c \quad\text{for a}\quad c>0. \tag{2.16}$$

(Here $N=N_j$ and $\eta=\eta_j$). In fact, we have $\hat{f}_j(\xi)=\eta_j^{-1/2}e^{-i\xi/j}\,h(\xi/\eta_j^{1/2})$.

Let $0\leq\theta(t)\leq 1$ be a $C_0^\infty((-1,\ 1))$ function such that $\theta=1$ in $|t|\leq 1/2$. Set $\chi_0(t)=\theta(t/5\eta^{1/2})$ and $\chi_1(t)=\theta(t/10\eta^{1/2})$. We are looking for a solution to

$$\{D_t^{2k}+\chi_0(t)\,\alpha(t,\ D_t)\,\chi_1(t)\}\,w(t)=0 \tag{2.17}$$

First we set $w_0(t)\equiv 1$. If $w(t)=w_0(t)+w_1(t)$ then

$$\{D_t^{2k}+A(t,\ D_t)\}\,w_1(t)=-\chi_0\alpha\chi_1\ (\equiv g(t)) \tag{2.18}$$

where $A(t,\ D_t)=\chi_0(t)\,\alpha(t,\ D_t)\chi_1(t)$. Consider this equation in the interval $I=(-10\eta^{1/2},\ 10\eta^{1/2})$ with the Dirichlet boundary condition

$$D_t^q w_1(\pm 10\eta^{1/2})=0,\ q=0,\ 1,\ \ldots,\ k-1. \tag{2.19}$$

It follows from (2.13) that $|(Au,\ u)|\leq C\eta^{\varepsilon-k-2}\|u\|^2$ and $\|D_t^k u\|^2\geq C\eta^{-k}\|u\|^2$. If G_η denotes the Green operator for this boundary value problem then

$$\|G_\eta f\|_{L^2(I)}\leq C\eta^k\|f\|_{L^2(I)} \quad\text{for}\quad f\in L^2\ (I). \tag{2.20}$$

Since $\|\chi_1\|_{L^2}\doteq O(\eta^{1/4})$ we have $\|g\|_{L^2}=O(\eta^{1/4-k-2+\varepsilon})$ by means of (2.13). It follows from (2.18) and (2.20) that $\|w_1\|_{L^2(I)}=O(\eta^{\varepsilon-7/4})$. By (2.18) and (2.19) we have

$$\|D_t^k w_1\|_{L^2(I)}^2+(A(t,\ D_t)w_1,\ w_1)=(g,\ w_1),$$

so that $\|D_t^k w_1\|_{L^2(I)}=O(\eta^{\varepsilon-7/4})$. If we extend w_1 outside of I by $w_1=0$ then $w_1\in C_0^k$ and by the interpolation $\|D_t^p w_1\|_{L^2(I)}=O(\eta^{\varepsilon-7/4})$ for $p=0,\ \ldots,\ k$. In case of $k\geq 2$, it follows from the Sobolev lemma that $\|w_1\|_{L^\infty}=O(\eta^{\varepsilon-7/4})$ and $w(0)=1+O(\eta^{-1})$. When $k=1$ it follows from (2.18) again that $\|D_t^2 w_1\|_{L^2(I)}=O(\eta^{1/4-3+\varepsilon})$ and hence $\|\chi_1 w_1\|_{L^\infty}=O(\eta^{\varepsilon-7/4})$. After all we see $w(0)=1+O(\eta^{-1})$. In view of (2.17) we have

$$\{D_t^{2k}+\chi_0(t)\,\alpha(t,\ D_t)\}\,\chi_1(t)\,w(t)=D_t^{2k}(\chi_1-1)w\ (\equiv F_1(t)).$$

Note that $F_1(t)=0$ for $|t|\leq 3\eta^{1/2}$. We set $\psi_1(D_t)=\theta(D_t\eta^{-\delta})$. Then

$$\{D_t^{2k}+\chi_0(t)\,\alpha(t,\ D_t)\}\,\psi_1(D_t)\chi_1(t)\,w(t)=\psi_1F_1-[\chi_0\alpha,\ \psi_1]\chi_1 w. \tag{2.21}$$

Here ψ_1F_1 satisfies $\psi_1F_1=O(\eta^{-\infty})$ on $|t|\leq\eta^{1/2}$. Set $v_1(t)=\psi_1(D_t)\chi_1(t)\,w(t)$.

Then we have

(2.22) $$\|D_t^p v_1\|_{L^2} = O(\eta^{\delta p+1/4}),\ p=0,\ 1,\ 2,\ \dots,$$

and $v_1(t) = O(\eta^{2\delta+1/4})$. Furthermore, $v_1(0) = 1 + O(\eta^{-1/2})$. In fact,

$$\begin{aligned}|(1-\psi_1(D_t))\chi_1(t)w(t)| &= \left|\iint e^{i(t-s)\tau}\tau^{-2}(1-\psi(\tau))(-D_s)^2\{\chi_1(s)w(s)\}\,ds\,d\tau\right| \\ &\le Const. \int_{\eta^\delta/2}^{\infty}\tau^{-2}d\tau\left\{\int |D_s^2\chi_1|ds + \int |D_s^2(\chi_1 w_1)|ds\right\} \\ &= Const.\,\eta^{-\delta-1/2}.\end{aligned}$$

If we set $g_2 \equiv [\chi_0\alpha,\ \psi_1]\chi_1 w$ then $\|g_2\|_{L^2} = O(\eta^{\varepsilon-2\delta-k-7/4})$. Let w_2 be $G_\eta g_2$, that is, a solution to

$$\{D_t^{2k}+A\}w_2 = g_2 \quad \text{in} \quad I$$

with the Dirichlet boundary condition. By the similar way as for w_1 we have $\|w_2\|_{L^2(I)} = O(\eta^{\varepsilon-2\delta-7/4})$. It follows from (2.21) that

$$\{D_t^{2k}+\chi_0(t)\,\alpha(t,\ D_t)\}v_1 = \psi_1 F_1 - g_2 = \psi_1 F_1 - \{D_t^{2k}+\chi_0(t)\,\alpha(t,\ D_t)\}\chi_1 w_2 + F_2,$$

where $F_2 = D_t^{2k}(\chi_1-1)w_2$. Set $\psi_q(D_t) = \theta(D_t\eta^{-\delta}/2^{q-1})$ for $q=2,\ 3,\ \dots$ If $v_2 = \psi_2(D_t)\chi_1(t)w_2(t)$ then we have

(2.23) $$\|D_t\, v_2(t)\|_{L^2} = O(\eta^{\delta(p-2)+\varepsilon-7/4}),\ p=0,\ 1,\ \dots.$$

Furthermore we have

$$\{D_t^{2k}+\chi_0(t)\,\alpha(t, D_t)\}(v_1+v_2) = -[\chi_0\alpha,\ \psi_2]\chi_1 w_2 + \psi_1 F_1 + \psi_2 F_2 + (\psi_2-1)\chi_0\alpha v_1.$$

Since $\psi_2 \supset\supset \psi_1$ the last term of the right hand side equals $O(\eta^{-\infty})$. If we set $g_3 = [\chi_0\alpha,\ \psi_2]\chi_1 w_2$ then $\|g_3\| = O(\eta^{2\varepsilon-4\delta-k-2-7/4})$. Set $w_3 = G_\eta g_3$ and $v_3 = \psi_3(D_t)\chi_1 w_3$. Repeat this procedure N_j times with $2^{N_j} < \eta_j^{\delta_0-\delta} \le 2^{N_j+1}$. Setting $v = \sum_{q=1}^{N_j} v_q$ we have

(2.24) $$\begin{aligned}&\{D_t^{2k}+\chi_0(t)\,\alpha(t,\ D_t)\}v(t) \\ &= -[\chi_0\alpha,\ \psi_{N_j}]\chi_1 w_{N_j} + \sum_{q=1}^{N_j}\psi_q F_q + \sum_{q=1}^{N_j-1}(\psi_{q+1}-1)\chi_0\alpha v_q \\ &\equiv -g_{N_j}(t) + \widetilde{h}_1(t) + \widetilde{h}_2(t).\end{aligned}$$

By checking the preceding argument carefully it is not difficult to see that there exists a $C_0 > 0$ independent of j such that

(2.25) $$\|D_t^p v_q\|_{L^2} \le C_0^q\,\eta^{p\delta+1/4+(\varepsilon-2\delta-2)(q-1)}$$
$$\text{for} \quad p=0,\ 1,\ \dots,\ 2k+2 \quad \text{and} \quad q=1,\ 2,\ \dots.$$

Since $C_0 \ll \eta = \eta_j$ for a sufficiently large j we see (2.15) by means of the Sobolev lemma. By the similar way as for (2.25) we have

$$\|g_{N_j}\|_{L^\infty} \leq C\|(1+D_t^2)g_{N_j}\|_{L^2} \leq CC_1^{N_j}\eta^{2\delta+1/4-k+(\varepsilon-2\delta-2)N_j}$$

for C, $C_1>0$ independent of j. Similarly, we see that

$$|\psi_q F_q(t)| \leq C_2^q \eta^{-N_j+(\varepsilon-2\delta-2)q} \quad \text{on} \quad |t| \leq \eta^{1/2}$$
$$\|(\psi_{q+1}-1)\chi_0 \alpha v_q\|_{L^\infty} \leq C_3^q \eta^{-N_j+(\varepsilon-2\delta-2)q}$$

for C_2, $C_3>0$ independent of j. Since $h=-g_{N_j}+\tilde{h}_1+\tilde{h}_2$ on $|t| \leq \eta^{1/2}$, from the above three estimates we obtain $|h(t)| \leq \eta^{-N_j}$ on $|t| \leq \eta^{1/2}$ if j is large enough. In view of (2.14), it follows from (2.25) that $\|h\|_{L^\infty} \leq Const.\,\eta^{(2k+2)\delta+1/4}$ because $\|(1+D_t^2)v\|_{L^2}=O(\eta^{2\delta+1})$ and $(1+D_t^2)\,\alpha(t, D_t)\,(1+D_t^2)^{-1}$ is L^2 bounded. Hence (2.16) is fulfilled. Now the proof of Theorem B is completed.

Remark. In the same way it is possible to prove the non-micro-hypoellipticity of the operator $D_x+ig_k(x)D_y^m$ with $m \geq 2k+2$ (cf. (0.3) of [5]).

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