Groupoid Dynamical Systems and Crossed Product, II—The Case of C*-Systems

By

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Abstract

By analogy with C*-dynamical system, we define a C*-groupoid dynamical system $(A, \Gamma, \rho)$ where $A$ is a C*-algebra, $\Gamma$ is a locally compact groupoid, and $\rho: \Gamma \to \text{Aut}(A)$ is a continuous groupoid homomorphism. The groupoid crossed product $A \times_\rho \Gamma$ is defined and is shown to have similar properties as the case of a group action. As a special case of this situation, if $\rho$ is a continuous homomorphism from $\Gamma$ to a locally compact group $G$, we obtain groupoid dynamical system $(C_0(G), \Gamma, \rho)$. In this case, there exists a co-action $\beta$ of $G$ on $C^*(\Gamma)$ and the groupoid crossed product $C_0(G) \times_\rho \Gamma$ is isomorphic to the co-crossed product $C^*(\Gamma) \rtimes G$ of $C^*(\Gamma)$ by $G$. The results in this paper is obtained by the analogy with our previous results for the case of $W^*$-systems.

§1. Introduction

In our previous paper [8], we defined a $W^*$-groupoid dynamical system and its groupoid crossed product based on the analogy with the case of a group action together with the several basic ideas. In this paper, we shall give the $C^*$-algebraic framework of groupoid dynamical system and its groupoid crossed product. Because we consider only the regular representation based on the canonical Hilbert $\Gamma$-bundle out of the transverse function (see [2]), all the crossed products are in the reduced category. The whole discussion is parallel to those of $W^*$-algebraic case.

In Section 2, we define $C^*$-groupoid dynamical system and its groupoid crossed product. In this section, we also describe the general properties of the groupoid crossed product. In Section 3, we shall discuss the $C^*$-groupoid dynamical system $(C_0(G), \Gamma, \rho)$ defined by a continuous groupoid homomorphism $\rho: \Gamma \to G$ for an auxiliary locally compact (not necessarily abelian) group.


G. For the examples, see [8], Section 7.

Throughout this paper, we use a non-commutative integration theory in terms of a locally compact topological groupoid which admits a faithful transverse function $\nu = \{ \nu^x \}_{x \in \Gamma}$ (see [2], [4], [10]).

\section{C*-Groupoid Dynamical Systems}

We shall start with the definition of a C*-groupoid dynamical system.

\textbf{Definition 2.1.} The triplet $(A, \Gamma, \rho)$ is called a C*-groupoid dynamical system (or C*-groupoid system, for short) if $A$ is a C*-algebra, $\Gamma$ is a locally compact groupoid with a faithful transverse function $\nu = \{ \nu^x \}_{x \in \Gamma}$, and $\rho: \Gamma \to \text{Aut}(A)$ is a continuous homomorphism.

The associated crossed product is defined as the completion of the set $C_c(\Gamma, A)$ of all $A$-valued continuous functions over $\Gamma$ with compact support by the C*-norm defined below. The set $C_c(\Gamma, A)$ is a *-algebra by:

\begin{equation}
(f_1 \ast f_2)(\tau) = \int_{\Gamma_x} \rho_\gamma(f_1(\gamma^{-1} \tau)) f_2(\gamma) \, d\nu^x(\gamma), \quad x = r(\tau),
\end{equation}

\begin{equation}
f^\#(\tau) = \rho_\gamma(f(\gamma^{-1})^*),
\end{equation}

where $f_1, f_2, f \in C_c(\Gamma, A)$. The C*-norm on $C_c(\Gamma, A)$ is defined by

\begin{equation}
\| f \| = \sup_{\tau \in \Gamma_{x(r)}} \| \pi_x(f) \|, \quad f \in C_c(\Gamma, A),
\end{equation}

where $H$ is a faithful representation Hilbert space of $A$. In view of (2.4), $\| \pi_x(f) \|$, $f \in C_c(\Gamma, A)$ is independent of the choice of representation Hilbert space $H$ so that the norm (2.3) is independent of the choice of representation Hilbert space $H$.

\textbf{Definition 2.2.} $A \times_r \Gamma$ or $C^*(A, \Gamma, \rho)$ denotes the C*-algebra obtained by the completion of $C_c(\Gamma, A)$ by the C*-norm given by (2.3).

\textbf{Example 2.3.} The definition of a groupoid algebra given by A. Connes is as follows. The set $C_c(\Gamma)$ is a *-algebra by

\begin{equation}
(f_1 \ast f_2)(\tau) = \int_{\Gamma_x} f_1(\gamma) f_2(\gamma^{-1} \tau) \, d\nu^x(\gamma), \quad x = r(\tau),
\end{equation}

\begin{equation}
f^\#(\tau) = \overline{f(\gamma^{-1})},
\end{equation}

where $H$ is a faithful representation Hilbert space of $A$.
where \( f_1, f_2, f \in C_c(\Gamma) \). The C*-algebra \( C^*(\Gamma) \) is defined by the completion of \( C_c(\Gamma) \) with respect to the norm on \( C_c(\Gamma) \) defined by

\[
\| f \| = \sup_{s \in F} \| \pi_s(f) \|, \quad f \in C_c(\Gamma),
\]

(2.7)

\[
[\pi_s(f) \xi](\tau) = \int_{\Gamma^s} f(\tau^{-1} \tau') \xi(\tau') d\nu_s(\tau'), \quad \tau \in \Gamma^s, \quad \xi \in L^2(\Gamma^s, \nu^s).
\]

(2.8)

Now, we define for \( A = C \) bijection \( R: C_c(\Gamma, A) \to C_c(\Gamma) \) by \( [Rf](\tau) = f(\tau^{-1}) \). Then, \( R \) is a *-algebra isomorphism between \( C_c(\Gamma, A) \) with \( A = C \) and \( C_c(\Gamma) \) preserving C*-norm (cf. (2.3), (2.4) and (2.7), (2.8)). So, our definition with \( A = C \) actually gives the usual Connes algebra \( C^*(\Gamma) \).

**Example 2.4.** Let \( (A, G, \alpha) \) be a C*-dynamical system. The crossed product \( A \times_a G \) associated with the C*-dynamical system \( (A, G, \alpha) \) is defined as the C*-completion of \( L^1(G, A) \) with the *-algebra operations defined by

\[
(f_1 \star f_2)(g) = \int G f_1(h) \alpha_g(f_2(h^{-1} g)) dh,
\]

(2.9)

\[
f^\alpha(g) = \Delta_g(g)^{-1} \alpha_g(f(g^{-1}))^*,
\]

(2.10)

for \( f_1, f_2, f \in L^1(G, A) \), and with the C*-norm defined through the *-representation

\[
[\pi(f) \xi](g) = \int G \alpha_g^{-1} f(h) \xi(h^{-1} g) dh,
\]

(2.11)

where \( f \in L^1(G, A) \), \( \xi \in L^2(G) \otimes H \) and \( H \) is any faithful representation Hilbert space of \( A \). It is known that due to inequality \( \| f \| \leq \| f \|_{L^1} \) (which follows from (2.11)), the C*-completion of \( C_c(G, A) \) gives \( A \times_a G \). For the purpose of comparison with our formulation, we define \( A \times_a G \) in a different manner. We define the *-algebra operations in \( C_c(G, A) \) by

\[
(f_1 \star f_2)(g) = \int G \alpha_g(f_1(h^{-1} g)) f_2(h) dh,
\]

(2.12)

\[
f^\alpha(g) = \alpha_g(f(g^{-1}))^*,
\]

(2.13)

for \( f_1, f_2, f \in C_c(G, A) \) and the C*-norm through the *-representation

\[
[\pi(f) \xi](g) = \int G \alpha_g(f(h^{-1} g)) \xi(h) dh,
\]

(2.14)

where \( f \in C_c(G, A) \), \( \xi \in L^2(G) \otimes H \). Then we obtain \( A \times_a G \) by taking the C*-completion of \( C_c(G, A) \). In fact, the mapping \( R \) defined by
is a $\ast$-isomorphism of $L^1(G, A)$ ($\ast$ and $\#$ given by $(2.9)$ and $(2.10)$) onto $L^1_{\text{sym}}(G, A)$ ($\ast$ and $\#$ given by $(2.12)$ and $(2.13)$), which is the $L^1$-space with respect to the symmetric Haar measure $d\mu(g) = d\phi(g)^{-1/2} dg$. The unitary mapping $\tilde{R} : L^2(G) \otimes H \to L^2(G) \otimes H$ defined by

$$[\tilde{R}\xi](g) = d\phi(g)^{-1/2} \xi(g^{-1}), \quad \xi \in L^2(G) \otimes H$$

implements this isomorphism and intertwines $\pi(f)$ ($\pi$ given by $(2.11)$) with $\{\pi_x \pi(R(f)) (\pi$ given by $(2.14)$).}

\textbf{Proposition 2.5.} (1) Let $f \in C_c(\Gamma, A)$. Then the family of operators $\{\pi_x(f)\}_{x \in \Gamma}^{\infty}$ is covariant in the sense that

$$((Ad_{\upsilon(\gamma)} \otimes \rho_\gamma)(\pi_x(f))) = \pi_x(f), \quad \gamma \in \Gamma_{\mathbb{R}},$$

where $Ad_{\upsilon(\gamma)} = U(\gamma) \cdot U(\gamma)^*$ and $[U(\gamma)](\gamma) = \xi(\gamma^{-1} \gamma), \xi \in \Gamma^\gamma, \xi \in L^2(\Gamma^\gamma, \nu^\gamma)$.\[\]

(2) If $\rho, \sigma : \Gamma \to \text{Aut}(A)$ are cohomologous in the sense that there exists a continuous mapping $\tau : \Gamma^{(0)} \to \text{Aut}(A)$ such that $\rho_{\gamma} = \tau_{\gamma} \circ \sigma_\gamma \circ \tau_{\gamma}^{-1}$. Then $A \times_{\rho} \Gamma \cong A \times_{\sigma} \Gamma$.

(3) If $\rho, \sigma : \Gamma \to \text{Aut}(A)$ are one-cocycle equivalent in the sense that there exists a unitary valued mapping $u : \Gamma \to M(A)$ such that $\tau \mapsto u_{\gamma} a$ and $\tau \mapsto au_{\gamma}$ are continuous for all $a \in A$ and

$$\rho_{\gamma}(a) = u_{\gamma} a u_{\gamma}^*, \quad a \in A,$$

$$u_{\gamma_1 \gamma_2} = u_{\gamma_1} u_{\gamma_2}, \quad s(\gamma_1) = r(\gamma_2),$$

then $A \times_{\rho} \Gamma \cong A \times_{\sigma} \Gamma$.

(4) If $\Gamma$ is the graph groupoid of topological transformation group $(X, G, \alpha)$, then $A \times_{\rho} \Gamma$ is isomorphic to a crossed product of $C_0(X) \otimes A$ by $G$ with the action

$$\rho_{\gamma}(f)(x) = \rho_{\gamma(x, g)}(f(\alpha_g^{-1}(x))), \quad f \in C_0(X) \otimes A, \quad g \in G.$$ (Note that $C_0(X) = C(X)$ if $X$ is compact.)

\textbf{Proof.} (1) Let $\gamma \in \Gamma_{\mathbb{R}}$ and $\xi \in L^2(\Gamma^\gamma, \nu^\gamma) \otimes H$. Then

$$[(Ad_{\upsilon(\gamma)} \otimes \rho_\gamma)(\pi_x(f)) \xi](\gamma) = \int_{\Gamma_{\mathbb{R}}} \rho_{\gamma} \circ \rho_\gamma (f(\gamma^{-1} (\gamma^{-1} \gamma))) \xi(\gamma \gamma) \nu^\gamma(\gamma)$$

$$= \int_{\Gamma_{\mathbb{R}}} \rho_{\gamma} \rho_\gamma (f(\gamma^{-1} (\gamma^{-1} \gamma))) \xi(\gamma \gamma) \nu^\gamma(\gamma)$$

$$= \int_{\Gamma_{\mathbb{R}}} \rho_{\gamma} \rho_\gamma (f(\gamma^{-1} (\gamma^{-1} \gamma))) \xi(\gamma \gamma) \nu^\gamma(\gamma)$$
This shows (2.17).

(2) We define a mapping $\Phi$ by

\[(2.22) \quad \Phi[f](r) = \tau_{r^{-1}}(f(r)), \quad f \in C_c(\Gamma, A).\]

Then $\Phi$ gives a bijective mapping of $C_c(\Gamma, A)$ onto itself and the following relations hold:

\[(2.23) \quad \Phi[f_1] \ast \sigma \Phi[f_2] = \Phi[f_1 \ast_r f_2], \quad f_1, f_2 \in C_c(\Gamma, A),\]

\[(2.24) \quad \Phi[f]^{(\ast, \sigma)} = \Phi[f^{(\ast, \rho)}], \quad f \in C_c(\Gamma, A),\]

where $\ast_r, (\ast, \sigma)$ and $\ast_r, (\ast, \rho)$ are convolution and involution of $C_c(\Gamma, A)$ with respect to the actions $\sigma$ and $\rho$ respectively. Moreover,

\[(2.25) \quad \pi_x^\sigma(\Phi[f]) \xi(r) = \int_{\Gamma^x} \sigma \tau_{r^{-1}}(f(\tilde{r}^{-1} r)) \xi(\tilde{r}) \, d\nu(\tilde{r}),\]

where $f \in C_c(\Gamma, A), \xi \in L^2(\Gamma^x, \nu^x) \otimes H$. Hence we obtain $||\pi_x^\sigma(\Phi[f])|| = ||\pi_x^\rho(f)||$ for any $x \in \Gamma^{(0)}$ where $\pi_x^\sigma$ and $\pi_x^\rho$ are representations relevant for $\sigma$ and $\rho$, respectively. This implies the desired isomorphism.

(3) We define a mapping $\Psi$ by

\[(2.26) \quad \Psi[f](r) = u_\sigma f(r), \quad f \in C_c(\Gamma, A).\]

This gives a bijective mapping of $C_c(\Gamma, A)$ onto itself and the following relations hold:

\[(2.27) \quad \Psi[f_1] \ast_r \Psi[f_2] = \Psi[f_1 \ast_r f_2], \quad f_1, f_2 \in C_c(\Gamma, A),\]

\[(2.28) \quad \Psi[f]^{(\ast, \rho)} = \Psi[f^{(\ast, \sigma)}], \quad f \in C_c(\Gamma, A),\]

where we use (2.18), (2.19) and $u_\sigma = 1$, which follows from (2.19). We define a family of unitary operators $U = \{U_x\}_{x \in \Gamma^{(0)}}$ by

\[(2.29) \quad [U_x \xi](r) = u_\sigma \xi(r), \quad \xi \in L^2(\Gamma^x, \nu^x) \otimes H.\]

Then, we obtain

\[(2.30) \quad \pi_x^\rho(\Psi[f]) \xi = U_x \pi_x^\sigma(f) U_x^* \xi, \quad f \in C_c(\Gamma, A), \quad \xi \in L^2(\Gamma^x, \nu^x) \otimes H,\]
for all $x \in \Gamma^\alpha$. Hence $||\pi_x^\alpha(f)|| = ||\pi_x^\alpha(f)||$ and we obtain the desired isomorphism.

(4) For $f_1, f_2, f \in C_c(\Gamma, A),$

$$\begin{align*}
(f_1 \ast_{\rho} f_2)(x, g) &= \int_G \rho_{(\alpha, h)}(f_1(\alpha_{h^{-1}}(x), h^{-1} g)) f_2(x, h) \, dh \\
&= \int_G [\rho_{h}[f_1](h^{-1} g)](x) f_2(x, h) \, dh \\
&= [(f_1 \ast_{\rho} f_2)(g)](x)
\end{align*}$$

where $\rho$ and $\rho$ in the subscripts for $*$ and in the subscripts for $\ast$ indicate the convolution and involution in $A \times_p \Gamma$ and in $C_0(X, A) \times \rho G$, respectively. Furthermore,

$$\begin{align*}
\pi_x^\alpha(f) \xi(x, g) &= \rho_{(\alpha, \xi)}(f(\alpha_{g^{-1}}(x), g^{-1} *)) \\
&= [\rho_{h}[f](g^{-1})]^{*}(x) \\
&= [\pi_x^\alpha(f) \xi](g)](x),
\end{align*}$$

for $\xi \in L^2(\Gamma, H) = L^2(G, L^2(X) \otimes H)$. In view of Example 2.4, these formulas agree with the defining relations (2.12), (2.13), (2.14) of $C^\ast$-crossed product $C_0(X, A) \times \rho G$ through the action (2.20). Hence we obtain the assertion by the density of $C_c(X, A)$ in $C_0(X, A)$. Q.E.D.

Remark 2.6. In the situation of (4), if $\rho: \Gamma \to \text{Aut}(A)$ is of $G$-split type (see Remark 4.8 of [8]), then the action (2.20) of $G$ on $C_0(X, A) \times \rho G$ is of product type.

Now remember the definition of a locally compact transformation groupoid which is introduced in analogy with the skew product, see [8], §5.

Lemma 2.7. Let $(\Omega, \Gamma, \rho)$ be a locally compact transformation groupoid and $\bar{\Gamma} = \Omega \times \rho \Gamma$ be the associated graph. Then,

$$C^\ast(\bar{\Gamma}) = C_c(\Omega) \times \rho \Gamma.$$  

(Note that $C_c(\Omega) = C(\Omega)$ if $\Omega$ is compact.)

Proof. By definition, $C_c(\Omega) \times \rho \Gamma$ is defined by the $C^\ast$-completion of the $*$-algebra $C_c(\Gamma, C_c(\Omega))$. By definition of the relevant $C^\ast$-norm, we may assume
that $C^0(\mathcal{G}) \times_\gamma \Gamma$ is generated by $C^0(\mathcal{G} \times \Gamma) = C^0(\Gamma, C^0(\mathcal{G})) \subset C^0(\Gamma, C^0(\mathcal{G}))$. In view of the definition of groupoid crossed product after Definition 2.1, the $*$-algebraic structure and the $C^*$-norm on $C^0(\mathcal{G} \times \Gamma)$ is

$$
(2.35) \quad (f_1 \ast f_2)(\omega, \gamma) = \int_{\mathcal{G} \times \Gamma} f_1(\rho_{\gamma^{-1}}(\omega), \bar{\gamma}^{-1} \gamma) f_2(\omega, \bar{\gamma}) \, d\nu^\gamma(\bar{\gamma}), \quad x = r(x),
$$

$$
(2.36) \quad f^*(\omega, \gamma) = f(\rho_{\gamma^{-1}}(\omega), \bar{\gamma}^{-1} \gamma),
$$

$$
(2.37) \quad ||f|| = \sup_{x \in \mathcal{G} \times \Gamma} ||\pi_x(f)||, \quad f \in C^0(\mathcal{G} \times \Gamma),
$$

$$
(2.38) \quad [\pi_x(f) \xi](\omega, \gamma) = \int_{\mathcal{G} \times \Gamma} f(\rho_{\gamma^{-1}}(\omega), \bar{\gamma}^{-1} \gamma) \xi(\omega, \bar{\gamma}) \, d\nu^\gamma(\bar{\gamma}),
$$

where $\xi \in L^2(\mathcal{G}) \otimes L^2(\Gamma^\gamma, \nu^\gamma)$ with respect to a suitable measure on $\mathcal{G}$.

$(C^0(\mathcal{G})$ is a concrete $C^*$-algebra on $L^2(\mathcal{G})$.) In view of (2.38), $||\pi_x(f)|| = \sup_{x \in \mathcal{G} \times \Gamma} ||\pi_{(x, z)}(f)||$, where

$$
(2.39) \quad [\pi_{(x, z)}(f) \xi](\gamma) = \int_{\mathcal{G} \times \Gamma} f(\rho_{\gamma^{-1}}(\omega), \bar{\gamma}^{-1} \gamma) \xi(\omega, \bar{\gamma}) \, d\nu^\gamma(\bar{\gamma}),
$$

where $\xi \in L^2(\Gamma^\gamma, \nu^\gamma)$. Hence $||f|| = \sup_{(x, z) \in \mathcal{G} \times \Gamma} ||\pi_{(x, z)}(f)||$. These expressions agree with the definition of the $*$-algebraic structure and the $C^*$-norm of $C^*(\mathcal{F})$, $\mathcal{F} = \mathcal{G} \times_\gamma \Gamma$.

Q.E.D.

§3. Groupoid Crossed Product and Co-action

In this section, we shall discuss the co-action on a groupoid algebra by a locally compact group arising from a groupoid homomorphism. First, we recall the definition of co-action and the associated crossed product in the $C^*$-algebraic framework (see [3], [5], [6], [7], [9]). Let $G$ be a locally compact group. The Kac-Takesaki operator $W$ is a unitary operator on $L^2(G \times G)$ defined by

$$
(3.1) \quad [W \xi](g, h) = \xi(g, gh), \quad \xi \in L^2(G \times G).
$$

Then we define an isomorphism $\delta_\gamma: W^\gamma_*(G) \rightarrow W^\gamma_*(G) \otimes W^\gamma_*(G)$ by

$$
(3.2) \quad \delta_\gamma(x) = W^*(x \otimes 1) W, \quad x \in W^\gamma_*(G)
$$

where $W^\gamma_*(G)$ is the $W^*$-algebra generated by the left regular representation $\lambda(g)$ of $g \in G$.

Definition 3.1. The co-action of $G$ on a $C^*$-algebra $A$ is defined as the
isomorphism

\[(3.3) \quad \delta: A \rightarrow \tilde{M}_L(A \otimes C^*_\phi(G))\]
satisfying

\[(3.4) \quad (\delta \otimes 1) \circ \delta = (1 \otimes \delta_c) \circ \delta\]

where

\[(3.5) \quad \tilde{M}_L(A \otimes_{\min} B) = \{a \in M(A \otimes_{\min} B): a(1 \otimes b) + (1 \otimes c) a \in A \otimes_{\min} B, \]
\[L_{\phi}(a) \in A \text{ for } b, c \in B, \phi \in B^*\}

and \(L_{\phi}\) denotes the left slice mapping by \(\phi\). The co-action \(\delta\) is said to be non-degenerate if \(\{L_{\phi}(\delta(a)): a \in A, \phi \in A(G)\}\) generates \(A\), where \(A(G)\) is the Fourier algebra of \(G\) (see Theorem 5 of [6]). If this is the case, the co-crossed product \(C^*\)-algebra \(A_{\omega G}\) by the co-action \(\delta\) is defined as the \(C^*\)-algebra generated by \((1 \otimes f) \delta(a)\) on \(H \otimes L^2(G), a \in A, f \in C_0(G)\), where \(H\) is a faithful representation Hilbert space of \(A\).

Now, let \(\Gamma\) be a locally compact topological groupoid with a faithful transverse function \(\nu = \{\nu^\gamma\}_{\gamma \in \Gamma_0}\) and \(G\) be a locally compact group.

**Theorem 3.2.** Let \(\rho: \Gamma \rightarrow G\) be a continuous homomorphism. Then there exists a continuous co-action \(\rho\) of \(G\) on \(C^*(\Gamma)\) such that the associated co-crossed product \(C^*(\Gamma) \ast_\rho G\) is isomorphic to the groupoid crossed product \(C_0(G) \times_\gamma \Gamma\).

**Proof.** By choosing a suitable faithful Borel measure \(\mu\) on \(\Gamma^{(0)}\), \(C^*(\Gamma)\) is a concrete \(C^*\)-algebra acting on a Hilbert space \(L^2(\Gamma, (\mu \circ \nu))\). The action of \(f \in C_\xi(\Gamma)\) on \(L^2(\Gamma, (\mu \circ \nu))\) is

\[(3.6) \quad [\pi(f) \xi](\gamma) = \int_{\Gamma^{(0)}} f(\gamma^{-1} \gamma) \xi(\gamma) d\nu^\gamma(\gamma), \quad x = r(\gamma),\]

where \(\xi \in L^2(\Gamma, (\mu \circ \nu))\). Now, we define a mapping \(\beta: C_\xi(\Gamma) \rightarrow B(L^2(\Gamma, (\mu \circ \nu)) \otimes L^2(G))\) by

\[(3.7) \quad [\beta(f) \xi](\gamma, g) = \int_{\Gamma^{(0)}} f(\gamma^{-1} \gamma) \xi(\gamma, \rho(\gamma^{-1} \gamma) g) d\nu^\gamma(\gamma), \quad x = r(\gamma),\]

where \(f \in C_\xi(\Gamma), \xi \in L^2(\Gamma, (\mu \circ \nu)) \otimes L^2(G)\). Then \(\beta(f) = W_\rho(\pi(f) \otimes 1) W_\rho^*\), where \(W_\rho\) is a unitary operator (analogue of Kac-Takesaki operator) defined by

\[(3.8) \quad [W_\rho \xi](\gamma, g) = \xi(\gamma, \rho(\gamma) g), \quad \xi \in L^2(\Gamma, (\mu \circ \nu)) \otimes L^2(G)\].

Hence \(\beta\) extends to a \(*\)-isomorphism \(C^*(\Gamma) \rightarrow B(L^2(\Gamma, (\mu \circ \nu)) \otimes L^2(G))\) which is
also denoted by $\beta$. To see that equality (3.4) holds, let $W^{(j)}_\rho, j=1, 2$ be unitary operators on $L^2(\Gamma, (\mu\circ\nu)) \otimes L^2(G \times G)$ defined by

\begin{align}
W^{(1)}_\rho \xi (r, g, h) &= \xi (r, \rho(r)g, h), \\
W^{(2)}_\rho \xi (r, g, h) &= \xi (r, g, \rho(r)h),
\end{align}

$\xi \in L^2(\Gamma, (\mu\circ\nu)) \otimes L^2(G \times G)$. Then $(\delta \otimes 1) \circ \beta(f) = W^{(1)}_\rho W^{(2)}_\rho (\pi(f) \otimes 1 \otimes 1)$ $W^{(2)*}_\rho$, $(1 \otimes \delta) \circ \beta(f) = (1 \otimes W^{*}) W^{(1)*}_\rho (\pi(f) \otimes 1 \otimes 1)$ $W^{(1)*}_\rho (1 \otimes W)$. Equality (3.4) for $\delta = \beta$ is obtained by the direct computation. Let $f \in C_c(\Gamma)$ and $\phi \in C_c(G)$. Then

\begin{align}
[\beta(f) (1 \otimes \lambda(\phi)) \xi] (r, g) &= \int_{r^*} f(\tilde{r}^{-1} r) \left[ \left((1 \otimes \lambda(\phi)) \xi \right)(\tilde{r}, \rho(\tilde{r}^{-1} r)g) \right] d\nu^\rho(\tilde{r}) \\
&= \int_{r^*} \left[ \int_G f(\tilde{r}^{-1} r) \phi(h) \xi(\tilde{r}, h^{-1} \rho(\tilde{r}^{-1} r)g) \right] d\nu(\tilde{r}) \\
&= \int_{r^*} \left[ \int_G f(\tilde{r}^{-1} r) \phi(\tilde{r}^{-1} r) \xi(\tilde{r}, k^{-1} g) \right] d\nu(\tilde{r}) \\
&= \left[ (\pi \otimes \lambda)(f \phi) \right] (r, g), \quad x = r(\tilde{r}),
\end{align}

where $\xi \in L^2(\Gamma, (\mu \circ \nu)) \otimes L^2(G)$. Hence $\beta(f) = (\pi \otimes \lambda)(f \phi)$. Similarly, $(1 \otimes \lambda(\phi)) \beta(f) = (\pi \otimes \lambda)(\phi \ast f)$ where $\phi \ast f \in C_c(\Gamma)$, $C_c(G)$) is defined by

\begin{equation}
(\phi \ast f)(r, g) = f(r)A(\rho(r))\phi(g \rho(r)).
\end{equation}

Hence $(1 \otimes b)\beta(a) + \beta(a)(1 \otimes c) \in C^*(G) \otimes C^*_\Gamma(G)$ for $a \in C^*(\Gamma)$, $b, c \in C^*_\Gamma(G)$. Next, let $f \in C_c(\Gamma)$ and $\psi \in C^*_\Gamma(G)$. Then,

\begin{align}
L_\psi(\beta(f)) &= \phi \ast f \in C_c(\Gamma), \\
\psi \ast f (r) &= \psi(\lambda(\rho(r)^{-1})) f(r).
\end{align}

This shows $L_\psi(\beta(C^*(\Gamma))) \subseteq C^*(\Gamma)$. Therefore $\beta(C^*(\Gamma)) \subseteq \bar{M}_L(C^*(\Gamma) \otimes C^*_\Gamma(G))$. By (3.15), the set $\{\psi \ast f : \psi \in C^*_\Gamma(G), f \in C_c(\Gamma)\}$ exhausts $C_c(\Gamma)$. Hence $\{L_\psi(\beta(f)) : \psi \in C^*_\Gamma(G), f \in C^*(\Gamma)\}$ generates $C^*(\Gamma)$ and the co-action is non-degenerate.

Lastly, we show $C^*(\Gamma) \rtimes G \cong C_\rho(G) \times G$. By the definition of the crossed product, $C^*(\Gamma) \rtimes G$ is the C*-algebra generated by $(1 \otimes \phi) \beta(f)$,
\( \phi \in C_0(G), f \in C^*(\Gamma) \) and hence generated by \( \beta(f) (1 \otimes \phi) \) by taking adjoint. By the density of \( C_c(\Gamma) \) in \( C^*(\Gamma) \), \( C^*(\Gamma) \ast G \) is generated by \( \beta(f) (1 \otimes \phi), f \in C_c(\Gamma) \), \( \phi \in C_c(G) \). Now, we define an unitary operator \( \tilde{W}_\rho : L^2(G) \otimes L^2(\Gamma, (\mu \circ \nu)) \to L^2 \), \( (\mu \circ \nu) \otimes L^2(G) \) by

\[
[\tilde{W}_\rho \xi](r, g) = \xi(\rho(\tau) g, r), \quad \xi \in L^2(G) \otimes L^2(\Gamma, (\mu \circ \nu)).
\]

Then,

\[
[\tilde{W}_\rho \beta(f)(1 \otimes \phi) \tilde{W}_\rho \xi](g, \tau) = \int_{P} f(\tau^{-1} r) \phi(\rho(\tau)^{-1} g) \xi(g, \tau) \, d\nu^x(\tau),
\]

where \( f \in C_c(\Gamma), \phi \in C_c(G), \xi \in L^2(G) \otimes L^2(\Gamma, (\mu \circ \nu)) \). Now, in view of the definition of \( C^*(\Gamma) \), \( \tilde{F} = G \times_{\rho} \Gamma \), the right hand side of (3.17) is equal to \( [\pi(\tilde{f}) \xi](g, \tau) \), where \( \pi \) is the representation of \( \tilde{F} \) on \( L^2(\tilde{F}), ((dg \otimes \mu) \circ \nu) = L^2(G) \otimes L^2(\Gamma, (\mu \circ \nu)) \) and \( \tilde{F}(g, r) = f(r) \phi(g) \). The linear combinations of such \( \tilde{f} \) belong to \( C_c(G) \otimes_{\text{alg}} C_c(\Gamma) \) which is dense in \( C_c(\tilde{F}) \) with respect to the \( L^1 \)-norm topology on \( C_c(\tilde{F}) \) defined by

\[
\|f\|_{L^1} = \max \left\{ \sup_{t \in \mathbb{T}} |f(t)| d\nu^x(t), \sup_{t \in \mathbb{R}^+} \int_{P} |f(t^{-1})| d\nu^x(t) \right\},
\]

where \( f \in C_c(\Gamma) \) (see [10]). This implies the norm density of \( \{\pi(\tilde{f}) : \tilde{F}(g, r) = f(r) \phi(g), f \in C_c(\Gamma), \phi \in C_c(G)\} \) in \( C^*(\tilde{F}) \). Q.E.D.

**Remark 3.3.** If \( G \) is abelian, then we obtain a \( C^* \)-dynamical system \((C^*(\Gamma), \hat{G}, \beta)\) where the \( \hat{G} \)-action \( \beta \) is defined by the relation

\[
\beta(f)(k)(r) = \langle \rho(\tau), k \rangle f(\tau), k \in \hat{G}, f \in C_c(\Gamma).
\]

By using \( L^1 \)-norm, this action is shown to be continuous. The continuity of the action also follows from the non-degeneracy of \( \beta \) as a co-action, see [6].

Now, we shall give an explicit correspondence between \( C^*(\tilde{F}), \tilde{F} = G \times_{\rho} \Gamma \) and \( C^*(\Gamma) \times_{\pi} \hat{G} \) for abelian \( G \). By the definition of a crossed product (see Example 2.4) and the density of \( C_c(\Gamma) \) in \( C^*(\Gamma) \), \( C^*(\Gamma) \times_{\pi} \hat{G} \) is given by the \( C^* \)-completions of \( C_c(\hat{G} \times \Gamma) \) with the \( * \)-algebraic structure

\[
(f_1 \ast f_2)(k, \tau) = \int_{\hat{G} \times \mathbb{T}^x} \langle \rho(\tau^{-1} \tau), t \rangle f_1(-l + k, \tau^{-1} \tau) f_2(l, \tau) \, dld\nu^x(\tau), \quad x = r(\tau),
\]
and with the $C^*$-norm given by the following regular representation

\[(3.22) \quad [\pi(f) \, \xi](k, r) = \int_{\mathcal{B} \times \mathcal{F}} \langle \varphi(\tilde{\varphi}^{-1} r), l \rangle f(-l+k, \tilde{\varphi}^{-1} r) \, \tilde{\rho}(l, \tilde{\varphi}) \, dl \, d\nu^r(\tilde{\varphi}),\]

\[x = r(\varphi),\]

where $f_1, f_2, f \in C_c(\widehat{G} \times \Gamma)$ and $\xi \in L^2(\widehat{G}) \otimes L^2(\Gamma, (\mu \circ \nu))$. Now, we define twisted inverse Plancherel transformations as follows:

\[(3.23) \quad [F \xi](g, r) = \int_{\mathcal{B}} \langle -\varphi(\tilde{\varphi}^{-1} r) + g, k \rangle f(k, r) \, dk, \quad f \in C_c(\widehat{G} \times \Gamma),\]

\[(3.24) \quad [\tilde{F} \xi](g, r) = \int_{\mathcal{B}} \langle -\varphi(\tilde{\varphi}^{-1} r) + g, k \rangle \tilde{\xi}(k, r) \, dk, \quad \xi \in L^2(\widehat{G}) \otimes L^2(\Gamma, (\mu \circ \nu)).\]

Then, $F(C_c(\widehat{G} \times \Gamma)) \subset C_c(\Gamma, C_0(G))$ (the image is dense in $L^1$-norm) and further $F$ gives a $*$-homomorphism, where $C_c(\Gamma, C_0(G))$ is a $*$-algebra by

\[(3.25) \quad (f_1 * f_2)(g, r) = \int_{\mathcal{B}} f_1(-\varphi(\tilde{\varphi}) + g, \tilde{\varphi}^{-1} r) f_2(g, \tilde{\varphi}) \, d\nu^r(\tilde{\varphi}),\]

\[x = r(\varphi),\]

\[(3.26) \quad f^\#(g, r) = \int_{\mathcal{B}} f(-\varphi(\tilde{\varphi}) + g, \tilde{\varphi}^{-1} r),\]

where $f_1, f_2, f \in C_c(\Gamma, C_0(G))$. It also holds that the unitary operator $\tilde{F}$ defined by (3.24) intertwines the $*$-representations of $C_c(\widehat{G} \times \Gamma)$ and $C_c(\Gamma, C_0(G))$ defined by (3.22) and

\[(3.27) \quad [\tilde{F}(f) \, \xi](g, r) = \int_{\mathcal{B}} f(-\varphi(\tilde{\varphi}) + g, \tilde{\varphi}^{-1} r) \, \xi(g, \tilde{\varphi}) \, d\nu^r(\tilde{\varphi}),\]

\[x = r(\varphi),\]

where $f \in C_c(\Gamma, C_0(G))$ and $\xi \in L^2(\mathcal{G}) \otimes L^2(\Gamma, (\mu \circ \nu))$. On the other hand, the operations (3.25), (3.26), (3.27) agree with the definition of $C_0(G) \times_\varphi \Gamma$. Hence the mapping $F$ defined by (3.23) gives the concrete isomorphism $C^*(\Gamma) \times_{\tilde{\varphi}} \widehat{G} \rightarrow C_0(G) \times_\varphi \Gamma$.

**Remark 3.4.** If we consider the case that $\Gamma$ is a locally compact abelian group, then $C_0(G) \times_\varphi \Gamma \cong C^*(\Gamma) \times_{\tilde{\varphi}} \widehat{G} \cong C_0(\widehat{\Gamma}) \times_\varphi \Gamma$. This duality can be viewed as the Plancherel transformation of abelian groupoid, see Bellissard-Testard [1] (see also Remark 6.2 (3) of [8]).

**Proposition 3.5.** Assume that $G$ is abelian. If $\varphi, \sigma : \Gamma \rightarrow G$ are cohomological in the sense that there exists a continuous mapping $\tau : \Gamma^{(0)} \rightarrow G$ such that $\varphi(r)$
The two $\hat{G}$-actions $\rho$, $\hat{\rho}$ are one-cocycle equivalent i.e. there exists a unitary valued mapping $u: \hat{G} \to M(C^*(\Gamma))$ such that $k \mapsto u_k a$ and $k \mapsto au_k$ are continuous for $a \in C^*(\Gamma)$ and

\begin{align}
\rho_k(a) &= \hat{u}_k \hat{\rho}(a) \hat{u}_k^*, \quad a \in C^*(\Gamma), \\
(3.28) \\
\hat{u}_{k+1} &= \hat{u}_k \hat{\rho}(u_k).
\end{align}

**Proof.** The unitary operator $u_k$ on $L^2(\Gamma, (\mu \circ \nu))$ defined by

\begin{equation}
[ u_k \xi ](\gamma) = < r(\gamma), k \nu \xi(\gamma), \xi \in L^2(\Gamma, (\mu \circ \nu))
\end{equation}

satisfies the condition. Q.E.D.

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**References**