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Singularities and Newton Polygons

by

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It was conjectured by V. I. Arnold that it should be possible to express all "reasonable" invariants associated to a holomorphic function, in terms of its Newton polygon, at least for "almost all" functions with a given polygon.

This conjecture has been worked out in many cases by the joint results of Arnold's school (D. Bernstein, A. Kouchnirenko, A. Varchenko, A. Xovanski). Their results will be the subject of this talk. In particular, the work of Varchenko concerning monodromy and oscillatory integrals should be of great interest in relation with the theory of b-functions.

§ 1. Newton Polygons; Genericity

1.1. Let f be a formal power series in k variables over $\boldsymbol{C}$:

$$f(x)=\sum_n a_n x^n,\ x=(x_1,\cdots,x_k),$$

$$n=(n_1,\cdots,n_k).$$

Definition 1. The *Newton polygon* of f is the union of all compact faces of the convex hull of

$$N(f)=\bigcup_n (n+\boldsymbol{R}_+{}^k),\ a_n\neq 0\,.$$

It is denoted by $\Gamma(f)$. The *principal part* of f is

$$\tilde{f}=\sum_{n\in\Gamma(f)} a_n x^n\,.$$

If γ is a face of $\Gamma(f)$, denote

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$$\tilde{f}_\gamma = \sum_{n\in\gamma} a_n x^n .$$

1.2. The above definition will apply to holomorphic functions. However, for a polynomial, a global analogue is needed:

Definition 2. Let P be a polynomial in k variables:

$$P(x) = \sum a_n x^n \quad \text{(finite)}.$$

The Newton polygon of f is the union of all closed faces not containing the origin of the convex hull of

$$N(P) = \{0, n, a_n \neq 0\}.$$

It is denoted $\Gamma(P)$. The principal part of P is defined in an obvious way, as well as $\tilde{P}_\gamma$.

1.3. With the above definitions, the genericity condition is the following:

Definition 3. The formal power series f (respectively the polynomial P) is *non-degenerate* if *for any closed face* γ *of* $\Gamma(f)$ (resp. $\Gamma(P)$) the functions

$$x_i \frac{\partial f_\gamma}{\partial x_i}, \quad i=1, \cdots, k \quad \left(\text{resp. } x_i \frac{\partial P_\gamma}{\partial x_i}\right)$$

have no common zeros in $(\boldsymbol{C}-0)^k$.

It is clear that for a given Newton diagram, the set of degenerate principal parts is an algebraic variety. From Sard-Bertini's theorem it follows that non-degenerate formal power series "are dense" in the Zariski topology (defined on their principal parts). A property true for such power series (for example) will be said to hold for "almost all" power series with Newton diagram fixed.

1.4. It will be essential to work with meromorphic functions of a particular type:

Definition 4. A *Laurent polynomial* is a finite sum of the form

$$f(x)=\sum a_m x^m,\ m\in \boldsymbol{Z}^k.$$

Such an f defines a function on $(\boldsymbol{C}-0)^k$, and non-degeneracy conditions can be also defined.

§ 2. Solutions of Analytic Equations, Milnor and Euler Numbers

Theorem 1. *If Γ is an integral polygon in $\boldsymbol{Z}^k$ (compact convex polygon with integral summits), then for almost all Laurent polynomials $(f_1, \cdots, f_k)$ with support Γ, the system of equations*

$$\begin{cases} f_1(x_1, \cdots, x_k)=0 \\ \quad\vdots \\ f_k(x_1, \cdots, x_k)=0 \end{cases} \qquad x_1\neq 0, \cdots, x_k\neq 0$$

has a finite number of solutions, equal to $k!V_k(\Gamma)$, where $V_k(\Gamma)$ is the k-volume of Γ.

This result is due to A. Kouchnirenko [1]. The proof uses the "Newton filtration" introduced earlier by V. I. Arnold. It has been generalized (D. Bernstein) to the case where the (f_i) are allowed various supports.

2.2. Though independent of the other results, I would like to quote the following striking result, due to D. Bernstein:

Theorem 2. *If $(\Gamma_1, \cdots, \Gamma_m)$ are integral polygons in $\boldsymbol{R}^k$, the number of integral points in $(i_1\Gamma_1+\cdots+i_m\Gamma_m)$ is a polynomial in the positive integral numbers $(i_1, \cdots, i_m)$.*

2.3.

Theorem 3. *Let f be a non-degenerate analytic function at the origin of $\boldsymbol{C}^k$, such that it contains for each i a term $\alpha x_i^{n_i}$ (α non-zero). Then the Milnor number of f*

$$\mu(f)=\dim_{\boldsymbol{C}}\boldsymbol{C}[[x]]/\left(\frac{\partial f}{\partial x}\right)$$

is equal to the alternating sum of the (V_i^-), where V_i^- denotes the *sum of i-volumes of all intersections with i-dimensional subspaces of coördinates with the set of points "under" the Newton polygon,* $\Gamma_{-}(f)$:

$$\Gamma_{-}(f) = \{\lambda x ;\, 0 \leqq \lambda \leqq 1,\, x \in \Gamma(f)\}.$$

This theorem is proved in [1]. The author has obtained a description "à la Milnor" in the generic case of the homotopy type of a regular fiber of a polynomial map.

2.4. The work on monodromy of Varchenko relies heavily on the following statement:

Theorem 4. *Suppose f is a polynomial in $\boldsymbol{C}^k$, satisfying the two conditions:*

The support of f is a convex integral polygon Γ, and for any face γ of Γ, the functions

$$\left(f_\gamma,\, x_i \frac{\partial f_\gamma}{\partial x_i},\ i=1, \cdots, k\right)$$

have no common zeros in $(\boldsymbol{C}-0)^k$, as well as the functions

$$\left(f,\, x_i \frac{\partial f_\gamma}{\partial x_i},\ i=1, \cdots, k\right).$$

Then the Euler characteristic of the set

$$X = \{x \in \boldsymbol{C}^k ; f(x) = 0,\, x_1 \neq 0,\, \cdots,\, x_k \neq 0\}$$

is equal to $(-1)^{k-1} k!\, V_k(\Gamma)$.

One proves here also that the condition of f is "generic".

2.5. The preceding results show that many informations are effectively computable from Newton combinatorial data. As a last example, let us mention that Xovanski has computed the arithmetic genus of the zero-locus V of a family of polynomials in terms of their Newton polygons, and he proved that, generically, if $\overline{V}$ is a compact birationally equivalent completion of V, there are no non-zero global differential forms on $\overline{V}$, except in top and zero dimensions.

2.6. It can very well be that a function be degenerate with respect to its Newton polygon, for any local choice of coördinates. Arnold asks if it is possible to make such a function equivalent to a non-degenerate one by adding a finite number of squares of the coördinate-functions.

§ 3. Zeta-Function of the Monodromy

Let f be a holomorphic function at the origin of $\boldsymbol{C}^n$, and suppose it belongs to $\mathfrak{m}^2$($\mathfrak{m}$ maximal ideal), and is non-degenerate (but it is not required to have an isolated singularity).

Consider the Milnor fibration (usual notations):

$$f\colon X\backslash X(0)\rightarrow T\backslash 0\,, \quad X(t)=f^{-1}(t)\cap B$$
$$T=\{t\in \boldsymbol{C};\, |t|\ll 1\}$$

and the monodromy $h(f)$ acting on $H^*(X(t), \boldsymbol{C})$. Then

$$\zeta_f(z)=\prod_{q\geqq 0}\det[Id.-zh(f), H^q(X(t), \boldsymbol{C})]^{(-1)^q}$$

is called the *Zeta-function of* $h(f)$. A. Varchenko has proved the following theorem:

Theorem 5. *Under the above assumptions,* $\zeta_f(z)$ *can be effectively computed from the Newton polygon of* f*, in the following way*:

Let $\theta(K)$, *for* K *an integral polyhedron in* $\boldsymbol{R}^k$, *be the product of all polynomials* $(1-z^{h(\Delta)})^{S(\Delta)}$, *where* Δ *is any* $(k-1)$*-face of* K, $S(\Delta)$ *its volume (normalized), and* $h(\Delta)$ *the cardinal of the quotient-group of* $\boldsymbol{Z}^K$ *by the subgroup generated by integer points belonging to affine space containing* Δ.

Define in the same way $\theta(K_i)$, $\theta(K_{ij})$, *by intersecting* K *with vector subspaces of coördinates. Then* $\zeta_f(Z)$ *is equal to*

$$\zeta_K(z)=\theta(K)\prod_i\theta(K_i)^{-1}\prod_{i<j}\theta(K_{ij})\cdots(-1)^{k-1}$$

The proof consists in applying the theory of toroidal imbeddings (Mumford) in order to get a "weak" resolution of singularities, *common* to all functions with a given Newton diagram; the theorem of A'Campo expresses ζ in terms of Euler characteristics of the divisors in a resolution of f, the final stroke is then Theorem 4.

§ 4. Oscillatory integrals

Let

$$I(\tau)=\int_{\boldsymbol{R}^k} e^{i\tau f(x)}\varphi(x)\,dx$$

the classical notation for the oscillatory integral, where φ is smooth with compact support, f real analytic. It is known that for τ tending to infinity,

$$I(\tau)\approx e^{i\tau f(0)}\sum_p\sum_{m=0}^{k-1}\tau^p(\log\tau)^m a_{p,m}(\varphi)$$

where p runs over a finite family of rational arithmetic progressions, independent of φ.

Definition 5. The *index of the singularity* defined by f is the supremum of the p's such that, for any open set containing 0, there exists a φ and an integer m such that

$$a_{p,m}(\varphi)\neq 0\,.$$

The number m will be called the *height* of $\beta(f)$.

Suppose f is non-degenerate (in the real-analytic sense), and let T denote the point of intersection of $\Gamma(f)$ with the diagonal in $\boldsymbol{R}^k$ $x_1=\cdots=x_k$;

$$T=(t,\cdots,t).$$

A. Varchenko has announced the following result:

Theorem 6. a) *If* $t>1$,

$$\beta(f)=\frac{1}{t}$$

and the height of $\beta(f)$ *is equal to the smallest dimension of faces containing* T.

b) *If* $0<t\leq 1$,

$$\beta(f)\geq\frac{1}{t},$$

but the inequality can be strict.

Theorem 7. a) *The singularity of a family of functions* f_α *of two variables with "*μ *constant" is constant.*

b) *Let* (f_α) *denote the following functions of* 3 *variables.*

$$f_\alpha(x, y, z)=(x^2+y^2+\alpha x^2+z^4)^2+x^n+y^n+z^n,$$

$$(n \text{ fixed}).$$

Then

$$\alpha>0 \quad \beta(f_\alpha)=3/4$$

$$\alpha=0 \quad \beta(f_0)=5/8$$

$$\alpha<0 \quad \beta(f_\alpha)=\frac{1}{2}+\gamma(n) \quad \lim_{n\to\infty}\gamma(n)=0.$$

The computation above gives a counter example to Arnold's conjecture on the semi-continuity of β.

References

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