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The Structure of Local Solutions of Partial Differential Equations of the Fuchsian Type

by

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Linear partial differential equations with regular singularity along a hypersurface were studied by several authors, say, Hasegawa [10][11], Baouendi-Goulaouic [3][4], Alinhac [1][2], Froim [7][8][9], Delache-Leray [6], Kashiwara-Oshima [13], Tsuno [15], etc$\cdots$ in various problems. In this note, we consider the hyperfunction solutions of certain type equations with regular singularity. The details of this note will be published in [14] anywhere else.

Let $(t, z) \in \boldsymbol{C} \times \boldsymbol{C}^n$ and let

$$P(t, z, D_t, D_z) = t^k D_t{}^m + P_1(t, z, D_z) t^{k-1} D_t{}^{m-1} + \cdots + P_k(t, z, D_z) D_t{}^{m-k} + \cdots + P_m(t, z, D_z)$$

be a linear differential operator whose coefficients are holomorphic functions defined in a neighbourhood of the origin such that

(A-i) $0 \leqq k \leqq m$

(A-ii) order of $P_j(t, z, D_z) \leqq j$ for $1 \leqq j \leqq m$

(A-iii) order of $P_j(0, z, D_z) \leqq 0$ for $1 \leqq j \leqq k$.

Then P is said *of the Fuchsian type with weight m-k with respect to t* (by [3]). By the condition (A-iii), $P_j(0, z, D_z)$ is a function. We set $P_j(0, z, D_z) = a_j(z)$ for $1 \leqq j \leqq k$. Then the indicial equation associated with P is defined by

$$\mathscr{C}(\lambda, z) = \lambda(\lambda-1)\cdots(\lambda-m+1) + a_1(z)\lambda(\lambda-1)\cdots(\lambda-m+2)$$

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$$+\cdots+a_k(z)\lambda(\lambda-1)\cdots(\lambda-m+k+1).$$

The roots, that we call the characteristic exponents of P, are denoted by $\lambda=0,1,\cdots,m-k-1,\ \rho_1(z),\cdots,\rho_k(z)$. They are functions of z.

We set

$\widetilde{\mathcal{O}}=$ the set of all the germs of multivalued holomorphic functions on $\boldsymbol{C}\times\boldsymbol{C}^n\backslash\{t=0\}$ at the origin.

Then we have the next theorem.

Theorem 1. *Assume that* $\rho_i(0),\ \rho_i(0)-\rho_j(0)\notin\boldsymbol{Z}$ *holds for* $1\leqq i\neq j\leqq k$. *Then the equation* $Pu=f$ *is always solvable in* $\widetilde{\mathcal{O}}$. *Moreover there exist holomorphic functions* $K_i(t,z,w)$ $(0\leqq i\leqq m-k-1)$, $L_j(t,z,w)$ $(1\leqq j\leqq k)$ *on*

$$U_\varepsilon=\{(t,z,w)\in\boldsymbol{C}\times\boldsymbol{C}^n\times\boldsymbol{C}^n;|t|,|z|,|w|<\varepsilon,\ |t|<M|z_i-w_i|^s,\ i=1,\cdots,n\}$$

$$s=min(m,k+1),\ M=constant$$

which satisfy the following conditions:

(1) *For any holomorphic functions* $\varphi_i(w)$, $\psi_j(w)$ *at the origin, we set*

$$u(t,z)=\sum_{i=0}^{m-k-1}\oint K_i(t,z,w)t^i\varphi_i(w)\,dw+\sum_{j=1}^{k}\oint L_j(t,z,w)t^{\rho_j(w)}\psi_j(w)\,dw$$

Then $u(t,z)$ *is a solution of the equation* $Pu=0$ *in* $\widetilde{\mathcal{O}}$.

(2) *If* $u(t,z)\in\widetilde{\mathcal{O}}$ *and* $Pu=0$ *holds, then* $u(t,z)$ *is uniquely expressed in the form* (1).

Next, we consider the equation in the real domain and investigate the structure of hyperfunction solutions. Let $(t,x)\in\boldsymbol{R}\times\boldsymbol{R}^n$ and let $P(t,x,D_t,D_x)$ be of the Fuchsian type with weight m-k with respect to t. Moreover we assume the following conditions on P:

(A-iv) $\sigma_m(P)$ has the form: $\sigma_m(P)(t,x,\tau,\xi)=t^k p_m(t,x,\tau,\xi)$

(A-v) All the roots $\tau(t,x,\xi)$ of the equation $p_m(t,x,\tau,\xi)=0$ are real, when t, x, ξ are real (near the origin).

Then we say that P is *a Funchsian hyperbolic operator with respect to t.* Note that if $k=0$, then P is nothing but a weakly hyperbolic operator in the direction dt ([5]).

Under these assumptions, we can give the meaning as hyperfunctions to the above $K_i(t,z,w)$, $L_j(t,z,w)$ in Theorem 1. We also denote these hyperfunctions by $K_i(t,x,y)$, $L_j(t,x,y)$ respectively. then K_i, L_j satisfy the following conditions:

$$\text{Supp } K_i,\ L_j \subset \{(t,x,y);|x-y|\leqq M|t|^{1/s}\}$$

$$S\text{-}S(K_i),\ (L_j)\subset\{(t,x,y,\sqrt{-1}(\tau,\xi,\eta)\infty);|x-y|\leqq M|t|^{1/s},$$
$$|\tau|\leqq M|\xi|,\ |\xi+\eta|\leqq M|\xi||t|^{1/s}\}.$$

Using these hyperfunctions, we have the next theorem.

Theorem 2. *Assume that* $\rho_i(0)$, $\rho_i(0)-\rho_j(0)\notin \boldsymbol{Z}$ *holds for* $1\leqq i\neq j\leqq k$. *Then the equation* $Pu=f$ *is always solvable in* $\mathcal{B}$ *(where* $\mathcal{B}$ *is the stalk of the sheaf of hyperfunctions at the origin). Moreover the above* $K_i(t,x,y)$ $(0\leqq i\leqq m-k-1)$, $L_j(t,x,y)$ $(1\leqq j\leqq k)$ *satisfy the following conditions*:

(1) *For any hyperfunctions* $\varphi_i(y)$, $\psi_j^{\pm}(y)$ *at the origin, we set*

$$u(t,x)=\sum_{i=0}^{m-k-1}\int K_i(t,x,y)\,t^i\varphi_i(y)\,dy$$
$$+\sum_{j=1}^{k}\sum_{\pm}\int L_j(t,x,y)\,(t\pm i0)^{\rho_j(y)}\psi_j^{\pm}(y)\,dy$$

or

$$u(t,x)=\sum_{i=0}^{m-k-1}\int K_i(t,x,y)\,t^i\varphi_i(y)\,dy$$
$$+\sum_{j=1}^{k}\sum_{\pm}\int L_j(t,x,y)\,t_{\pm}^{\rho_j(y)}\psi_j^{\pm}(y)\,dy$$

Then $u(t,x)$ *is a solution of the equation* $Pu=0$ *in* $\mathcal{B}$.

(2) *If* $u(t,x)\in\mathcal{B}$ *and* $Pu=0$ *holds, then* $u(t,x)$ *is uniquely expressed in the form* (1).

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