Local Cohomology of Analytic Spaces

by

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The purpose of this paper is to show that the local cohomology of a complex analytic space embedded in a complex manifold is a holonomic system of linear differential equations of infinite order and its holomorphic solution sheaves are a resolution of the constant sheaf $\mathcal{C}$ in this space which provides the Poincaré lemma. The proof relies on the theories of the $\mathcal{b}$-function and holonomic systems due to M. Kashiwara ([2] and [3]) and A. Grothendieck’s theorem on the De Rham cohomology of an algebraic variety ([1]). I am very much indebted to M. Kashiwara from whose papers I learned so much.

Notations

We use the following notations:

$(X, \mathcal{O}_X)$ : complex smooth manifold.
$Y$ : reduce analytic subspace of $X$.
$I$ : coherent ideal sheaf defining $Y$.
$\mathcal{D}_x^\infty = \mathcal{D}^\infty$ : sheaf of differential operators on $X$.
$\mathcal{D}_x = \mathcal{D}$ : sheaf of differential operators of finite order.
$D(\mathcal{A})$ : derived category of the category of $\mathcal{A}$-modules if $\mathcal{A}$ is any sheaf of rings.

A complex means a bounded complex. The sheaf $\mathcal{D}$ is coherent and the sheaf $\mathcal{D}^\infty$ is flat over $\mathcal{D}$.

§ 1. Main Theorems

The local cohomology $R\Gamma_Y(\mathcal{D}_X)$ of $Y$ is an object of $D(\mathcal{D}^\infty)$ because any injective $\mathcal{D}^\infty$-module is flabby. The algebraic local cohomology of $Y$ is the object of $D(\mathcal{D})$ defined intrinsically by

$$R\Gamma_Y(\mathcal{O}_X = R \lim_{\rightarrow} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/I^k; \mathcal{O}_X).$$

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Theorem 1.1  
i) The local algebraic cohomology is a complex with $\mathcal{D}$-holonomic cohomology.  
ii) We have the canonical morphism in $D(C_x)$

\[ R \hom_{\mathcal{D}}(R\Gamma_{\mathcal{I}}(\mathcal{O}_x); \mathcal{O}_x \cong C_y). \]

The natural morphism $R\Gamma_{\mathcal{I}}(\mathcal{O}_x) \to R\Gamma_y(\mathcal{O}_x)$ induces a morphism in $D(\mathcal{D}^\omega)$

\[ \mathcal{D}^\omega \otimes R\Gamma_{\mathcal{I}}(\mathcal{O}_x) \to \mathcal{D}^\omega \otimes R\Gamma_{\mathcal{I}}(\mathcal{O}) \to R\Gamma_y(\mathcal{O}_x). \]

Theorem 1.2  The morphism (*) is an isomorphism in $D(\mathcal{D}^\omega)$ and the local cohomology sheaves of $Y$ are $\mathcal{D}^\omega$-holonomic and admissible modules.

The theorem (1, 1) is the Poincaré lemma because it gives a resolution of the constant sheaf $C_Y$ in terms of analytic structure of $Y$. The flatness of $\mathcal{D}^\omega$ over $\mathcal{D}$ gives the following formula for every $p$:

\[ H^p_y(\mathcal{O}_x) = \mathcal{D}^\omega \otimes \lim_{\underset{k}{\longrightarrow}} \mathcal{E}_{\mathcal{I}}^p C_x(\mathcal{O}_x/\mathcal{D}_x^k; \mathcal{O}_x). \]

We get an expression of the local cohomology of a space in terms of the extension sheaves in analytic geometry which is useful in applications, for example in the proof of the theorem $B$. We have also the resolution

\[ R \hom_{\mathcal{D}^\omega}(R\Gamma_y(\mathcal{O}_x); \mathcal{O}_x \cong C_y). \]

We give the sketch of the proofs.

§ 2. The Algebraic Local Cohomology

To be coherent with the notations [S.K.K.] we denote by $\mathcal{B}_{\mathcal{I}}^* x$ the cohomology of $R\Gamma_{\mathcal{I}}(\mathcal{O}_x)[\text{codim } Y]$. To show that $\mathcal{B}_{\mathcal{I}}^* x$ are $\mathcal{D}$-holonomic we construct or canonical complex $L_{\mathcal{I}}(\mathcal{O}_x)$ on $X$ which has the same cohomology as $R\Gamma_{\mathcal{I}}(\mathcal{O}_x)$. This complex reduces to the dualizing complex $L^*(\mathcal{O}_x)$ of J. P. Ramis and G. Ruget [8] if $Y = X$. We first suppose that $\text{codim}(Y) = 1$. In this case

\[ \text{(1)} \]

I was told by J. P. Ramis that he gets this formula with B. Malgrange by using cristalline cohomology.
\[ L'_{\gamma Y}(\mathcal{O}_X) = R\Gamma_{\gamma Y}(\mathcal{O}_X) \cong \lim_{\rightarrow k} \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X/I^k; \mathcal{O}_X) [-1] = \mathcal{B}_{\gamma Y}|_X [-1] \]

To see that \( \mathcal{B}_{\gamma Y}|_X \) is \( \mathcal{D} \)-holonomic system we can suppose that \( Y = f^{-1}(0) \) where \( f \in \Gamma(X, \mathcal{O}_X) \) and
\[ \mathcal{B}_{\gamma Y}|_X \simeq \mathcal{O}_X[f^{-1}]/\mathcal{O}_X. \]

The singular supports \( SS(\mathcal{O}_X[f^{-1}]) \) and \( SS(\mathcal{B}_{\gamma Y}|_X) \) are the same because \( SS(\mathcal{O}_X) \) is empty and it is enough to show that \( \mathcal{O}_X[f^{-1}] \) is \( \mathcal{D} \)-holonomic. But it is just a consequence of the fundamental theorem of M. Kashiwara ([3]) which says that the \( \mathcal{D} \)-module \( \mathcal{R} = \mathcal{D}[s]f^* \) is a coherent purely \( (n-1) \)-dimensional \( \mathcal{D} \)-module if \( n = \dim X \).

Indeed, this theorem proves the the existence of the \( b \)-function of \( f \) and this \( b \)-function gives
\[ \mathcal{O}_X[f^{-1}] = \mathcal{D} \cdot f^{-N} \]
for a natural number \( N \) large enough. We have the exact sequence
\[ 0 \to (s+N) \mathcal{D}[s] f^* \to \mathcal{D}[s] f^* \to \mathcal{D} \cdot f^{-N} \to 0. \]

This sequence shows that \( \mathcal{O}_X[f^{-1}] \) is a coherent \( \mathcal{D} \)-module and a classical fact in dimension and multiplicity implies that \( \dim SS(\mathcal{O}_X[f^{-1}]) = n-1 \) which means that \( \mathcal{O}_X[f^{-1}] \) is \( \mathcal{D} \)-holonomic. Let us define \( L'_{\gamma Y}(\mathcal{O}_X) \) when \( \text{codim}(Y) \geq 2 \). Remember that for any regular noetherian scheme \((X, \mathcal{O}_X)\) over \( \mathcal{C} \) the cousin complex,
\[ L'((\mathcal{O}_X) = \mathcal{H}_{\mathcal{O}_X}(\mathcal{O}_X) \]

is an injective hence a flabby resolution of \( \mathcal{O}_X \). For a compact \( K \subset X \) we denote by \((\check{X}(K), \mathcal{O}_{\check{X}(K)})\) the affine scheme defined by \( \mathcal{O}(K) \) and by \( \check{Y}(K) \) the subscheme defined by the ideal \( I(K) \) of the vanishing functions on \( Y \) in a neighborhood of \( K \). For a point \( x \in X \) the fiber of \( L'_{\gamma Y}(\mathcal{O}_X) \) at \( x \) will be \( \Gamma_{\check{Y}(x)}(L'((\mathcal{O}_{\check{X}(x)}))) \) which is just the local cohomology of \( \check{Y}(x) \) in the local scheme \( \check{X}(x) \). To glue the different fibers we take a compact polycylinder \( K \), the ring \( \mathcal{O}(K) \) is noetherian and the cousin complex \( L'((\mathcal{O}_{\check{X}(K)}) \) is a flabby resolution of \( \mathcal{O}_{\check{X}(K)} \). If \( x \in \check{K} \), the morphism \( \mathcal{O}(K) \to \mathcal{O}_x \) gives a morphism
\[ \Gamma_{\mathcal{O}_x}(L'((\mathcal{O}_{\check{X}(K)}))) \to \Gamma_{\check{Y}(x)}(L'((\mathcal{O}_{\check{X}(x)}))) \).

**Lemma 2.1** When \( K \) runs over the neighborhoods of \( x \), the morphism
This lemma glues together the fibers and the complex \( L'_{\mathcal{V}(\mathcal{O}_X)} \) will have the same cohomology as \( R\Gamma_{\mathcal{V}(\mathcal{O}_X)} \) because of the expression of the local cohomology of a closed space in a noetherian scheme in terms of the extensions. Now to prove that \( \mathcal{B}_{\mathcal{V}(\mathcal{O}_X)} \) are coherent \( \mathcal{D} \)-modules it is enough to prove that for any small polycylinder \( K' (K, \mathcal{B}_{\mathcal{V}(\mathcal{O}_X)}) \) are \( \mathcal{D}(K) \)-module of finite type and that the morphism

\[
\Gamma(K, \mathcal{B}_{\mathcal{V}(\mathcal{O}_X)}) \otimes \mathcal{O}_x \to \mathcal{B}_{\mathcal{V}(\mathcal{O}_X)},
\]

is an isomorphism for every \( x \in \hat{K} \) (see [6]). If \( f_1, \ldots, f_q (q \geq 2) \) are functions of \( \mathcal{O}(K) \) defining \( Y \) in a neighborhood of \( K \) we can compute \( \Gamma(K, \mathcal{B}_{\mathcal{V}(\mathcal{O}_X)}) \) as the \( \check{\text{C}} \)ech cohomology of the Zariski-\( \check{\text{C}} \)ech covering \( U = \bigcup U_i \) of \( \check{X}(K) \setminus \check{Y}(K) \) where \( U_i = \check{X}(K) \setminus V(f_i) \). The \( \check{\text{C}} \)ech complex \( \check{C}^\cdot (U, \mathcal{O}_\check{X}(K)) \) is a complex of \( \mathcal{D}(K) \)-module of finite type in virtue of the codimension one so \( \Gamma(K, \mathcal{B}_{\mathcal{V}(\mathcal{O}_X)}) \) are of finite type. The module \( \mathcal{O}_x \) is flat over \( \mathcal{O}(K) \) if \( x \in \hat{K} \) so the tensor product with \( \mathcal{O}_x \) over \( \mathcal{O}(K) \) commutes with the cohomology. But we have in virtue of the one codimensionality the following isomorphism,

\[
\check{C}^\cdot (U, \mathcal{O}_\check{X}(K)) \otimes \mathcal{O}_x \cong \check{C}^\cdot (U, \mathcal{O}_\check{X}(K)).
\]

Taking the cohomology in the both hand side we have the isomorphism

\[
\Gamma(K, \mathcal{B}_{\mathcal{V}(\mathcal{O}_X)}) \otimes \mathcal{O}_x \cong \mathcal{B}_{\mathcal{V}(\mathcal{O}_X)},
\]

and the \( \mathcal{D} \)-modules \( \mathcal{B}_{\mathcal{V}(\mathcal{O}_X)} \) are coherent. To see that these modules are holonomic we just notice that the object of the complex \( \Gamma_{\mathcal{V}(\mathcal{O}_X)} (L' (\mathcal{O}_\check{X}(K))) \) have dimensions \( n-1 \) as \( \mathcal{D}_x \)-modules if we compute it by the \( \check{\text{C}} \)ech-Zariski cohomology.

§ 3. The De Rham Complex of \( R\Gamma_{\mathcal{V}(\mathcal{D}_X)} \).

We denote by \( T \) the tangent vector bundle of \( X \) and by \( \mathcal{O}_X = \mathcal{O} \) the De Rham complex of \( X \). The first Spencer sequence \( \mathcal{D} \otimes \Lambda^\cdot (T) \) is a projective resolution of \( \mathcal{O}_X \) in the category of \( \mathcal{D} \)-modules. For any complex \( \mathcal{M}^\cdot \) of \( \mathcal{D} \)-modules we have
These isomorphisms hold in $D(C_X)$. The complex $R\hom_{\mathcal{O}}(O_x; M')$ is called the De Rham complex of $M'$ and is denoted by $DR(M')$. By the Poincaré lemma we have $R\Gamma_{\mathcal{O}}(O_x; O_x) \cong L \otimes C_X$ in $D(C_X)$. The natural injection $R\Gamma_{\mathcal{O}}(O_x) \to R\Gamma_{\mathcal{O}}(O_x)$ gives a morphism

\[
R\hom_{\mathcal{O}}(O_x; R\Gamma_{\mathcal{O}}(O_x)) \rightarrow R\hom_{\mathcal{O}}(O_x, O_x) \rightarrow R\Gamma_{\mathcal{O}}(O_x).
\]

**Theorem 3.1** The composed morphism $R\hom_{\mathcal{O}}(O_x; R\Gamma_{\mathcal{O}}(O_x)) \rightarrow R\Gamma_{\mathcal{O}}(O_x)$ is an isomorphism in $D(C_X)$.

The question is local. We can suppose that $\mathcal{O}_x = (\mathfrak{f}_1, \mathfrak{f}_2 \cdots, \mathfrak{f}_q)$. Let $\mathcal{I}_1 = (\mathfrak{f}_1, \mathfrak{f}_2 \cdots, \mathfrak{f}_{q-1})$ and $Y_1$ and $Y_2$ the spaces defined by $\mathcal{I}_1$ and $\mathcal{I}_2$. We have $Y = Y_1 \cap Y_2$ and $Y_1 \cup Y_2$ is defined by $(f_1, f_2, \cdots, f_{q-1}, f_q)$. We have a triangle

\[
R\Gamma_{\mathcal{O}}(O_x) \rightarrow R\Gamma_{\mathcal{O}}(O_x) \bigoplus R\Gamma_{\mathcal{O}}(O_x)
\]

To see that (1) is a triangle in $D(O)$, it is enough to see it on each fiber because of the nature of the Cousin complex. But if $x \in X$ the triangle

\[
R\Gamma_{\mathcal{O}}(O_x), x \rightarrow R\Gamma_{\mathcal{O}}(O_x), x \bigoplus R\Gamma_{\mathcal{O}}(O_x), x
\]

is just the Mayer-Vietoris sequence of the subspaces $Y_1(x)$ and $Y_2(x)$ in the scheme $X(x)$. We can also use Artin-Rees lemma and cofinality. The functor $R\hom_{\mathcal{O}}(O_x; *)$ from $D(O)$ to $D(C_X)$ is a $\partial$-functor and transforms triangle (1) into the triangle (2)

\[
DR(R\Gamma_{\mathcal{O}}(O_x)) \rightarrow DR(R\Gamma_{\mathcal{O}}(O_x)) \bigoplus DR(R\Gamma_{\mathcal{O}}(O_x)).
\]
The Mayer-Vietoris sequence of $Y_1$ and $Y_2$ in $X$ gives the triangle (3)
\[(3) \quad +1 \quad \xrightarrow{\text{\text{R}}\Gamma_{Y_1\cup Y_2}(C_X)} \quad \xrightarrow{\text{\text{R}}\Gamma_Y(C_X)} \quad \xrightarrow{\text{\text{R}}\Gamma_{Y_1}(C_X) \oplus \text{\text{R}}\Gamma_{Y_2}(C_X)}\]
The morphism of the theorem 3.1 is a morphism of the triangle (2) to triangle (3). By induction on $q$ the proof of the theorem 3.1 is reduced to the case $\mathcal{J} = (f)$. If $Y = f^{-1}(0)$ let $U$ be $X \setminus Y$ and $j$ the injection of $U$ in $X$. In this case
\[\text{\text{R}}\Gamma_{\mathcal{R}j_!}(\mathcal{O}_X) = \mathcal{B}_{Y\setminus X}[-1] = \mathcal{O}_X[f^{-1}]/\mathcal{O}_X[-1]\]
We have the triangle in $D(C_X)$
\[\xrightarrow{DR(\mathcal{B}_{Y\setminus X})} \quad +1 \quad \xrightarrow{\text{\text{R}}\Gamma_Y(C_X)} \quad \xrightarrow{DR(\mathcal{O}_X[-1])}\]
and the triangle in $D(C_X)$
\[\xrightarrow{\text{\text{R}}\Gamma_Y(C_X)[1]} \quad +1 \quad \xrightarrow{\text{\text{R}}j_*C_U}\]
But the following composed morphism
\[DR(\mathcal{O}_X[f^{-1}]) \to j_*\mathcal{O}_U \to \text{\text{R}}j_*\mathcal{O}_U \to \text{\text{R}}j_*C_U\]
is an isomorphism by the Grothendieck's theorem [1]. Finally we get
\[\text{\text{R}}\text{Hom}_{D}(\mathcal{O}_X; \text{\text{R}}\Gamma_{\mathcal{R}j_!}(\mathcal{O}_X)) \to \text{\text{R}}\Gamma_Y(C_X)\]
and the proof of theorem (3.1) is complete.

§ 4. Verdier Duality

Remember that a $C$-analytic finitistic sheaf of $C$ vector spaces ($C$-analytiquement constructible in French) is a sheaf $\mathcal{F}$ of finite $C$-vector spaces such that there exists a stratification $\bigcup X_i$ of $X$ and the restriction $F|_{X_i}$ on each stratum is locally constant. By M. Kashiwara [2] the complex $\text{\text{R}}\text{Hom}_{D}(\mathcal{M}, \mathcal{O}_X)$ has finitistic cohomology if $\mathcal{M}$ is a $\mathcal{D}$-holonomic system. Because the category of finitistic sheaves is a stick subcategory
of $\mathcal{C}$-vector spaces the complex $\mathbb{R} \text{Hom}_{\mathcal{C}}(\mathcal{M}^*, \mathcal{O}_x)$ has finitistic cohomology if $\mathcal{M}^*$ is a complex with $\mathcal{D}$-holonomic cohomology. We call a complex with finitistic cohomology a finitistic complex. So by the first part of theorem 1.1 the complex $\mathbb{R} \text{Hom}_{\mathcal{O}}(\mathbb{R} \Gamma_{\mathcal{M}^*}(\mathcal{O}_x); \mathcal{O}_x)$ is finitistic. Using the topological duality of Verdier and devissage [12] we can see that $\mathbb{R} \text{Hom}_{\mathcal{C}_x}(\mathcal{F}^*, \mathcal{C}_x)$ is finitistic if $\mathcal{F}^*$ is a finitistic complex and the natural morphism

$$\mathcal{F}^* \to \mathbb{R} \text{Hom}_{\mathcal{C}_x}(\mathbb{R} \text{Hom}_{\mathcal{C}_x}(\mathcal{F}^*, \mathcal{C}_x); \mathcal{C}_x)$$

is an isomorphism in $D(\mathcal{C}_x)$. The sheaf $\mathcal{C}_y$ is finitistic and we have

$$\mathcal{C}_y \to \mathbb{R} \text{Hom}_{\mathcal{C}_x}(\mathbb{R} \text{Hom}_{\mathcal{C}_x}(\mathcal{C}_y; \mathcal{C}_x); \mathcal{C}_x).$$

We have also

$$\mathbb{R} \text{Hom}_{\mathcal{O}}(\mathbb{R} \Gamma_{\mathcal{M}^*}(\mathcal{O}_x); \mathcal{O}_x) \cong \mathbb{R} \text{Hom}_{\mathcal{C}_x}(\mathbb{R} \text{Hom}_{\mathcal{C}_x}(\mathbb{R} \text{Hom}_{\mathcal{O}}(\mathbb{R} \Gamma_{\mathcal{M}^*}(\mathcal{O}_x); \mathcal{O}_x); \mathcal{C}_x); \mathcal{C}_x).$$

To prove that $\mathcal{C}_y \cong \mathbb{R} \text{Hom}_{\mathcal{O}}(\mathbb{R} \Gamma_{\mathcal{M}^*}(\mathcal{O}_x); \mathcal{O}_x)$ it suffices to prove that $\mathbb{R} \text{Hom}_{\mathcal{C}_x}(\mathcal{C}_y; \mathcal{C}_x) = \mathbb{R} \Gamma_{\mathcal{M}^*}(\mathcal{C}_x) \cong \mathbb{R} \text{Hom}_{\mathcal{C}_x}(\mathbb{R} \text{Hom}_{\mathcal{O}}(\mathbb{R} \Gamma_{\mathcal{M}^*}(\mathcal{O}_x); \mathcal{O}_x); \mathcal{C}_x).$ By theorem 3.1 we have $\mathbb{R} \Gamma_{\mathcal{M}^*}(\mathcal{C}_x) = \mathbb{R} \text{Hom}_{\mathcal{O}}(\mathcal{C}_x; \mathbb{R} \Gamma_{\mathcal{M}^*}(\mathcal{O}_x))$ and we must prove that

$$\mathbb{R} \text{Hom}_{\mathcal{O}}(\mathcal{O}_x; \mathbb{R} \Gamma_{\mathcal{M}^*}(\mathcal{O}_x)) \cong \mathbb{R} \text{Hom}_{\mathcal{C}_x}(\mathbb{R} \text{Hom}_{\mathcal{O}}(\mathbb{R} \Gamma_{\mathcal{M}^*}(\mathcal{O}_x); \mathcal{O}_x); \mathcal{C}_x).$$

This can be done by the following theorem which completes the proof of the theorem (1, 1):

**Theorem 4.1** Let $\mathcal{M}^*$ be a complex with $\mathcal{D}$-holonomic cohomology then we have a canonical isomorphism in $D(\mathcal{C}_x)$

$$\mathbb{R} \text{Hom}_{\mathcal{O}}(\mathcal{O}_x; \mathcal{M}^*) \cong \mathbb{R} \text{Hom}_{\mathcal{C}_x}(\mathbb{R} \text{Hom}_{\mathcal{O}}(\mathcal{M}^*; \mathcal{O}_x); \mathcal{C}_x).$$

§ 5. T.V.S. Homological Algebra

To prove theorem 4.1 and theorem 1.2 we need to define the functor $\mathbb{R} \text{Hom}_{\mathcal{C}_x}(\mathbb{F}; \mathcal{G})$ if $\mathcal{F}^*$ and $\mathcal{G}^*$ are complex of $\mathcal{D}_x$-module locally free with differential operators of finite order. Roughly speaking it is the "derived" functor of $\text{Hom}_{\mathcal{C}_x}(\mathbb{F}; \mathcal{G})$ which represents the continuous homomorphisms of Fréchet-nuclear sheaves. This category
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is not abelian. J. P. Ramis had noticed [10] and [11] that $C_x$ is just the Libermann complex $B'$ [5] and using the graded ring $B'$ he could define $R \operatorname{Hom}_{C_x}(\mathcal{F}; \mathcal{O})$. We do not give here the precise definition but we recall the formula

$$R \operatorname{Hom}_{C_x}(\mathcal{O}_x; \mathcal{O}) = R \operatorname{Hom}_{C_x}(\mathcal{I}_x; \mathcal{O}_x)$$

where $\mathcal{I}_x$ is the sheaf of infinite jets; see [9]. We list the properties of this functor in the following theorem:

**Theorem 5.1** We have the following isomorphisms:

a) $R \operatorname{Hom}_{C_x}(\mathcal{O}_x; \mathcal{O}) \cong \mathcal{D}^\infty$

b) $R \operatorname{Hom}_{C_x}(\mathcal{O}_x; \mathcal{O}) \cong \mathcal{O}[-n]$

c) $R \operatorname{Hom}_{C_x}(\mathcal{I}_x; \mathcal{O}) \cong R \operatorname{Hom}_{C_x}(\mathcal{I}_x; \mathcal{O})$ if $\mathcal{I}_x$ finitistic.

We can now prove theorem 4.1. The natural morphism

$$\operatorname{Hom}_{C_x}(\mathcal{F}; \mathcal{O}) \otimes \mathcal{M} \rightarrow \operatorname{Hom}_{C_x}(\mathcal{M}; \mathcal{F}; \mathcal{O}),$$

where $\mathcal{F}$ and $\mathcal{M}$ are left $\mathcal{D}$-modules, gives rise to the morphism of functors

$$R \operatorname{Hom}_{C_x}(\mathcal{F}; \mathcal{O}) \otimes \mathcal{M} \rightarrow R \operatorname{Hom}_{C_x}(\mathcal{M}; \mathcal{F}; \mathcal{O}).$$

Notice that the structure of right $\mathcal{D}$-module of $\operatorname{Hom}_{C_x}(\mathcal{F}; \mathcal{O})$ comes from the structure of left $\mathcal{D}$-module $\mathcal{F}$. This morphism of functors is an isomorphism if $\mathcal{M}$ has $\mathcal{D}$-coherent cohomology by the way out “left” functor lemma. We have a natural morphism in $\mathcal{D}(C_x)$

$$R \operatorname{Hom}_{C_x}(\mathcal{O}_x; \mathcal{O}) \rightarrow R \operatorname{Hom}_{C_x}(\mathcal{O}_x; \mathcal{O}),$$

which give a morphism by composition with the last one

$$(*) \quad R \operatorname{Hom}_{C_x}(\mathcal{O}_x; \mathcal{O}) \otimes \mathcal{M} \rightarrow R \operatorname{Hom}_{C_x}(\mathcal{M}; \mathcal{O}_x; \mathcal{O}).$$

**Theorem 5.2** Let $\mathcal{M}$ a complex with $\mathcal{D}$-holonomic cohomology then $(*)$ is an isomorphism in $\mathcal{D}(C_x)$.

The question is local. We can suppose that $\mathcal{M}$ is a single holono-
mic $\mathcal{D}$-module admitting a free resolution. In this case $\mathbf{R} \text{Hom}_\mathcal{D}(\mathcal{M}; \mathcal{O}_X)$ is a complex of free $\mathcal{O}_X$-modules with differential being differential operators of finite order and $\mathbf{R} \text{Hom}_{\mathcal{D}_x}(\mathbf{R} \text{Hom}_\mathcal{D}(\mathcal{M}; \mathcal{O}_X); \mathcal{O}^*)$ has a meaning. The morphism $(\ast)$ transit via the morphism $\mathbf{R} \text{Hom}_{\mathcal{D}_x} \times (\mathcal{O}_X; \mathcal{O}^*) \xrightarrow{L} \mathcal{M} \rightarrow \mathbf{R} \text{Hom}_{\mathcal{D}_x}(\mathbf{R} \text{Hom}_\mathcal{D}(\mathcal{M}; \mathcal{O}_X); \mathcal{O}^*)$. The last morphism is an isomorphism by the technique of the way out "left" functor lemma. The theorem 5.2 is a consequence of the property c) of theorem (5.1) because $\mathbf{R} \text{Hom}_\mathcal{D}(\mathcal{M}'; \mathcal{O}_X)$ is finitistic. In $D(C_x)$ we have $\mathcal{O}^* \cong C_x$ and $\mathbf{R} \text{Hom}_{\mathcal{D}_x}(\mathcal{O}_X; \mathcal{O}^*) \cong \mathcal{O}^*[-n]$, from the isomorphism $(\ast)$ we have

$$\mathcal{O}^* \otimes \mathcal{M}^*[-n] \xrightarrow{L} \mathbf{R} \text{Hom}_{\mathcal{D}_x}(\mathbf{R} \text{Hom}_\mathcal{D}(\mathcal{M}'; \mathcal{O}_X); C_x)$$

and the theorem 4.1 follows if we notice that

$$\mathcal{O}^* \otimes \mathcal{M}^*[-n] \xrightarrow{L} \mathbf{R} \text{Hom}_\mathcal{D}(\mathcal{O}_X; \mathcal{M}').$$

### §6. Local Cohomology of $Y$

The natural morphism of functors in the argument $\mathcal{M}'$

$$\mathbf{R} \text{Hom}_{\mathcal{D}_x}(\mathcal{O}_X; \mathcal{O}_X) \xrightarrow{L} \mathcal{M}' \rightarrow \mathbf{R} \text{Hom}_{\mathcal{D}_x}(\mathbf{R} \text{Hom}_\mathcal{D}(\mathcal{M}'; \mathcal{O}_X); \mathcal{O}_X)$$

is an isomorphism if $\mathcal{M}'$ has $\mathcal{D}$-coherent cohomology by the way out left functor. The natural morphism

$$\mathbf{R} \text{Hom}_{\mathcal{D}_x}(\mathcal{O}_X; \mathcal{O}_X) \rightarrow \mathbf{R} \text{Hom}_{\mathcal{D}_x}(\mathcal{O}_X; \mathcal{O}_X)$$

gives the morphism

$$\mathbf{R} \text{Hom}_{\mathcal{D}_x}(\mathcal{O}_X; \mathcal{O}_X) \xrightarrow{L} \mathcal{M}' \rightarrow \mathbf{R} \text{Hom}_{\mathcal{D}_x}(\mathbf{R} \text{Hom}_\mathcal{D}(\mathcal{M}'; \mathcal{O}_X); \mathcal{O}_X).$$

### Proposition 6.1

The morphism $(\ast)$ is an isomorphism in $D(C_x)$ if $\mathcal{M}'$ has $\mathcal{D}$-holonomic cohomology.

The question is local. We can suppose that $\mathcal{M}'$ is a single $\mathcal{D}$-holonomic system admitting a free resolution. We finish the proof in the same way as in the last section. We apply this situation to $\mathcal{M}' = RF_{\mathcal{U}}(\mathcal{O}_X)$. We have
and by theorem 1.1 \( R \text{Hom}_x(R \Gamma(Y)(O_x); O_x) \simeq R \text{Hom}_x(O_x) \), So

\[
\mathcal{D}^m \otimes \mathcal{R} \Gamma_Y(O_x) \simeq \mathcal{D}^m \otimes \mathcal{R} \Gamma_Y(O_x) \simeq R \text{Hom}_x(O_x) \simeq \mathcal{R} \Gamma_Y(O_x).
\]

The proof of theorem 1.2 is over. The details will appear elsewhere.

References


