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Local Cohomology of Analytic Spaces

by

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The purpose of this paper is to show that the local cohomology of a complex analytic space embedded in a complex manifold is a holonomic system of linear differential equations of infinite order and its holomorphic solution sheaves are a resolution of the constant sheaf $\boldsymbol{C}$ in this space which provides the Poincaré lemma. The proof relies on the theories of the b-function and holonomic systems due to M. Kashiwara ([2] and [3]) and A. Grothendieck's theorem on the De Rham cohomology of an algebraic variety ([1]). I am very much indebted to M. Kashiwara from whose papers I learned so much.

Notations

We use the following notations:

$(X, \mathcal{O}_X)$	:	complex smooth manifold.
Y	:	reduce analytic subspace of X.
I	:	coherent ideal sheaf defining Y.
$\mathcal{D}_X{}^\infty = \mathcal{D}^\infty$	:	sheaf of differential operators on X.
$\mathcal{D}_X = \mathcal{D}$	:	sheaf of differential operators of finite order.
$D(\mathcal{A})$	:	derived category of the category of $\mathcal{A}$-modules if $\mathcal{A}$ is any sheaf of rings.

A complex means a bounded complex. The sheaf $\mathcal{D}$ is coherent and the sheaf $\mathcal{D}^\infty$ is flat over $\mathcal{D}$.

§ 1. Main Theorems

The local cohomology $\boldsymbol{R}\Gamma_Y(\mathcal{D}_X)$ of Y is an object of $D(\mathcal{D}^\infty)$ because any injective $\mathcal{D}^\infty$-module is flabby. The algebraic local cohomology of Y is the object of $D(\mathcal{D})$ defined instrinsically by

$$\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X = \boldsymbol{R} \varinjlim_{k} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X/\mathcal{J}^k; \mathcal{O}_X).$$

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Theorem 1.1 i) *The local algebraic cohomology is a complex with $\mathcal{D}$-holonomic cohomology.* ii) *We have the canonical morphism in $D(\boldsymbol{C}_X)$*

$$\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X);\mathcal{O}_X \xrightarrow{\sim} \boldsymbol{C}_Y.$$ [1]

The natural morphism $\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X) \to \boldsymbol{R}\Gamma_Y(\mathcal{O}_X)$ induces a morphism in $D(\mathcal{D}^\infty)$

$$(*) \qquad \mathcal{D}^\infty \overset{L}{\underset{\mathcal{D}}{\otimes}} \boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X) = \mathcal{D}^\infty \underset{\mathcal{D}}{\otimes} \boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}) \to \boldsymbol{R}\Gamma_Y(\mathcal{O}_X).$$

Theorem 1.2 *The morphism* $(*)$ *is an isomorphism in* $D(\mathcal{D}^\infty)$ *and the local cohomology sheaves of* Y *are* $\mathcal{D}^\infty$*-holonomic and admissible modules.*

The theorem (1.1) is the Poincaré lemma because it gives a resolution of the constant sheaf $\boldsymbol{C}_Y$ in terms of analytic structure of Y. The flatness of $\mathcal{D}^\infty$ over $\mathcal{D}$ gives the following formula for every p:

$$\mathcal{H}^p{}_Y(\mathcal{O}_X) = \mathcal{D}^\infty \underset{\mathcal{D}}{\otimes} \varinjlim_k \mathcal{E}xt^p{}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}^k;\mathcal{O}_X).$$

We get an expression of the local cohomology of a space in terms of the extension sheaves in analytic geometry which is useful in applications, for example in the proof of the theorem B. We have also the resolution

$$\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}_\infty}(\boldsymbol{R}\Gamma_Y(\mathcal{O}_X);\mathcal{O}_X \xrightarrow{\sim} \boldsymbol{C}_Y.$$

We give the sketch of the proofs.

§ 2. The Algebraic Local Cohomology

To be coherent with the notations [S.K.K.] we denote by $\mathfrak{B}^*_{Y|X}$ the cohomology of $\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X)$ [codim Y]. To show that $\mathfrak{B}^*_{Y|X}$ are $\mathcal{D}$-holonomic we construct or canonical complex $L^\cdot_{[Y]}(\mathcal{O}_X)$ on X which has the same cohomology as $\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X)$. This complex reduces to the dualizing complex $L^\cdot(\mathcal{O}_X)$ of J. P. Ramis and G. Ruget [8] if $Y=X$. We first suppose that $\operatorname{codim}(Y)=1$. In this case

[1] I was told by J. P. Ramis that he gets this formula with B. Malgrange by using cristalline cohomology.

$$L^{\cdot}_{[Y]}(\mathcal{O}_X) = \boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X) \xrightarrow{\sim} \varinjlim_k \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}^k; \mathcal{O}_X)[-1] = \mathfrak{B}^1_{Y|X}[-1]$$

To see that $\mathfrak{B}^1_{Y|X}$ is $\mathcal{D}$-holonomic system we can suppose that $Y = f^{-1}(0)$ where $f \in \Gamma(X, \mathcal{O}_X)$ and

$$\mathfrak{B}^1_{Y|X} \xrightarrow{\sim} \mathcal{O}_X[f^{-1}]/\mathcal{O}_X .$$

The singular supports $SS(\mathcal{O}_X[f^{-1}])$ and $SS(\mathfrak{B}^1_{Y|X})$ are the same because $SS(\mathcal{O}_X)$ is empty and it is enough to show that $\mathcal{O}_X[f^{-1}]$ is $\mathcal{D}$-holonomic. But it is just a consequence of the fundamental theorem of M. Kashiwara ([3]) which says that the $\mathcal{D}$-module $\mathcal{N} = \mathcal{D}[s]f^s$ is a coherent purely $(n-1)$-dimensional $\mathcal{D}$-module if $n = \dim X$.

Indeed, this theorem proves the the existence of the b-function of f and this b-function gives

$$\mathcal{O}_X[f^{-1}] = \mathcal{D}.f^{-N}$$

for a natural number N large enough. We have the exact sequence

$$0 \to (s+N)\mathcal{D}[s]f^s \to \mathcal{D}[s]f^s \to \mathcal{D}.f^{-N} \to 0 .$$

This sequence shows that $\mathcal{O}_X[f^{-1}]$ is a coherent $\mathcal{D}$-module and a classical fact in dimension and multiplicity implies that $\dim SS(\mathcal{O}_X[f^{-1}]) = n-1$ which means that $\mathcal{O}_X[f^{-1}]$ is $\mathcal{D}$-holonomic. Let us define $L^{\cdot}_{[Y]}(\mathcal{O}_X)$ when $\operatorname{codim}(Y) \geq 2$. Remember that for any regular noetherian scheme $(\tilde{X}, \mathcal{O}_{\tilde{X}})$ over $\boldsymbol{C}$ the cousin complex,

$$L^{\cdot}(\mathcal{O}_{\tilde{X}}) = \mathcal{H}_{Z.|Z.+1}(\mathcal{O}_{\tilde{X}})$$

is an injective hence a flabby resolution of $\mathcal{O}_{\tilde{X}}$. For a compact $K \subset X$ we denote by $(\tilde{X}(K), \mathcal{O}_{\tilde{X}(K)})$ the affine scheme defined by $\mathcal{O}(K)$ and by $\tilde{Y}(K)$ the subscheme defined by the ideal $I(K)$ of the vanishing functions on Y in a neighborhood of K. For a point $x \in X$ the fiber of $L^{\cdot}_{[Y]}(\mathcal{O}_X)$ at x will be $\Gamma_{\tilde{Y}(x)}(L^{\cdot}(\mathcal{O}_{\tilde{X}(x)}))$ which is just the local cohomology of $\tilde{Y}(x)$ in the local scheme $\tilde{X}(x)$. To glue the different fibers we take a compact polycylinder K, the ring $\mathcal{O}(K)$ is noetherian and the cousin complex $L^{\cdot}(\mathcal{O}_{\tilde{X}(K)})$ is a flabby resolution of $\mathcal{O}_{\tilde{X}(K)}$. If $x \in \mathring{K}$, the morphism $\mathcal{O}(K) \to \mathcal{O}_x$ gives a morphism

$$\Gamma_{\tilde{X}(K)}(L^{\cdot}(\mathcal{O}_{\tilde{X}(K)})) \to \Gamma_{\tilde{Y}(x)}(L^{\cdot}(\mathcal{O}_{\tilde{X}(x)})) .$$

Lemma 2.1 *When K runs over the neighborhoods of x, the morphism*

$\varinjlim_{K\ni x} \Gamma_{\tilde{Y}(K)}(L^{\cdot}(\mathcal{O}_{\tilde{X}(K)})) \to \Gamma_{\tilde{Y}(x)}(L^{\cdot}(\mathcal{O}_{\tilde{X}(x)}))$ *is an isomorphism.*

This lemma glues together the fibers and and the complex $L^{\cdot}_{[Y]}(\mathcal{O}_X)$ will have the same cohomology as $\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X)$ because of the expression of the local cohomology of a closed space in a noetherian scheme in terms of the extensions. Now to prove that $\mathfrak{B}^{*}_{Y|X}$ are coherent $\mathcal{D}$-modules it is enough to prove that for any small polycylinder $K\Gamma(K, \mathfrak{B}^{*}_{Y|X})$ are $\mathcal{D}(K)$-module of finite type and that the morphism

$$\Gamma(K, \mathfrak{B}^{*}_{Y|X}) \underset{\mathcal{O}(K)}{\otimes} \mathcal{O}_x \to \mathfrak{B}^{*}_{Y|X,x}$$

is an isomorphism for every $x \in \mathring{K}$ (see [6]). If $f_1, \cdots, f_q$ ($q \geqq 2$) are functions of $\mathcal{O}(K)$ defining Y in a neighborhood of K we can compute $\Gamma(K, \mathfrak{B}^{*}_{Y|X})$ as the Čech cohomology of the Zariski-Čech covering $\mathfrak{U} = \bigcup_i U_i$ of $\tilde{X}(K) \setminus \tilde{Y}(K)$ where $U_i = \tilde{X}(K) \setminus V(f_i)$. The Čech complex $C^{\cdot}(\mathfrak{U}, \mathcal{O}_{\tilde{X}(K)})$ is a complex of $\mathcal{D}(K)$-module of finite type in vertue of the codimension one so $\Gamma(K, \mathfrak{B}^{*}_{Y|X})$ are of finite type. The module $\mathcal{O}_x$ is flat over $\mathcal{O}(K)$ if $x \in \mathring{K}$ so the tensor product with $\mathcal{O}_x$ over $\mathcal{O}(K)$ commutes with the cohomology. But we have in vertue of the one codimensionality the following isomorphism,

$$C^{\cdot}(\mathfrak{U}, \mathcal{O}_{\tilde{X}(K)}) \underset{\mathcal{O}(K)}{\otimes} \mathcal{O}_x \xrightarrow{\sim} C^{\cdot}(\mathfrak{U}, \mathcal{O}_{\tilde{X}(x)}).$$

Taking the cohomology in the both hand side we have the isomorphism

$$\Gamma(K, \mathfrak{B}^{*}_{Y|X}) \underset{\mathcal{O}(K)}{\otimes} \mathcal{O}_x \xrightarrow{\sim} \mathfrak{B}^{*}_{Y|X,x},$$

and the $\mathcal{D}$-modules $\mathfrak{B}^{*}_{Y|X}$ are coherent. To see that these modules are holonomic we just notice that the object of the complex $\Gamma_{\tilde{Y}(x)}(L^{\cdot}(\mathcal{O}_{\tilde{X}(x)}))$ have dimensions $n-1$ as $\mathcal{D}_x$-modules if we compute it by the Čech-Zariski cohomology.

§ 3. The De Rham Complex of $\boldsymbol{R}\Gamma_{[Y]}(\mathcal{D}_X)$.

We denote by T the tangent vector bundle of X and by $\Omega^{\cdot}_X = \Omega^{\cdot}$ the De Rham complex of X. The first Spencer sequence $\mathcal{D} \underset{\mathcal{O}_X}{\otimes} \Lambda^{\cdot}(T)$ is a projective resolution of $\mathcal{O}_X$ in the category of $\mathcal{D}$-modules. For any complex $\mathcal{M}^{\cdot}$ of $\mathcal{D}$-modules we have

$$\boldsymbol{R}\,\mathcal{H}om^{\cdot}_{\mathcal{D}}(\mathcal{O}_X;\mathcal{M}^{\cdot})\xrightarrow{\sim}\mathcal{H}om^{\cdot}_{\mathcal{D}}(\mathcal{D}\underset{\mathcal{O}_X}{\otimes}\Lambda^{\cdot}(T);\mathcal{M}^{\cdot})\xrightarrow{\sim}\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(\Lambda^{\cdot}(T);\mathcal{M}^{\cdot})$$
$$=\Omega^{\cdot}{}_{\mathcal{O}_X}\otimes\mathcal{M}^{\cdot}.$$

These isomorphisms hold in $D(\boldsymbol{C}_X)$. The complex $\boldsymbol{R}\,\mathcal{H}om^{\cdot}_{\mathcal{D}}(\mathcal{O}_X;\mathcal{M}^{\cdot})$ is called the De Rham complex of $\mathcal{M}^{\cdot}$ and is denoted by $DR(\mathcal{M}^{\cdot})$ By the Poincaré lemma we have $\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X;\mathcal{O}_X)\xrightarrow{\sim}\Omega^{\cdot}\xrightarrow{\sim}\boldsymbol{C}_X$ in $D(\boldsymbol{C}_X)$. The naturel injection $\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X)\to\boldsymbol{R}\Gamma_Y(\mathcal{O}_X)$ gives a morphism

$$(*)\qquad \boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X;\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X))\to\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X;\boldsymbol{R}\Gamma_Y(\mathcal{O}_X))$$
$$\xrightarrow{\sim}\boldsymbol{R}\Gamma_Y\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X,\mathcal{O}_X)\xrightarrow{\sim}\boldsymbol{R}\Gamma_Y(\boldsymbol{C}_X).$$

Theorem 3.1 *The composed morphism* $\boldsymbol{R}\,\mathcal{H}om^{\cdot}_{\mathcal{D}}(\mathcal{O}_X;\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X))\to\boldsymbol{R}\Gamma_Y(\boldsymbol{C}_X)$ *is an isomorphism in* $D(\boldsymbol{C}_X)$.

The question is local. We can suppose that $\mathcal{I}=(f_1,\cdots,f_q)$. Let $\mathcal{I}_1=(f_1,\cdots,f_{q-1})$, $\mathcal{I}_2=(f_q)$ and Y_1 and Y_2 the spaces defined by $\mathcal{I}_1$ and $\mathcal{I}_2$. We have $Y=Y_1\cap Y_2$ and $Y_1\cup Y_2$ is defined by $(f_1f_q,\cdots,f_{q-1}f_q)$. We have a triangle

$$(1)\qquad \begin{array}{ccc} & \boldsymbol{R}\Gamma_{[Y_1\cup Y_2]}(\mathcal{O}_X) & \\ {}^{+1}\swarrow & & \nwarrow \\ \boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X) & \longrightarrow & \boldsymbol{R}\Gamma_{[Y_1]}(\mathcal{O}_X)\oplus\boldsymbol{R}\Gamma_{[Y_2]}(\mathcal{O}_X) \end{array}$$

To see that (1) is a triangle in $D(\mathcal{D})$, it is enough to see it on each fiber because of the nature of the Cousin complex. But if $x\in X$ the triangle

$$\begin{array}{ccc} & \boldsymbol{R}\Gamma_{[Y_1\cup Y_2]}(\mathcal{O}_X),x & \\ {}^{+1}\swarrow & & \nwarrow \\ \boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X),_x & \longrightarrow & \boldsymbol{R}\Gamma_{[Y_1]}(\mathcal{O}_X),_x\oplus\boldsymbol{R}\Gamma_{[Y_2]}(\mathcal{O}_X),_x \end{array}$$

is just the Mayer-Vietoris sequence of the subspaces $\widetilde{Y}_1(x)$ and $\widetilde{Y}_2(x)$ in the scheme $\widetilde{X}(x)$. We can also use Artin-Rees lemma and cofinality. The functor $\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X;*)$ from $D(\mathcal{D})$ to $D(\boldsymbol{C}_X)$ is a ∂-functor and transforms triangle (1) into the triangle (2)

$$(2)\qquad \begin{array}{ccc} & DR(\boldsymbol{R}\Gamma_{[Y_1\cup Y_2]}(\mathcal{O}_X)) & \\ {}^{+1}\swarrow & & \nwarrow \\ DR(\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X)) & \longrightarrow & DR(\boldsymbol{R}\Gamma_{[Y_1]}(\mathcal{O}_X))\oplus DR(\boldsymbol{R}\Gamma_{[Y_2]}(\mathcal{O}_X)). \end{array}$$

The Mayer-Vietoris sequence of Y_1 and Y_2 in X gives the triangle (3)

$$\begin{array}{ccc} & \boldsymbol{R}\Gamma_{Y_1\cup Y_2}(\boldsymbol{C}_X) & \\ {}^{+1}\swarrow & & \nwarrow \\ \boldsymbol{R}\Gamma_Y(\boldsymbol{C}_X) & \longrightarrow & \boldsymbol{R}\Gamma_{Y_1}(\boldsymbol{C}_X)\oplus\boldsymbol{R}\Gamma_{Y_2}(\boldsymbol{C}_X) \end{array} \tag{3}$$

The morphism of the theorem 3.1 is a morphism of the triangle (2) to triangle (3). By induction on q the proof of the theorem 3.1 is reduced to the case $\mathcal{J}=(f)$. If $Y=f^{-1}(0)$ let U be $X\backslash Y$ and j the injection of U in X. In this case

$$\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X)=\mathfrak{B}^1_{Y|X}[-1]=\mathcal{O}_X[f^{-1}]/\mathcal{O}_X[-1]$$

We have the triangle in $D(\boldsymbol{C}_X)$

$$\begin{array}{ccc} & DR(\mathfrak{B}^1_{Y|X}) & \\ {}^{+1}\swarrow & & \nwarrow \\ DR(\mathcal{O}_X) & \longrightarrow & DR(\mathcal{O}_X[f^{-1}]) \end{array}$$

and the triangle in $D(\boldsymbol{C}_X)$

$$\begin{array}{ccc} & \boldsymbol{R}\Gamma_Y(\boldsymbol{C}_X)[1] & \\ {}^{+1}\swarrow & & \nwarrow \\ \boldsymbol{C}_X & \longrightarrow & \boldsymbol{R}j_*\boldsymbol{C}_U \end{array}$$

But the following composed morphism

$$DR(\mathcal{O}_X[f^{-1}])\to j_*\Omega^{\cdot}_U\xrightarrow{\sim}\boldsymbol{R}j_*\Omega^{\cdot}_U\to\boldsymbol{R}j_*\boldsymbol{C}_U$$

is an isomorphism by the Grothendieck's theorem [1]. Finally we get

$$\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X;\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X))\xrightarrow{\sim}\boldsymbol{R}\Gamma_Y(\boldsymbol{C}_X)$$

and the proof of theorem (3·1) is complete.

§ 4. Verdier Duality

Remember that a $\boldsymbol{C}$-analytic finitistic sheaf of $\boldsymbol{C}$ vector spaces ($\boldsymbol{C}$-analytiquement constructible in French) is a sheaf $\mathcal{F}$ of finite $\boldsymbol{C}$-vector spaces such that there exists a stratification $\bigcup_i X_i$ of X and the restriction $F|_{X_i}$ on each stratum is locally constant. By M. Kashiwara [2] the complex $\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\mathcal{M},\mathcal{O}_X)$ has finitistic cohomology if $\mathcal{M}$ is a $\mathcal{D}$-holonomic system. Because the category of finitistic sheaves is a stick subcategory of $\boldsymbol{C}$-vector spaces the complex $\boldsymbol{R}\ \mathcal{H}om_{\mathcal{D}}(\mathcal{M}^{\cdot}, \mathcal{O}_X)$ has finitistic cohomology if $\mathcal{M}^{\cdot}$ is a complex with $\mathcal{D}$-holonomic cohomology, We call a complex with finitistic cohomology a finitistic complex. So by the first part of theorem 1.1 the complex $\boldsymbol{R}\ \mathcal{H}om_{\mathcal{D}}(\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X); \mathcal{O}_X)$ is finitistic. Using the topological duality of Verdier and devissage [12] we can see that $\boldsymbol{R}\ \mathcal{H}om_{\boldsymbol{C}_X}(\mathcal{F}^{\cdot}, \boldsymbol{C}_X)$ is finitistic if $\mathcal{F}^{\cdot}$ is a finitistic complex and the natural morphism

$$\mathcal{F}^{\cdot} \to \boldsymbol{R}\ \mathcal{H}om_{\boldsymbol{C}_X}(\boldsymbol{R}\ \mathcal{H}om_{\boldsymbol{C}_X}(\mathcal{F}^{\cdot}, \boldsymbol{C}_X); \boldsymbol{C}_X)$$

is an isomorphism in $D(\boldsymbol{C}_X)$. The sheaf $\boldsymbol{C}_Y$ is finitistic and we have

$$\boldsymbol{C}_Y \xrightarrow{\sim} \boldsymbol{R}\ \mathcal{H}om_{\boldsymbol{C}_X}(\boldsymbol{R}\ \mathcal{H}om_{\boldsymbol{C}_X}(\boldsymbol{C}_Y; \boldsymbol{C}_X); \boldsymbol{C}_X).$$

We have also

$$\boldsymbol{R}\ \mathcal{H}om_{\mathcal{D}}(\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X); \mathcal{O}_X)$$
$$\xrightarrow{\sim} \boldsymbol{R}\ \mathcal{H}om_{\boldsymbol{C}_X}(\boldsymbol{R}\ \mathcal{H}om_{\boldsymbol{C}_X}(\boldsymbol{R}\ \mathcal{H}om_{\mathcal{D}}(\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X); \mathcal{O}_X); \boldsymbol{C}_X); \boldsymbol{C}_X)$$

To prove that $\boldsymbol{C}_Y \xrightarrow{\sim} \boldsymbol{R}\ \mathcal{H}om_{\mathcal{D}}(\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X); \mathcal{O}_X)$ it suffices to prove that $\boldsymbol{R}\ \mathcal{H}om_{\boldsymbol{C}_X}(\boldsymbol{C}_Y; \boldsymbol{C}_X) = \boldsymbol{R}\Gamma_Y(\boldsymbol{C}_X) \xrightarrow{\sim} \boldsymbol{R}\ \mathcal{H}om_{\boldsymbol{C}_X}(\boldsymbol{R}\ \mathcal{H}om_{\mathcal{D}}(\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X); \mathcal{O}_X); \boldsymbol{C}_X)$. By theorem 3.1 we have $\boldsymbol{R}\Gamma_Y(\boldsymbol{C}_X) = \boldsymbol{R}\ \mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X; \boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X))$ and we must prove that

$$\boldsymbol{R}\ \mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X; \boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X)) \xrightarrow{\sim} \boldsymbol{R}\ \mathcal{H}om_{\boldsymbol{C}_X}(\boldsymbol{R}\ \mathcal{H}om_{\mathcal{D}}(\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X); \mathcal{O}_X); \boldsymbol{C}_X).$$

This can be done by the following theorem which completes the proof of the theorem (1, 1):

Theorem 4.1 *Let $\mathcal{M}^{\cdot}$ be a complex with $\mathcal{D}$-holonomic cohomology then we have a canonical isomorphism in $D(\boldsymbol{C}_X)$*

$$\boldsymbol{R}\ \mathcal{H}om_{\mathcal{D}}(\mathcal{O}_X; \mathcal{M}^{\cdot}) \xrightarrow{\sim} \boldsymbol{R}\ \mathcal{H}om_{\boldsymbol{C}_X}(\boldsymbol{R}\ \mathcal{H}om_{\mathcal{D}}(\mathcal{M}; \mathcal{O}_X); \boldsymbol{C}_X).$$

§5. T.V.S. Homological Algebra

To prove theorem 4.1 and theorem 1.2 we need to define the functor $\boldsymbol{R}\ \mathcal{H}omtop^{\cdot}_{\boldsymbol{C}_X}(\mathcal{F}; \mathcal{G}^{\cdot})$ if $\mathcal{F}^{\cdot}$ and $\mathcal{G}^{\cdot}$ are complex of $\mathcal{D}_X$-module locally free with differential operators of finite order. Roughly speeking it is the "derived" functor of $\mathcal{H}omtop_{\boldsymbol{C}_X}(\mathcal{F}; \mathcal{G})$ which represents the continuous homomorphisms of Fréchet-nuclear sheaves. This category is not abelian. J. P. Ramis had noticed [10] and [11] that $\boldsymbol{C}_X$ is just the Libermann complex $\boldsymbol{B}^\cdot$ [5] and using the graded ring $\boldsymbol{B}^\cdot$ he could define $\boldsymbol{R}\,\mathcal{H}om top_{\boldsymbol{C}_X}(\mathcal{F};\mathcal{G})$. We do not give here the precise definition but we recall the formula

$$\boldsymbol{R}\,\mathcal{H}om top_{\boldsymbol{C}_X}(\mathcal{O}_X;\mathcal{O}_X)=\boldsymbol{R}\,\mathcal{H}om top_{\mathcal{O}_X}(\mathcal{J}_\infty;\mathcal{O}_X)$$

where $\mathcal{J}_\infty$ is the sheaf of infinite jets; see [9]. We list the properties of this functor in the following theorem:

Theorem 5.1 *We have the following isomorphisms;*

a) $\boldsymbol{R}\,\mathcal{H}om top_{\boldsymbol{C}_X}(\mathcal{O}_X;\mathcal{O}_X)\overset{\sim}{\to}\mathcal{D}^\infty$

b) $\boldsymbol{R}\,\mathcal{H}om top_{\boldsymbol{C}_X}(\mathcal{O}_X;\Omega^\cdot)\overset{\sim}{\to}\Omega^n[-n]$

c) $\boldsymbol{R}\,\mathcal{H}om top_{\boldsymbol{C}_X}(\mathcal{F}^\cdot;\mathcal{G})\overset{\sim}{\to}\boldsymbol{R}\,\mathcal{H}om_{\boldsymbol{C}_X}(\mathcal{F}^\cdot;\mathcal{G}^\cdot)$ if $\mathcal{F}^\cdot$ finitistic.

We can now prove theorem 4.1. The natural morphism

$$\mathcal{H}om_{\boldsymbol{C}_X}(\mathcal{F};\mathcal{G})\underset{\mathcal{D}}{\otimes}\mathcal{M}^\cdot\to\mathcal{H}om_{\boldsymbol{C}_X}(\mathcal{H}om_{\mathcal{D}}(\mathcal{M}^\cdot;\mathcal{F});\mathcal{G}),$$

where $\mathcal{F}^\cdot$ and $\mathcal{M}^\cdot$ are left $\mathcal{D}$-modules, gives rise to the morphism of functors

$$\boldsymbol{R}\,\mathcal{H}om_{\boldsymbol{C}_X}(\mathcal{F}^\cdot;\mathcal{G}^\cdot)\overset{L}{\underset{\mathcal{D}}{\otimes}}\mathcal{M}^\cdot\to\boldsymbol{R}\,\mathcal{H}om_{\boldsymbol{C}_X}(\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\mathcal{M}^\cdot;\mathcal{F}^\cdot);\mathcal{G}^\cdot).$$

Notice that the structure of right $\mathcal{D}$-module of $\mathcal{H}om_{\boldsymbol{C}_X}(\mathcal{F}^\cdot;\mathcal{G}^\cdot)$ comes from the structure of left $\mathcal{D}$-module $\mathcal{F}^\cdot$. This morphism of functors is an isomorphism if $\mathcal{M}^\cdot$ has $\mathcal{D}$-coherent cohomology by the way out "left" functor lemma. We have a natural morphism in $D(\boldsymbol{C}_X)$

$$\boldsymbol{R}\,\mathcal{H}om top_{\boldsymbol{C}_X}(\mathcal{O}_X;\Omega^\cdot)\to\boldsymbol{R}\,\mathcal{H}om_{\boldsymbol{C}_X}(\mathcal{O}_X;\Omega^\cdot)$$

which give a morphism by composition with the last one

$$(*)\qquad \boldsymbol{R}\,\mathcal{H}om top_{\boldsymbol{C}_X}(\mathcal{O}_X;\Omega^\cdot)\overset{L}{\underset{\mathcal{D}}{\otimes}}\mathcal{M}^\cdot\to\boldsymbol{R}\,\mathcal{H}om_{C_X}(\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\mathcal{M}^\cdot;\mathcal{O}_X);\Omega^\cdot).$$

Theorem 5.2 *Let $\mathcal{M}^\cdot$ a complex with $\mathcal{D}$-holonomic cohomology then* (*) *is an isomorphism in* $D(\boldsymbol{C}_X)$.

The question is local. We can suppose that $\mathcal{M}^\cdot$ is a single holonomic $\mathcal{D}$-module admitting a free resolution. In this case $\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\mathcal{M}\,;\mathcal{O}_X)$ is a complex of free $\mathcal{O}_X$-modules with differential being differential operators of finite order and $\boldsymbol{R}\,\mathcal{H}omtop_{\boldsymbol{C}_X}(\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\mathcal{M}\,;\mathcal{O}_X)\,;\Omega^{\cdot})$ has a meaning. The morphism (*) transit via the morphism $\boldsymbol{R}\,\mathcal{H}omtop_{\boldsymbol{C}_X} \times (\mathcal{O}_X;\ \Omega^{\cdot}) \overset{L}{\underset{\mathcal{D}}{\otimes}} \mathcal{M} \to \boldsymbol{R}\,\mathcal{H}omtop_{\boldsymbol{C}_X}(\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\mathcal{M}\,;\mathcal{O}_X)\,;\Omega^{\cdot})$. The last morphism is an isomorphism by the technique of the way out "left" functor lemma. The theorem 5.2 is a consequence of the property c) of theorem (5.1) because $\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\mathcal{M}^{\cdot};\mathcal{O}_X)$ is finitistic. In $D(\boldsymbol{C}_X)$ we have $\Omega^{\cdot} \xrightarrow{\sim} \boldsymbol{C}_X$ and $\boldsymbol{R}\,\mathcal{H}omtop_{\boldsymbol{C}_X}(\mathcal{O}_X;\Omega^{\cdot}) \xrightarrow{\sim} \Omega^n[-n]$, from the isomorphism (*) we have

$$\Omega^n \overset{L}{\underset{\mathcal{D}}{\otimes}} \mathcal{M}^{\cdot}[-n] \xrightarrow{\sim} \boldsymbol{R}\,\mathcal{H}om^{\cdot}{}_{\boldsymbol{C}_X}(\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\mathcal{M}.;\mathcal{O}_X);\boldsymbol{C}_X)$$

and the theorem 4.1 follows if we notice that

$$\Omega^n \overset{L}{\underset{\mathcal{D}}{\otimes}} \mathcal{M}^{\cdot}[-n] \xrightarrow{\sim} DR(\mathcal{M}^{\cdot}) = \boldsymbol{R}\,\mathcal{H}om^{\cdot}{}_{\mathcal{D}}(\mathcal{O}_X;\mathcal{M}^{\cdot}).$$

§ 6. Local Cohomology of Y.

The natural morphism of functors in the argument $\mathcal{M}^{\cdot}$

$$\boldsymbol{R}\,\mathcal{H}om_{\boldsymbol{C}_X}(\mathcal{O}_X;\mathcal{O}_X) \overset{L}{\underset{\mathcal{D}}{\otimes}} \mathcal{M}^{\cdot} \to \boldsymbol{R}\,\mathcal{H}om_{\boldsymbol{C}_X}(\boldsymbol{R}\,\mathcal{H}om^{\cdot}{}_{\mathcal{D}}(\mathcal{M}^{\cdot};\mathcal{O}_X);\mathcal{O}_X)$$

is an isomorphism if $\mathcal{M}^{\cdot}$ has $\mathcal{D}$-coherent cohomology by the way out left functor. The natural morphism

$$\boldsymbol{R}\,\mathcal{H}omtop_{\boldsymbol{C}_X}(\mathcal{O}_X;\mathcal{O}_X) \to \boldsymbol{R}\,\mathcal{H}om_{\boldsymbol{C}_X}(\mathcal{O}_X;\mathcal{O}_X)$$

gives the morphism

$$(*) \qquad \boldsymbol{R}\,\mathcal{H}omtop_{\boldsymbol{C}_X}(\mathcal{O}_X;\mathcal{O}_X) \overset{L}{\underset{\mathcal{D}}{\otimes}} \mathcal{M}^{\cdot} \to \boldsymbol{R}\,\mathcal{H}om_{\boldsymbol{C}_X}(\boldsymbol{R}\,\mathcal{H}om_{\mathcal{D}}(\mathcal{M}^{\cdot};\mathcal{O}_X);\mathcal{O}_X).$$

Proposition 6.1 *The morphism* (*) *is an isomorphism in* $D(\boldsymbol{C}_X)$ *if* $\mathcal{M}^{\cdot}$ *has* $\mathcal{D}$*-holonomic cohomology.*

The question is local. We can suppose that $\mathcal{M}^{\cdot}$ is a single $\mathcal{D}$-holonomic system admitting a free resolution. We finish the proof in the same way as in the last section We apply this situation to $\mathcal{M}^{\cdot} = \boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X)$. We have

$$\mathcal{D}^{\infty} \underset{\mathcal{D}}{\otimes} \boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X) \overset{\sim}{\to} \mathcal{D}^{\infty} \overset{L}{\underset{\mathcal{D}}{\otimes}} \boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X) \overset{\sim}{\to} \boldsymbol{R}\ \mathcal{H}om top_{\boldsymbol{C}_X}(\mathcal{O}_X; \mathcal{O}_X) \overset{L}{\underset{\mathcal{D}}{\otimes}} \mathcal{M}.$$

$$\overset{\sim}{\to} \boldsymbol{R}\ \mathcal{H}om_{\boldsymbol{C}_X}(\boldsymbol{R}\ \mathcal{H}om_{\mathcal{D}}(\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X); \mathcal{O}_X); \mathcal{O}_X)$$

and by theorem 1.1 $\boldsymbol{R}\ \mathcal{H}om_{\mathcal{D}}(\boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X); \mathcal{O}_X) \overset{\sim}{\to} \boldsymbol{C}_Y$. So

$$\mathcal{D}^{\infty} \underset{\mathcal{D}}{\otimes} \boldsymbol{R}\Gamma_{[Y]}(\mathcal{O}_X) \overset{\sim}{\to} \boldsymbol{R}\ \mathcal{H}om_{\boldsymbol{C}_X}(\boldsymbol{C}_Y; \mathcal{O}_X) = \boldsymbol{R}\Gamma_Y(\mathcal{O}_X).$$

The proof of theorem 1.2 is over. The details will appear elsewhere.

References

(S.K.K.) Sato, M., Kawai, T. and Kashiwara, M., Microfunctions and pseudo-differential equations, *Lecture note in Math.*, **287**, Springer, Heidelberg-New York 265–529, 1973.

[1] Grothendieck, A., On the De Rham Cohomology of algebraic varieties, *Publ. Math. I.H.E.S*; **29** (1966), 95–103.

[2] Kashiwara, M., On the maximally overdetermined systems of linear differential equation I*. *Publ. R.I.M.S. Kyoto Univ.* **10** (1975), 563–579.

[3] Kashiwara, M., Lettre à Malgrange Janvier (1975). On the rationality of the roots of *b*-functions.

[4] Libermann, D. and Herrera, M., Duatity and the De Rham Cohomology of infinitesimal neighborhoods, *Invent. Math.*, **13** (1971), 97–326.

[5] Libermann, D., Generalizations of the De Rham Complex with applications to duality theory and the cohomology of singular varieties, *Proc. conf. of Complex Analysis*, Rice, 1972.

[6] Malgrange, B., Pseudo-differentiels operateurs—*Seminaire Grenoble* (1976).

[7] Malgrange, B. et Ramis, J. P., (to appear)

[8] Ramis J. P. et Ruget. G., Complex dualisant en geometrie analytique, *Publ. I.H.E.S.*, **38**, 77–91 (1971).

[9] Ramis, J. P. et Ruget, G., Dualité et résidu, *Invent. Math.*, **26** Fasc 2, (1974), 89–131.

[10] Ramis, J. P., Lettre à Malgrange Janvier 1976)

[11] Ramis, J. P., Lettre à Verdier (Février 1976)

[12] Verdier, J. L., Classe d'homologie d'un cycle séminaire Douady-Verdier E.N.S. (1975).