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On an inequality in $l_p(\mathbb{C})$ involving Basel problem

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The following conjecture was proposed by Z. Retkes on the Open Problem Garden [2]:

Conjecture. *For all* $\alpha = (\alpha_1, \alpha_2, \ldots) \in l_2(\mathbb{C})$ *the following inequality holds*

$$\sum_{n\geq 1} |\alpha_n|^2 \geq \frac{6}{\pi^2} \sum_{k\geq 0} \Big| \sum_{l\geq 0} \frac{1}{l+1} \alpha_{2^k\cdot(2l+1)} \Big|^2. \tag{1}$$

The aim of this short note is to verify this conjecture as an application of the Basel problem together with the Cauchy–Schwarz inequality and give a possible extension to $\alpha \in l_p(\mathbb{C})$, $p > 1$. Since it was the Basel problem which helped us to figure out the proof of the conjecture quickly, we would like to discuss it in a bit more detail. The Basel

Der Riemannsche Umordnungssatz besagt, dass man die Summanden einer bedingt konvergenten Reihe so umsortieren kann, dass die neue Summe einen beliebig vorgeschriebenen Wert annimmt. Dies ist also wahr, wenn die Reihe der Absolutbeträge der Summanden nicht konvergiert. Ist die Reihe hingegen absolut konvergent, so hat das Umsortieren auf den Wert der Reihe keinen Einfluss. In der vorliegenden Arbeit zeigen die Autoren eine geschickte Umordnungsvariante mit möglicher Gewichtung der einzelnen Summanden auf. Insbesondere beweisen sie unter Verwendung von elementaren Techniken, dass die neue Summe durch die unendliche Summe (von Potenzen) der ursprünglichen Summanden beschränkt ist, wobei die Ungleichung eine multiplikative Konstante führt, die im Spezialfall mit Eulers Basler Problem zusammenhängt. Sie lösen damit eine Aufgabe aus dem *Open Problem Garden*.

problem is the problem of finding the exact value of the sum

$$\sum_{m\geq 1}\frac{1}{m^2},$$

which was posed by Pietro Mengoli in 1644 and solved by Leonhard Euler in 1735, who showed that the series sums to $\frac{\pi^2}{6}$. The problem is named after Basel, hometown of Leonhard Euler as well as of the Bernoulli family, who unsuccessfully attacked the problem. Nowadays, in literature one can find dozens of methods of calculating this series; for example, see [1], [3] and the references therein.

Proof of the conjecture. Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|_2$ denote the usual inner product and the norm, respectively, in the Hilbert space $l_2(\mathbb{C})$. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ be any element of $l_2(\mathbb{C})$ and $k \in \mathbb{N}_0$ be fixed. Consider the vectors

$$x = \left(1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\right), \quad \text{and} \quad y_k = (\alpha_{2^k}, \alpha_{2^k\cdot 3}, \alpha_{2^k\cdot 5}, \ldots).$$

We have

$$\sum_{k\geq 0}|\langle x, y_k\rangle|^2 = \sum_{k\geq 0}\Big|\sum_{l\geq 0}\frac{1}{l+1}\alpha_{2^k\cdot(2l+1)}\Big|^2. \tag{2}$$

Since $\alpha \in l_2(\mathbb{C})$, it is obvious that $y_k \in l_2(\mathbb{C})$ for each $k \in \mathbb{N}_0$. Moreover, $x \in l_2(\mathbb{C})$ since

$$\|x\|_2^2 = \sum_{l\geq 0}\frac{1}{(l+1)^2} = \frac{\pi^2}{6}.$$

Now the Cauchy–Schwarz inequality yields

$$\begin{aligned}\sum_{k\geq 0}|\langle x, y_k\rangle|^2 &\leq \sum_{k\geq 0}\|x\|_2^2\|y_k\|_2^2\\ &= \|x\|_2^2\sum_{k\geq 0}\|y_k\|_2^2 = \frac{\pi^2}{6}\sum_{k\geq 0}\|y_k\|_2^2.\end{aligned}$$

On the other hand

$$\begin{aligned}\sum_{k\geq 0}\|y_k\|_2^2 &= \sum_{k\geq 0}\sum_{l\geq 0}|\alpha_{2^k\cdot(2l+1)}|^2\\ &= \sum_{n\geq 1}|\alpha_n|^2,\end{aligned}$$

where we have used the elementary fact that every positive integer n can be written in the form $n = 2^s r$ uniquely, where $s \in \mathbb{N}_0$ and r is an odd positive integer. Therefore,

$$\sum_{k\geq 0}|\langle x, y_k\rangle|^2 \leq \frac{\pi^2}{6}\sum_{n\geq 1}|\alpha_n|^2. \tag{3}$$

Now (3) together with (2) implies (1), and we are done. □

It is easy to see from the above proof that (1) can be generalized further:

Let $x = (x_0, x_1, \dots) \in l_q(\mathbb{C})$ be a general sequence. Considering the q-norm of x, the p-norm of y_k and replacing the 2nd powers with pth, the Hölder inequality applied to each summand on the left-hand side of the identity (2) leads to the following result.

Theorem. *Let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for all $\alpha = (\alpha_1, \alpha_2, \dots) \in l_p(\mathbb{C})$ and $x = (x_0, x_1, \dots) \in l_q(\mathbb{C})$ the following inequality holds*

$$\sum_{n\geq 1} |\alpha_n|^p \geq \Big(\sum_{l\geq 0} |x_l|^q\Big)^{-\frac{p}{q}} \Big(\sum_{k\geq 0} \Big| \sum_{l\geq 0} x_l \alpha_{2^k(2l+1)} \Big|^p\Big).$$

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