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Effective bounds in E. Hopf rigidity for billiards and geodesic flows

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Abstract. In this paper we show that in some cases the E. Hopf rigidity phenomenon allows quantitative interpretation. More precisely, we estimate from above the measure of the set $\mathcal{M}$ swept by minimal orbits. These estimates are sharp, i.e. if $\mathcal{M}$ occupies the whole phase space we recover the E. Hopf rigidity. We give these estimates in two cases: the first is the case of convex billiards in the plane, sphere or hyperbolic plane. The second is the case of conformally flat Riemannian metrics on a torus. It seems to be a challenging question to understand such a quantitative bound for Burago–Ivanov theorem.

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1. Introduction and the result

In this paper we estimate from above the measure of the set $\mathcal{M}$ in the phase space which is occupied by minimal orbits of a Hamiltonian system. These bounds are of obvious importance for dynamics because all "rotational" invariant torii, as well as Aubry–Mather sets, are filled by minimal orbits.

These estimates provide the quantitative refinement of the E. Hopf rigidity. We prove these bounds for two Hamiltonian systems. The first system is a symplectic map of the cylinder corresponding to the billiard ball motion inside a convex curve γ lying on a surface Σ of constant curvature $0, \pm 1$. The second system is a geodesic flow on a torus with conformally flat Riemannian metric.

Nowadays there are many cases and approaches where E. Hopf rigidity phenomenon is established. It is an important problem to understand which ones can be made quantitative. In particular, it seems to be a challenging question whether it is possible to give a quantitative version for the Burago–Ivanov proof [5] of the E. Hopf conjecture.

Throughout the paper we denote by Ω the phase space of the Hamiltonian system in question. For the billiard in a convex domain bounded by closed curve γ, the phase

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space Ω is a cylinder: $\Omega = \gamma \times (-1, 1)$ equipped with the standard symplectic form $dx \wedge d(\cos\varphi)$ giving the invariant measure $d\mu = \sin\varphi dx d\varphi$. Here and later the billiard map will be denoted by T, x will denote arclength parameter on γ, and φ is an inward angle. As for geodesic flow on the torus, the phase space Ω is a unit tangent bundle $\Omega = T_1\mathbf{T}^n$ equipped with the Liouville measure.

We will use the following definition in this paper:

Definition 1.1. A geodesic will be called m-geodesic if it has no conjugate points. A billiard configuration $\{x_n\}$ will be called m-configuration if the second variation is negative definite between any two end points. The corresponding orbits in the phase space will be called m-orbits.

Here follow a couple of remarks explaining the definition. By Morse theory, for a geodesic to be without conjugate points is equivalent to having second variation positive definite between any two points. For billiards, any discrete Jacobi field along every m-configuration vanishes not more than once and moreover changes sign not more than once (see [1] and [8]).

We shall denote by $\mathcal{M} \subseteq \Omega$ the invariant subset of the phase space consisting of all m-orbits. It then follows that $\mathcal{M}$ is a closed set (see [13] for the discrete case).

We shall introduce the following notation for the portion of the phase space occupied by the set

$$\Delta = \Omega \setminus \mathcal{M}, \ \delta = \mu(\Delta)/\mu(\Omega),$$

where $\mu(\Omega)$ is the total measure of the phase space. Notice that the total measure equals $2P$ for the case of billiards (here and later P denotes the length of the boundary curve γ and A the area bounded by γ) and equals $\omega_{n-1}Vol_g(\mathbf{T}^n)$ for a Riemannian metric g on the torus (here and below ω_{n-1} is the volume of the standard unit sphere $\mathbf{S}^{n-1} \subset \mathbf{R}^n$). So by the definition, $\delta \in [0, 1]$ is dimensionless constant and the case $\delta = 0$ is the case when all the orbits are m-orbits, which corresponds to the rigidity case. The purpose of this paper is to estimate δ from below.

We first formulate the bounds for the case of billiards:

Theorem 1.2. *Let γ be a simple closed strictly convex curve on Σ. Denote by k_{min}, k_{max} the minimum and the maximum of the geodesic curvature function k. The following estimates hold true:*

1. *For the Euclidean plane, $\Sigma = \mathbf{R}^2$:*

$$\delta \geq \frac{\pi(P^2 - 4\pi A)}{4P(P + \sqrt{4\pi A})} \geq \frac{\pi(P^2 - 4\pi A)}{8P^2}, \tag{1.1}$$

and also

$$\delta \geq \frac{(P^2 - 4\pi A)k_{\min}}{8P}. \tag{1.2}$$

2. *For the Hemisphere, $\Sigma = \mathbf{S}^2$, for a curve γ lying entirely in the hemisphere:*

$$\delta \geq \frac{\pi}{2\arctan(\frac{1}{k_{\min}})} \frac{P^2 + A^2 - 4\pi A}{P(2\pi + \sqrt{P^2 + (2\pi - A)^2})}. \tag{1.3}$$

3. *For the Hyperbolic plane, $\Sigma = \mathbf{H}^2$, provided the boundary curve γ is convex with respect to horocycles, that is $k_{\min} > 1$:*

$$\delta \geq \frac{\pi}{2\operatorname{arctanh}(\frac{1}{\mathrm{k}_{\min}})} \frac{P^2 - A^2 - 4\pi A}{P(2\pi + \sqrt{(2\pi + A)^2 - P^2})}. \tag{1.4}$$

The following remarks are in order.

Remark 1.3. Notice that the numerators of the bounds of the theorem contain the isoperimetric defect and therefore $\delta = 0$ implies the curve γ is a circle on Σ. Moreover, it follows from Bonnesen type inequalities (see [4]) that for small δ the curve is close to a circle in the sense of Hausdorff distance.

I would also like to mention a somewhat related result of [9] where a quantitative version of a theorem by Mather is given estimating the area free from caustics inside the domain bounded by γ.

Remark 1.4. The estimate (1.1) uses the method of [1] where the Hopf rigidity for billiards was found. The bounds (1.2),(1.3),(1.4) on δ are obtained using the so called Mirror equation. The proof of E. Hopf rigidity for plane billiards using Mirror equation was obtained in [14] and later in [2] for the Sphere and Hyperbolic plane. Strangely the estimates (1.1) and (1.2) are incomparable, for some curves (1.1) is better and for others (1.2) is better. Let us mention that it remains unclear how to push the approach of [1] to work for Sphere and Hyperbolic plane.

Remark 1.5. Let me point out that in (1.4) for the $\mathbf{H}^2$ we need an extra assumption on γ to have $k_{\min} > 1$. For the case of rigidity when all the orbits are m-orbits this assumption is redundant as it is proven in [2]. However, in the general case it is not clear how to get rid of it.

Proof of Theorem 1.2 is given in Sections 2, 3.

Let me state now the result for geodesic flow. We consider Riemannian metric on the torus $\mathbf{T}^n = \mathbf{R}^n/\Gamma$ of the form $g = fg_0$ where g_0 is standard Euclidean metric on $\mathbf{R}^n$ and $f > 0$ is a conformal factor. Hopf rigidity in this case was proven in [12] (and later in [6] by another method) generalizing the original proof of E. Hopf [11] and L. Green [10]. Our purpose is to make their approach quantitative and to estimate the Liouville measure δ from below. To do this one needs a refinement of the original Hopf method, because a straightforward application of the method does not lead to any estimate on δ (it is especially clear for the case $n = 2$). For the proof below, some of the earlier ideas of [3] on rigidity of Newton equations are used.

We shall split the result into two cases, $n = 2$ and $n > 2$.

Theorem 1.6.

1. *For $n = 2$, let $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ be any positive smooth function. Denote by $\Psi(f) = \psi'(f)\left(\frac{4}{f} - \frac{\psi'(f)}{\psi(f)}\right)$. Then the following estimate holds true:*
$$\delta \geq \frac{\pi \int_{\mathbf{T}^2} \Psi(f)|grad_{g_0} f|^2_{g_0}\, dVol_{g_o}}{4\|K\|_{C^0}\|\psi(f)\|_{C^0} Vol(\mathbf{T}^2, g)},$$
where K is the curvature of the metric g.
2. *For $n > 2$, for any positive function $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ introduce*
$$\Psi(f) = \Psi(f) = f^{\frac{n}{2}-1}\psi'(f)\left(\frac{4}{f} - \frac{\psi'(f)}{\psi(f)}\right) + (n-2) f^{\frac{n}{2}-3}\psi(f).$$
Then the following estimate holds
$$\delta \geq \frac{(n-1)\omega_{n-1} \int_{\mathbf{T}^n} \Psi(f)|grad_{g_0} f|^2_{g_0}\, dVol_{g_o}}{4n\|Ric\|_{C^0}\|\psi(f)\|_{C^0} Vol(\mathbf{T}^n, g)},$$
where Ric stands for the Ricci tensor of g.

Obviously this statement makes sense only if Ψ is positive function. It turns out to be positive for many choices of ψ.

Corollary 1.7. *For the particular choice of $\psi(f) = f^{\alpha}$ we have:*

1. *For $n = 2$ and for every α in the range $0 < \alpha < 4$ it follows $\Psi(f) = \alpha(4-\alpha) f^{\alpha - 2}$ and thus*
$$\delta \geq \frac{\pi\alpha(4-\alpha) \int_{\mathbf{T}^2} f^{\alpha-2}(f^2_{x_1} + f^2_{x_2}) dx_1 dx_2}{4\|K\|_{C^0}\|f\|^{\alpha}_{C^0} \int f dx_1 dx_2}.$$
2. *For $n > 2$ and for every α in the range where $(n-2) + \alpha(4-\alpha) > 0$ it follows*
$$\Psi(f) = ((n-2) + \alpha(4-\alpha)) f^{\frac{n}{2}-3+\alpha},$$
and thus
$$\delta \geq ((n-2) + \alpha(4-\alpha)) \frac{(n-1)\omega_{n-1} \int_{\mathbf{T}^n} f^{\frac{n}{2}-3+\alpha}|grad_{g_0} f|^2_{g_0}\, dVol_{g_o}}{4n\|Ric\|_{C^0}\|f^{\alpha}\|_{C^0} Vol(\mathbf{T}^n, g)}.$$

Example 1.8. For $n = 2$ and $\alpha = 2$ one has
$$\delta \geq \frac{\pi \int_{\mathbf{T}^2} (f^2_{x_1} + f^2_{x_2}) dx_1 dx_2}{\|K\|_{C^0}\|f\|^2_{C^0} \int f dx_1 dx_2}.$$
As for $n > 2$ and $\alpha = 2$ one has:
$$\delta \geq \frac{(n+2)(n-1)\omega_{n-1} \int_{\mathbf{T}^n} f^{\frac{n}{2}-1}|grad_{g_0} f|^2_{g_0}\, dVol_{g_o}}{4n\|Ric\|_{C^0}\|f\|^2_{C^0} Vol(\mathbf{T}^n, g)}.$$

Remark 1.9. For both cases $n = 2$ and $n > 2$ and for $\alpha \geq 2$ one gets the strongest estimate in the Corollary for $\alpha = 2$, because then the value of $\alpha(4-\alpha)$ becomes maximal. Analogously, for the case $n > 2$ the estimate of the Corollary for $\alpha \leq 0$ is best possible for $\alpha = 0$. Thus the meaningful range for α in the Corollary is $\alpha \in (0, 2]$, for $n = 2$ and $\alpha \in [0, 2]$ for $n \geq 2$. Apart from these remarks, the estimates for different values of α seem to be incomparable. Let me also point out that unlike the case $n = 2$, for $n > 2$ the choice $\alpha = 0$ is allowed, and corresponds to the inequality considered by A. Knauf.

Proofs of Theorem 1.6 are given in Sections 4, 5.

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2. Estimates for planar billiards.

It follows from [1] that along any m-configuration one can construct a positive discrete Jacobi field and then by using this field is able to define a bounded measurable function $\omega : \mathcal{M} \to \mathbf{R}$, satisfying the inequality:

$$\omega(y, \psi) - \omega(x, \varphi) \geq L_{11}(x, y) + 2L_{12}(x, y) + L_{22}(x, y). \tag{2.1}$$

Both here and below $T : (x, cos\varphi) \mapsto (y, \cos\psi)$; L denotes the distance between $\gamma(x)$ and $\gamma(y)$; x is an arclength parameter on γ and subindexes of L stand for partial derivatives with respect to x, y respectively.

Integrating against the invariant measure μ inequality (2.1) over the set $\mathcal{M}$ of all m-orbits. We get:

$$0 \geq \int_{\mathcal{M}} (L_{11}(x, y) + 2L_{12}(x, y) + L_{22}(x, y)) d\mu.$$

After computation this leads to the inequality:

$$\int_{\mathcal{M}} \frac{(\sin\varphi + \sin\psi)^2}{L} d\mu \leq \int_{\mathcal{M}} (k(x)\sin\varphi + k(y)\sin\psi)\, d\mu. \tag{2.2}$$

The LHS of (2.2) can be estimated from below by Cauchy–Schwartz inequality and Santalo formulas:

$$\text{LHS} \geq \frac{(\int_{\mathcal{M}} (\sin\varphi + \sin\psi) d\mu)^2}{\int_{\mathcal{M}} L d\mu} \geq \frac{(2\int_{\mathcal{M}} \sin\varphi\, d\mu)^2}{\int_{\Omega} L d\mu} = \frac{(2\int_{\mathcal{M}} \sin\varphi\, d\mu)^2}{2\pi A}.$$

The RHS of (2.2) can be estimated:

$$\text{RHS} = 2\int_{\mathcal{M}} k(x)\sin\varphi\, d\mu \leq 2\int_{\Omega} k(x)\sin\varphi\, d\mu = 2\pi^2.$$

Therefore (2.2) gives the following

$$\frac{2\int_{\mathcal{M}} \sin\varphi \, d\mu}{\sqrt{2\pi A}} \leq \sqrt{2\pi^2}.$$

Therefore

$$2\int_{\mathcal{M}} \sin\varphi \, d\mu \leq \pi\sqrt{4\pi A}.$$

Estimating the left hand side of the last inequality we get:

$$\pi P - 4\delta P \leq 2\int_{\Omega} \sin\varphi \, d\mu - 2\int_{\Delta} \sin\varphi d\mu = 2\int_{\mathcal{M}} \sin\varphi \, d\mu \leq \pi\sqrt{4\pi A}.$$

Thus

$$\frac{\sqrt{4\pi A}}{P} \geq 1 - \frac{4}{\pi}\delta,$$

so that

$$\frac{\pi}{4}\left(1 - \frac{\sqrt{4\pi A}}{P}\right) \leq \delta.$$

Then

$$\frac{\pi}{4}\left(\frac{P^2 - 4\pi A}{P(2P)}\right) \leq \frac{\pi}{4}\left(\frac{P^2 - 4\pi A}{P(P + \sqrt{4\pi A})}\right) \leq \delta.$$

This proves (1.1).

In order to prove (1.2) we use another measurable function defined on the subset filled by m-orbits

$$a : \mathcal{M} \to \mathbf{R}, \ 0 < a(x,\varphi) < L(x,\varphi),$$

which is related in fact to the function ω discussed in the proof of (1.1)(see [2]). This function satisfies the Mirror equation for any point $(x,\varphi) \in \mathcal{M}$:

$$\frac{1}{a(x,\varphi)} + \frac{1}{L(T^{-1}(x,\varphi)) - a(T^{-1}(x,\varphi))} = \frac{2k(x)}{\sin\varphi}. \tag{2.3}$$

Then it follows

$$\frac{a(x,\varphi) + (L(T^{-1}(x,\varphi)) - a(T^{-1}(x,\varphi))}{2} \geq \frac{\sin\varphi}{k(x)}.$$

Integrate this inequality against the invariant measure $d\mu$ over the set $\mathcal{M}$. We have:

$$\frac{1}{2}\int_{\mathcal{M}} L d\mu \geq \int_{\mathcal{M}} \frac{\sin\varphi}{k(x)}. \tag{2.4}$$

The LHS of (2.4) can be estimated using Santalo formula:

$$\pi A = \frac{1}{2}\int_{\Omega} L d\mu \geq \frac{1}{2}\int_{\mathcal{M}} L d\mu.$$

And for the RHS using Cauchy–Schwartz inequality we have:

$$\int_{\mathcal{M}} \frac{\sin\varphi}{k(x)} = \int_{\Omega} \frac{\sin\varphi}{k(x)} - \int_{\Delta} \frac{\sin\varphi}{k(x)} \geq \frac{\pi}{2} \int_0^P \frac{1}{k(x)} dx - \frac{2\delta P}{k_{\min}} \geq \frac{\pi}{2} \cdot \frac{P^2}{2\pi} - \frac{2\delta P}{k_{\min}}.$$

Therefore (2.4) yields:

$$\pi A \geq \frac{P^2}{4} - \frac{2\delta P}{k_{\min}},$$

which is equivalent to (1.2). This completes the proof of (1.2).

3. Billiard on the Sphere and the Hyperbolic plane

The Mirror equation for billiards on Hemisphere and Hyperbolic plane is obtained in [2]. For the Hemisphere, there exists a measurable function

$$a : \mathcal{M} \to \mathbf{R},\ 0 < a(x,\varphi) < L(x,\varphi)$$

such that for any point $(x,\varphi) \in \mathcal{M}$ the following holds:

$$\cot\left(a(x,\varphi)\right) + \cot\left(L(T^{-1}(x,\varphi)) - a(T^{-1}(x,\varphi))\right) = \frac{2k(x)}{\sin\varphi}. \tag{3.1}$$

This implies:

$$\cot \frac{a(x,\varphi) + L(T^{-1}(x,\varphi)) - a(T^{-1}(x,\varphi))}{2} \leq \frac{k(x)}{\sin\varphi}.$$

Equivalently

$$\frac{a(x,\varphi) + L(T^{-1}(x,\varphi)) - a(T^{-1}(x,\varphi))}{2} \geq \arctan\left(\frac{\sin\varphi}{k(x)}\right).$$

Integrating over $\mathcal{M}$ with respect to the invariant measure $d\mu = \sin\varphi\, dx d\varphi$ we get:

$$\int_{\mathcal{M}} L\, d\mu \ \geq\ 2\int_{\mathcal{M}} \arctan\left(\frac{\sin\varphi}{k(x)}\right) d\mu. \tag{3.2}$$

For the LHS of (3.2) we have:

$$\int_{\mathcal{M}} L\, d\mu \leq \int_{\Omega} L\, d\mu = 2\pi A.$$

As for the RHS of (3.2), we compute and use the Gauss–Bonnet formula to get:

$$\begin{aligned}
&2\int_{\mathcal{M}} \arctan\left(\frac{\sin\varphi}{k(x)}\right) d\mu \\
&\quad = 2\int_{\Omega} \arctan\left(\frac{\sin\varphi}{k(x)}\right) d\mu - 2\int_{\Delta} \arctan\left(\frac{\sin\varphi}{k(x)}\right) d\mu \\
&\quad \geq 2\int_0^P dx \int_0^\pi \arctan\left(\frac{\sin\varphi}{k(x)}\right) \sin\varphi \, d\varphi - 4P\delta \arctan\left(\frac{1}{k_{\min}}\right) \\
&\quad = 2\pi \int_0^P (\sqrt{k^2(x)+1} - k(x))dx - 4P\delta \arctan\left(\frac{1}{k_{\min}}\right) \\
&\quad = 2\pi \int_0^P \sqrt{k^2(x)+1}dx - 2\pi(2\pi - A) - 4P\delta \arctan\left(\frac{1}{k_{\min}}\right).
\end{aligned}$$

Substitute now the estimates back into (3.2):

$$\int_0^P \sqrt{k^2(x)+1}dx \leq 2\pi + \frac{2P\delta}{\pi} \arctan\left(\frac{1}{k_{\min}}\right).$$

And then the following two inequalities follow. The first one:

$$\int_0^P (\sqrt{k^2(x)+1} - 1)dx \leq 2\pi - P + \frac{2P\delta}{\pi} \arctan\left(\frac{1}{k_{\min}}\right).$$

And the second:

$$\int_0^P (\sqrt{k^2(x)+1} + 1)dx \leq 2\pi + P + \frac{2P\delta}{\pi} \arctan\left(\frac{1}{k_{\min}}\right).$$

Multiplying the two and using Cauchy–Schwartz inequality we get:

$$(2\pi - A)^2 = \left(\int_0^P k(x)dx\right)^2 \leq \left(2\pi + \frac{2P\delta}{\pi} \arctan\left(\frac{1}{k_{\min}}\right)\right)^2 - P^2.$$

Therefore

$$\sqrt{P^2 + (2\pi - A)^2} \leq 2\pi + \frac{2P\delta}{\pi} \arctan\left(\frac{1}{k_{\min}}\right).$$

And thus

$$\begin{aligned}
\frac{P^2 - 4\pi A + A^2}{\sqrt{P^2 + (2\pi - A)^2} + 2\pi} &= \sqrt{P^2 + (2\pi - A)^2} - 2\pi \\
&\leq \frac{2P\delta}{\pi} \arctan\left(\frac{1}{k_{\min}}\right).
\end{aligned}$$

This is exactly (1.3), so the proof for the Hemisphere is finished.

For the Hyperbolic plane the proof is similar. Let us sketch the main steps. Let's first recall that for the Hyperbolic case we need an additional requirement $k(x) \geq k_{\min} > 1$. In particular this implies

$$P < k_{\min} P \leq \int_0^P k(x)dx = 2\pi + A.$$

We start again with a measurable function

$$a : \mathcal{M} \to \mathbf{R},\ 0 < a(x,\varphi) < L(x,\varphi)$$

such that for any point $(x,\varphi) \in \mathcal{M}$ the Mirror equation holds:

$$\coth\left(a(x,\varphi)\right) + \coth\left(L(T^{-1}(x,\varphi)) - a(T^{-1}(x,\varphi))\right) = \frac{2k(x)}{\sin\varphi}. \tag{3.3}$$

This leads to the inequality:

$$a(x,\varphi) + L(T^{-1}(x,\varphi)) - a(T^{-1}(x,\varphi)) \geq 2\text{arctanh}\left(\frac{\sin\varphi}{k(x)}\right).$$

Integrating over $\mathcal{M}$ we get

$$\int_{\mathcal{M}} L\ d\mu\ \geq\ 2\int_{\mathcal{M}} \text{arctanh}\left(\frac{\sin\varphi}{k(x)}\right) d\mu,$$

which leads to the inequality:

$$2\pi A \geq 2\int_{\mathcal{M}} L\ d\mu\ \geq \pi\int_0^P (k(x) - \sqrt{k^2(x)-1})dx - 4P\delta\text{arctanh}\left(\frac{1}{\mathrm{k}_{\min}}\right).$$

This implies using Gauss–Bonnet formula:

$$\int_0^P \sqrt{k^2(x)-1}dx \geq 2\pi - \frac{2P\delta}{\pi}\text{arctanh}\left(\frac{1}{\mathrm{k}_{\min}}\right).$$

By Cauchy Schwartz inequality we have:

$$\begin{aligned}\int_0^P \sqrt{k^2(x)-1}dx &\leq \left(\int_0^P (k(x)-1)dx \int_0^P (k(x)+1)dx\right)^{\frac{1}{2}}\\ &= \sqrt{(A+2\pi)^2 - P^2}.\end{aligned}$$

Thus we get:

$$\sqrt{(A+2\pi)^2 - P^2} \geq 2\pi - \frac{2P\delta}{\pi}\text{arctanh}\left(\frac{1}{\mathrm{k}_{\min}}\right).$$

This completes the proof.

4. Proof of the estimates for geodesic flows in $n = 2$.

The original E. Hopf method needs a modification in order to get bounds on the measure of the set of m-geodesics. This is accomplished as follows.

First, following E. Hopf, for every geodesic with no conjugate points one constructs by a limiting procedure, a positive solution of the Jacobi equation and then a measurable bounded function $\omega : \mathcal{M} \to \mathbf{R}$ which is smooth along the orbits of the geodesic flow satisfying the Riccati equation:

$$\dot{\omega} + \omega^2 + K = 0. \tag{4.1}$$

Here the derivative is taken in the direction of the vector of the geodesic flow in $T_1\mathbf{T}^2$, and K is the curvature of the conformal metric $g = f(dx_1^2 + dx_2^2)$. Let's recall that $\mathcal{M}$ is a closed subset of the phase space $\Omega = T_1\mathbf{T}^2$ invariant under the geodesic flow.

Multiplying both sides of the equation by a positive factor $\psi(f)$ we get:

$$\frac{d}{dt}(\psi\omega) - \omega\frac{d}{dt}(\psi(f)) + \psi(f)\omega^2 + \psi(f)K = 0. \tag{4.2}$$

Which leads to

$$\frac{d}{dt}(\psi(f)\omega) - \psi'(f)(f_{x_1}\dot{x_1} + f_{x_2}\dot{x_2})\omega + \psi(f)\omega^2 + \psi(f)K = 0. \tag{4.3}$$

For $T_1\mathbf{T}^2$ we have $\dot{x_1} = \frac{1}{\sqrt{f}}\cos\varphi, \dot{x_2} = \frac{1}{\sqrt{f}}\sin\varphi$ therefore

$$\frac{d}{dt}(\psi(f)\omega) - \psi'(f)\left(\frac{f_{x_1}}{\sqrt{f}}\cos\varphi + \frac{f_{x_2}}{\sqrt{f}}\sin\varphi\right)\omega + \psi(f)\omega^2 + \psi(f)K = 0. \tag{4.4}$$

Integrating the last equation over the set $\mathcal{M}$ against the Liouville measure $d\mu = f dx_1 dx_2 d\varphi$ and using its invariance under the geodesic flow we get:

$$-\int_{\mathcal{M}} \psi'\left(\frac{f_{x_1}}{\sqrt{f}}\cos\varphi + \frac{f_{x_2}}{\sqrt{f}}\sin\varphi\right)\omega d\mu + \int_{\mathcal{M}} \psi\omega^2 d\mu + \int_{\mathcal{M}} \psi K d\mu = 0. \tag{4.5}$$

Denote the first and the last term in equation (4.5) by A and C respectively. Then, by the Cauchy–Schwartz inequality for A we have:

$$\begin{aligned} A &\geq -\left(\int_{\mathcal{M}} \frac{(\psi')^2}{f\psi} \left(f_{x_1}\cos\varphi + f_{x_2}\sin\varphi\right)^2 d\mu\right)^{\frac{1}{2}} \left(\int_{\mathcal{M}} \psi\omega^2 d\mu\right)^{\frac{1}{2}} \\ &\geq -\left(\int_{\Omega} \frac{(\psi')^2}{f\psi} \left(f_{x_1}\cos\varphi + f_{x_2}\sin\varphi\right)^2 d\mu\right)^{\frac{1}{2}} \left(\int_{\mathcal{M}} \psi\omega^2 d\mu\right)^{\frac{1}{2}} \\ &= -\left(\pi \int_{\mathbf{T}^2} \frac{(\psi')^2}{\psi} (f_{x_1}^2 + f_{x_2}^2) dx_1 dx_2\right)^{\frac{1}{2}} \left(\int_{\mathcal{M}} \psi\omega^2 d\mu\right)^{\frac{1}{2}}. \end{aligned}$$

The third term C can be written as follows:

$$\begin{aligned} C &= \int_{\Omega} \psi K d\mu - \int_{\Delta} \psi K d\mu \\ &\geq \int_{\Omega} \psi K d\mu - \|\psi(f)\|_{C_0} \|K\|_{C_0} \mu(\Delta) \\ &= 2\pi \int_{\mathbf{T}^2} \psi(f) K f dx_1 dx_2 - \|\psi(f)\|_{C_0} \|K\|_{C_0} \cdot \delta \cdot Vol(\mathbf{T}^2, g). \end{aligned}$$

Substituting the explicit expression for $K = -\frac{\Delta(\log f)}{2f}$ and integrating by parts we get:

$$C \geq \pi \int_{\mathbf{T}^2} \frac{\psi'(f)}{f} (f_{x_1}^2 + f_{x_2}^2) dx_1 dx_2 - \|\psi(f)\|_{C_0} \|K\|_{C_0} \delta \cdot Vol(\mathbf{T}^2, g).$$

Using the estimates of the terms A and C in the equation (4.5) we have:

$$\begin{aligned} &-\left(\pi \int_{\mathbf{T}^2} \frac{(\psi')^2}{\psi} (f_{x_1}^2 + f_{x_2}^2) dx_1 dx_2\right)^{\frac{1}{2}} \cdot X + X^2 \\ &+ \pi \int_{\mathbf{T}^2} \frac{\psi'(f)}{f} (f_{x_1}^2 + f_{x_2}^2) dx_1 dx_2 - \|\psi(f)\|_{C_0} \|K\|_{C_0} \cdot \delta \cdot Vol(\mathbf{T}^2, g) \leq 0. \end{aligned} \tag{4.6}$$

Where we denoted X by

$$X = \left(\int_{\mathcal{M}} \psi\omega^2 d\mu\right)^{\frac{1}{2}}.$$

Next, notice that (4.6) is a quadratic inequality in X and therefore the discriminant must be non-negative:

$$\pi \int_{\mathbf{T}^2} \frac{(\psi')^2}{\psi}(f_{x_1}^2 + f_{x_2}^2)dx_1dx_2 - 4\pi \int_{\mathbf{T}^2} \frac{\psi'(f)}{f}(f_{x_1}^2 + f_{x_2}^2)dx_1dx_2 + 4\|\psi(f)\|_{C_0}\|K\|_{C_0} \cdot \delta \cdot Vol(\mathbf{T}^2, g) \geq 0.$$

And this leads to precisely the inequality which is claimed. This completes the proof for $n = 2$.

5. Proof of the estimates for geodesic flows in $n > 2$.

In this case we modify the approach of L. Green and A. Knauf in a way similar to what we did for the case $n = 2$. We start with a measurable bounded function (see [10] or [7] for the construction) $\omega : \mathcal{M} \to \mathbf{R}$ which satisfies the differential inequality:

$$\frac{d}{dt}\omega + \frac{\omega^2}{n-1} + R \leq 0,$$

where the derivative is along the geodesic flow and R is a function

$$R : \Omega \to \mathbf{R},\ R(v) = Ric(v, v).$$

Multiplying both sides of the inequality by a positive factor $\psi(f)$ we get:

$$\frac{d}{dt}(\psi\omega) - \psi'(f)\dot{f}\omega + \psi(f)\frac{\omega^2}{n-1} + \psi(f)R \leq 0. \tag{5.1}$$

Integrating against the invariant measure $d\mu = f^{\frac{n}{2}}dxdo$ over the set $\mathcal{M} \subseteq \Omega$ (where dx, do are the standard measures on Euclidean space and on the unit sphere) we get:

$$-\int_{\mathcal{M}} \psi'(f)\dot{f}\omega d\mu + \int_{\mathcal{M}} \frac{\psi\omega^2}{n-1}d\mu + \int_{\mathcal{M}} \psi R d\mu \leq 0. \tag{5.2}$$

We can now estimate the first term A and the last term C using the Cauchy–Schwartz inequality as follows:

$$
\begin{aligned}
A &\geq -\left(\int_{\mathcal{M}} \frac{(\psi')^2}{\psi}(\dot{f})^2 d\mu\right)^{\frac{1}{2}} \left(\int_{\mathcal{M}} \psi\omega^2 d\mu\right)^{\frac{1}{2}} \\
&\geq -\left(\int_{\Omega} \frac{(\psi')^2}{\psi}(\dot{f})^2 d\mu\right)^{\frac{1}{2}} \left(\int_{\mathcal{M}} \psi\omega^2 d\mu\right)^{\frac{1}{2}} \\
&= -\left(\int_{\Omega} \frac{(\psi')^2}{f\psi} < grad_{g_0} f, f\dot{x} >_{g_0}^2 f^{\frac{n}{2}} dx do\right)^{\frac{1}{2}} \left(\int_{\mathcal{M}} \psi\omega^2 d\mu\right)^{\frac{1}{2}} \\
&= -\left(\frac{\omega_{n-1}}{n} \int_{\mathbf{T}^n} \frac{(\psi')^2}{\psi} \|grad_{g_0} f\|_{g_0}^2 f^{\frac{n}{2}-1} dx\right)^{\frac{1}{2}} \left(\int_{\mathcal{M}} \psi\omega^2 d\mu\right)^{\frac{1}{2}}.
\end{aligned}
$$

For the last term C we have:

$$
\begin{aligned}
C &= \int_{\Omega} \psi R d\mu - \int_{\Delta} \psi R d\mu \geq \int_{\Omega} \psi R d\mu - \|\psi(f)\|_{C_0} \|R\|_{C_0} \mu(\Delta) \\
&= \frac{\omega_{n-1}}{n} \int_{\mathbf{T}^n} \psi(f) Scal(g) f^{\frac{n}{2}} dx - \|\psi(f)\|_{C_0} \|R\|_{C_0} \cdot \delta \cdot Vol(\mathbf{T}^n, g),
\end{aligned}
$$

where $Scal(g)$ is the Scalar curvature of g. Substituting the explicit expression for $Scal$,

$$
Scal(g) = (1-n) f^{-2} \Delta f + \frac{(1-n)(n-6)}{4} f^{-3} \|grad_{g_0} f\|_{g_0}^2,
$$

and integrating by parts the term with the Laplacian we get:

$$
\begin{aligned}
&C \geq \\
&\frac{\omega_{n-1}(1-n)}{n} \left(\int_{\mathbf{T}^n} \psi(f) f^{\frac{n}{2}-2} \Delta f dx + \int_{\mathbf{T}^n} \frac{(n-6)}{4} \psi(f) f^{\frac{n}{2}-3} \|grad_{g_0} f\|_{g_0}^2 dx\right) \\
&\qquad - \|\psi(f)\|_{C_0} \|R\|_{C_0} \cdot \delta \cdot Vol(\mathbf{T}^n, g) \\
&\geq \frac{\omega_{n-1}(1-n)}{n} \int_{\mathbf{T}^n} \left(-(\psi(f) f^{\frac{n}{2}-2})' + \frac{(n-6)}{4} \psi(f) f^{\frac{n}{2}-3}\right) \|grad_{g_0} f\|_{g_0}^2 dx \\
&\qquad - \|\psi(f)\|_{C_0} \|R\|_{C_0} \cdot \delta \cdot Vol(\mathbf{T}^n, g) =: \tilde{C}.
\end{aligned}
$$

Substituting into (5.2) the estimates on A, C and using the notation $X = \left(\int_{\mathcal{M}} \psi(f)\omega^2 d\mu\right)^{\frac{1}{2}}$ we get the quadratic inequality:

$$
-X \cdot \left(\frac{\omega_{n-1}}{n} \int_{\mathbf{T}^n} \frac{(\psi')^2}{\psi} \|grad_{g_0} f\|_{g_0}^2 f^{\frac{n}{2}-1} dx\right)^{\frac{1}{2}} + \frac{1}{n-1} X^2 + \tilde{C} \leq 0.
$$

Therefore the discriminant of this quadratic polynomial must be non-negative, which leads to

$$\frac{\omega_{n-1}}{n}\int_{\mathbf{T}^n}\frac{(\psi')^2}{\psi}\|grad_{g_0} f\|^2_{g_0} f^{\frac{n}{2}-1}dx - \frac{4}{n-1}\tilde{C} \geq 0.$$

Then

$$\frac{4}{n-1}\|\psi(f)\|_{C_0}\|R\|_{C_0}\cdot\delta\cdot Vol(\mathbf{T}^n, g)$$
$$\geq -\frac{\omega_{n-1}}{n}\int_{\mathbf{T}^n}\left(\frac{(\psi')^2}{\psi}f^{\frac{n}{2}-1} - 4(\psi(f)f^{\frac{n}{2}-2})' + (n-6)\psi(f)f^{\frac{n}{2}-3}\right)$$
$$\cdot\|grad_{g_0} f\|^2_{g_0}dx.$$

By the definition of $\Psi(f)$ this is equivalent to:

$$\frac{4}{n-1}\|\psi(f)\|_{C_0}\|R\|_{C_0}\cdot\delta\cdot Vol(\mathbf{T}^n, g) \geq \frac{\omega_{n-1}}{n}\int_{\mathbf{T}^n}\Psi(f)\|grad_{g_0} f\|^2_{g_0}dx.$$

This proves the claim for $n > 2$.

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