

## Current twisting and nonsingular matrices

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**Abstract.** We show that for  $k \geq 3$ , given any matrix in  $\mathrm{GL}(k, \mathbb{Z})$ , there is a hyperbolic fully irreducible automorphism of the free group of rank  $k$  whose induced action on  $\mathbb{Z}^k$  is the given matrix.

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### 1. Introduction

Considerable progress has been made in understanding the dynamics of elements of the outer automorphism group of a nonabelian free group of rank  $k$ ,  $\mathrm{Out} F_k$ , by considering the corresponding situation for the mapping class group of a compact oriented surface of genus  $g$ ,  $\mathrm{MCG}(S_g)$ . Indeed, some of the most fruitful examples of this pedagogy include the Culler–Vogtmann Outer space  $CV_k$  [16], as well as the Bestvina–Handel train-track representatives [7].

As a consequence of the Thurston classification of elements in  $\mathrm{MCG}(S_g)$ , the most important elements to understand are the *pseudo-Anosov mapping classes* [31]. Such elements are characterized as those mapping classes for which no isotopy class of a simple closed curve in  $S_g$  is periodic. If a mapping class fixes the isotopy class of a simple closed curve, then it restricts to a mapping class on the subsurface obtained by cutting along the simple closed curve. In general, if  $f \in \mathrm{MCG}(S_g)$ , then  $S_g$  decomposes into subsurfaces (which only intersect along their boundaries) such that for some  $n$ , the element  $f^n$  can be represented by a homeomorphism that restricts to each subsurface as either the identity or a pseudo-Anosov map and acts as a Dehn twist in a neighborhood of intersection of the subsurfaces.

An element  $\phi \in \mathrm{Out} F_k$  is *fully irreducible*, also called *irreducible with irreducible powers (iwip)*, if no conjugacy class of a proper free factor of  $F_k$  is periodic. As above, if  $\phi$  is not fully irreducible, then  $F_k$  has a free factor  $F_{k'}$  such that for some  $n$ , the element  $\phi^n$  restricts to an element of  $\mathrm{Out} F_{k'}$ . However, it is not the case that  $\phi^n$

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preserves some free factorization of  $F_k$ . The dynamics of iterating a fully irreducible element on a conjugacy class of an element of  $F_k$  are similar to the dynamics of iterating a pseudo-Anosov mapping class on a simple closed curve [7].

Thurston also characterized pseudo-Anosov mapping classes as those elements  $f \in \text{MCG}(S_g)$  whose mapping torus  $S_g \times [0, 1]/(x, 0) \sim (f(x), 1)$  admits a hyperbolic metric [31]. However the analogous characterization for fully irreducible elements does not hold as the mapping torus  $F_k \rtimes_{\Phi} \mathbb{Z}$  is not necessarily a hyperbolic group when  $\Phi \in \text{Aut } F_k$  represents a fully irreducible element of  $\text{Out } F_k$ . Automorphisms of  $F_k$  such that the mapping torus  $F_k \rtimes_{\Phi} \mathbb{Z}$  is hyperbolic are precisely those for which no nontrivial element of  $F_k$  is periodic [3], [10], [18]. Using this correspondence, we say an element  $\phi \in \text{Out } F_k$  is *hyperbolic* if no conjugacy class of a nontrivial element of  $F_k$  is periodic. In the literature, such elements have also been called *atoriodal*. We remark that there are hyperbolic elements that are not fully irreducible and fully irreducible elements that are not hyperbolic. However, fully irreducible elements that are not hyperbolic have a power that is realized by a pseudo-Anosov mapping class on a surface with a single boundary component [7]. When  $k = 2$ , no element of  $\text{Out } F_k$  is hyperbolic as  $\text{Out } F_2 \cong \text{MCG}^{\pm}(S_{1,1})$  where  $S_{1,1}$  is the torus with a single puncture.

One method to understand an element of  $\text{MCG}(S_g)$  is to examine its action on the first homology of the surface,  $H_1(S_g, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ . Any such element preserves the algebraic intersection number between curves on  $S_g$ , giving the short exact sequence

$$1 \rightarrow \mathcal{I}_g \rightarrow \text{MCG}(S_g) \xrightarrow{f \mapsto f_*} \text{Sp}(2g, \mathbb{Z}) \rightarrow 1.$$

Similarly, the action of an outer automorphism on  $H_1(F_k, \mathbb{Z}) \cong \mathbb{Z}^k$  leads to the following short exact sequence:

$$1 \rightarrow \text{IA}_k \rightarrow \text{Out } F_k \xrightarrow{\phi \mapsto \phi_*} \text{GL}(k, \mathbb{Z}) \rightarrow 1.$$

There are various homological criteria that ensure that a given element of the mapping class group is pseudo-Anosov [11], [25], [27] or, in the free group setting, that a given element of  $\text{Out } F_k$  is hyperbolic and fully irreducible [19]. The main goal of this paper is to generalize to the free group setting a theorem of Papadopoulos showing that there is no homological obstruction for an element to be pseudo-Anosov [29], i.e., for any  $A \in \text{Sp}(2g, \mathbb{Z})$ , there is a pseudo-Anosov mapping class  $f \in \text{MCG}(S)$  such that  $f_* = A$ .

**Theorem 6.1.** *Suppose  $k \geq 3$ . For any  $A \in \text{GL}(k, \mathbb{Z})$ , there is a hyperbolic fully irreducible outer automorphism  $\phi \in \text{Out } F_k$  such that  $\phi_* = A$ .*

**Remark 1.1.** For  $k = 2$ , the function  $\phi \mapsto \phi_*$  is an isomorphism and hence there are matrices  $A \in \text{GL}(2, \mathbb{Z})$  that are not represented by fully irreducible automorphisms.

Papadopoulos relies on the characterization of pseudo-Anosov mapping classes in terms of their dynamics on the Thurston boundary of Teichmüller space. The Teichmüller space for a surface  $S_g$  is the space of marked hyperbolic structures on  $S_g$ ; Thurston compactified Teichmüller space using the space of projectivized measured laminations. Pseudo-Anosovs are precisely the mapping classes with exactly two fixed points in the compactified Teichmüller space [31]. Using this characterization, Papadopoulos shows that if  $f, h \in \text{MCG}(S_g)$  where  $f$  is pseudo-Anosov and  $f$  and  $h$  satisfy an additional hypothesis, then for large enough  $m$ , the mapping class  $f^m h$  is pseudo-Anosov [29].

Our approach for proving Theorem 6.1 is similar to that of Papadopoulos. Namely, we show that if  $\phi$  is hyperbolic and fully irreducible, and  $\phi$  and  $\psi \in \text{Out } F_k$  satisfy a certain hypothesis, then for large enough  $m$ , the element  $\phi^m \psi$  is hyperbolic and fully irreducible (Propositions 3.1 and 4.5). As such, one needs a space where the dynamics of an element dictate its type, as with the action of a pseudo-Anosov on the Thurston boundary of Teichmüller space.

Since the properties of being hyperbolic and of being fully irreducible are independent, it is perhaps of no surprise that two different spaces are used in verifying each property for  $\phi^m \psi$ . We consider the action on the space of measured geodesic currents,  $\text{Curr}(F_k)$ , as defined by Bonahon [8] (Section 2.4). This space is the completion of the space of conjugacy classes for  $F_k$ , and thus is natural for testing hyperbolicity. We also consider a new complex defined by Bestvina and Feighn for  $\text{Out } F_k$  that has the useful property of being  $\delta$ -hyperbolic [5] (Section 2.5). Stabilizers in  $\text{Out } F_k$  of conjugacy classes of proper free factors have bounded orbits in this complex, and thus it provides a natural setting for checking fully irreducibility.

Once we establish that  $\phi^m \psi$  is a hyperbolic fully irreducible element under a certain hypothesis, our problem is reduced to finding for any  $\psi \in \text{Out } F_k$  a hyperbolic fully irreducible element  $\phi \in \text{IA}_k$  which, together with  $\psi$ , satisfies the hypothesis. To build such elements we apply a construction from our earlier work [12]; namely, we use *Dehn twist automorphisms* to build customized hyperbolic fully irreducible elements of  $\text{Out } F_k$ . Satisfying the hypothesis then requires that we understand the stable and unstable currents in  $\mathbb{P}\text{Curr}(F_k)$  associated to a product of Dehn twists. This is our other main result, with definitions appearing in Section 2.

**Theorem 5.2.** *Let  $T_1$  and  $T_2$  be very small cyclic trees that fill, with edge stabilizers  $c_1$  and  $c_2$ , and with associated Dehn twist automorphisms  $\delta_1$  and  $\delta_2$ . Let  $N \geq 0$  be such that for  $n \geq N$ , we have that  $\delta_1^n \delta_2^{-n}$  is a hyperbolic fully irreducible outer automorphism with stable and unstable currents  $[\mu_+^n]$  and  $[\mu_-^n]$  in  $\mathbb{P}\text{Curr}(F_k)$ . Then*

$$\lim_{n \rightarrow \infty} [\mu_+^n] = [\eta_{c_1}] \quad \text{and} \quad \lim_{n \rightarrow \infty} [\mu_-^n] = [\eta_{c_2}].$$

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application of our construction of hyperbolic fully irreducible outer automorphisms. We are also grateful to the referee for thoughtful and interesting suggestions concerning our results.

## 2. Preliminaries

**2.1. Bounded cancellation.** When working with free groups, the following lemma due to Cooper is indispensable. For a basis  $\mathcal{A}$ , let  $|x|_{\mathcal{A}}$  denote the word length of  $x \in F_k$  with respect to  $\mathcal{A}$  and  $\ell_{\mathcal{A}}(x)$  the length of the cyclic word determined by  $x$ .

**Lemma 2.1** ([14], Bounded cancellation lemma). *Suppose  $\mathcal{A}$  and  $\mathcal{B}$  are bases for the free group  $F_k$ . There is a constant  $C = C(\mathcal{A}, \mathcal{B})$  such that if  $w$  and  $w'$  are two elements of  $F_k$ , where*

$$|w|_{\mathcal{A}} + |w'|_{\mathcal{A}} = |ww'|_{\mathcal{A}},$$

then

$$|w|_{\mathcal{B}} + |w'|_{\mathcal{B}} - |ww'|_{\mathcal{B}} \leq 2C.$$

We denote by  $\text{BCC}(\mathcal{A}, \mathcal{B})$  the bounded cancellation constant; that is, the minimal constant  $C$  satisfying the lemma for  $\mathcal{A}$  and  $\mathcal{B}$ . In other words, if  $ww'$  is a reduced word in  $\mathcal{A}$ , and we can write  $w = \prod_{i=1}^m x_i$  and  $w' = \prod_{i=1}^{m'} x'_i$  where  $x_i, x'_i \in \mathcal{B}$ , then for  $C = \text{BCC}(\mathcal{A}, \mathcal{B})$  the subwords  $x_1 \cdots x_{m-C-1}$  and  $x'_{C+1} \cdots x'_{m'}$  appear as subwords of  $ww'$  when considered as a word in  $\mathcal{B}$ . Applying the bounded cancellation lemma to  $w^2$  where  $w$  is a cyclically reduced word with respect to  $\mathcal{A}$ , we see that  $w$  is “almost cyclically reduced” with respect to  $\mathcal{B}$ , i.e.,  $w = zxz^{-1}$  where  $x$  is cyclically reduced with respect to  $\mathcal{B}$  and  $|z|_{\mathcal{B}} \leq \text{BCC}(\mathcal{A}, \mathcal{B})$ .

**2.2. Culler–Vogtmann Outer space.** Equally indispensable to the study of  $\text{Out } F_k$  is the *Culler–Vogtmann Outer space*  $CV_k$  [16]. This is the projectivized space of minimal discrete free actions of  $F_k$  on  $\mathbb{R}$ -trees and is analogous to the Teichmüller space for a surface. There is a compactification  $\overline{CV}_k$  [15] that is precisely the projectivized space of minimal very small actions of  $F_k$  on  $\mathbb{R}$ -trees [4], [13]. Recall that an action on an  $\mathbb{R}$ -tree is minimal if there is no invariant subtree; it is very small if the stabilizer of an arc is either trivial or a maximal cyclic subgroup, and if the stabilizer of any tripod is trivial. We consider the unprojectivized versions  $cv_k$  and  $\overline{cv}_k$  as well.

The group  $\text{Out } F_k$  acts on either of the above spaces on the right by pre-composing the action homomorphism. Fully irreducible elements act on  $\overline{CV}_k$  with North-South dynamics.

**Theorem 2.2** ([26], Theorem 1.1). *Every fully irreducible element  $\phi \in \text{Out } F_k$  acts on  $\overline{CV}_k$  with exactly two fixed points  $[T_+]$  and  $[T_-]$ . Further, for any  $[T] \in \overline{CV}_k$  such*

that  $[T] \neq [T_-]$  it holds that

$$\lim_{m \rightarrow \infty} [T\phi^m] = [T_+].$$

The trees  $[T_+]$  and  $[T_-]$  are called the *stable* and *unstable* trees of  $\phi$  respectively. The stable and unstable trees of  $\phi^{-1}$  are  $[T_-]$  and  $[T_+]$ , respectively.

**2.3. Dehn twists.** As mentioned in the introduction, we build customized hyperbolic fully irreducible elements of  $\text{Out } F_k$  using Dehn twist automorphisms. These are defined analogously to a Dehn twist homeomorphism of a surface. Specifically, given a splitting  $F_k = A *_c B$ , we define an automorphism by

$$\begin{aligned} \delta(a) &= a && \text{for all } a \in A, \\ \delta(b) &= bcb^{-1} && \text{for all } b \in B. \end{aligned}$$

The automorphism  $\delta$  acts trivially on homology and therefore belongs to the subgroup  $\text{IA}_k$ . A Dehn twist automorphism arising from amalgamations over  $\mathbb{Z}$  is analogous to a Dehn twist around a separating simple closed curve on a surface.

We similarly obtain an automorphism  $\delta$  from an HNN-extension of the form

$$F_k = A *_\mathbb{Z} = \langle A, t \mid t^{-1}a_0t = a_1 \rangle$$

for  $a_0, a_1 \in A$  by

$$\begin{aligned} \delta(a) &= a && \text{for all } a \in A, \\ \delta(t) &= a_0t. \end{aligned}$$

An automorphism arising from an HNN-extension should be compared to a Dehn twist around a nonseparating curve on a surface.

From Bass–Serre theory, a splitting of  $F_k$  over  $\mathbb{Z}$  defines an action of  $F_k$  on a tree  $T$ , the *Bass–Serre tree* of the splitting (see [2] or [30]). We will refer to such  $F_k$ -trees as *cyclic*. In a certain sense, cyclic trees for  $F_k$  correspond to simple closed curves on a surface; as in the mapping class group, the Dehn twist automorphisms associated to cyclic trees generate an index two subgroup of  $\text{Aut } F_k$  (the subgroup which induces an action of  $\text{SL}_k(\mathbb{Z})$  on homology). Note that if  $\delta$  is the Dehn twist automorphism associated to the cyclic tree  $T$ , then  $\delta$  preserves the action of  $F_k$  on  $T$ , i.e., there is an isometry  $h_\delta: T \rightarrow T$  such that for all  $g \in F_k$  and all  $x \in T$  we have  $h_\delta(gx) = \delta(g)h_\delta(x)$ . In particular,  $\ell_T(\delta(g)) = \ell_T(g)$  for all  $g \in F_k$ .

We are primarily interested in the *outer* automorphism group of  $F_k$ , and so in the sequel a Dehn twist will refer to an element of  $\text{Out } F_k$  which is induced by a Dehn twist automorphism in  $\text{Aut } F_k$ .

The role of intersection number of simple closed curves is played by *free volume*.

**Definition 2.3** (Free volume). Suppose  $X$  is a finitely generated free group that acts on a simplicial tree  $T$  such that the stabilizer of an edge is either trivial or cyclic. The *free volume*  $\text{vol}_T(X)$  of  $X$  with respect to  $T$  is the number of edges in the graph of groups decomposition  $T^X/X$  with trivial stabilizer. Here  $T^X$  denotes the smallest  $X$ -invariant subtree.

In the case that  $X = \langle x \rangle$ , the free volume  $\text{vol}_T(X)$  is just the *translation length*  $\ell_T(x)$  of  $x$  in  $T$ .

We say two cyclic trees *fill* if

$$\text{vol}_{T_1}(X) + \text{vol}_{T_2}(X) > 0$$

for every proper free factor or cyclic subgroup  $X \subset F_k$ . With these notions we have shown the following analog to a classical theorem of Thurston:

**Theorem 2.4** ([12], Theorem 5.3). *Let  $\delta_1$  and  $\delta_2$  be the Dehn twist automorphisms of  $F_k$  for two filling cyclic trees of  $F_k$ . Then there exists  $N = N(\delta_1, \delta_2)$  such that for all  $m, n \geq N$ :*

- (1)  $\langle \delta_1^m, \delta_2^n \rangle$  is isomorphic to the free group on two generators; and
- (2) if  $\phi \in \langle \delta_1^m, \delta_2^n \rangle$  is not conjugate to a power of either  $\delta_1^m$  or  $\delta_2^n$ , then  $\phi$  is a hyperbolic fully irreducible element of  $\text{Out } F_k$ .

Key to our analysis in [12] and Section 5 of the present paper is the following theorem, which measures how the free volume changes upon twisting.

**Theorem 2.5** ([12], Theorem 4.6). *Let  $\delta_2$  be a Dehn twist automorphism corresponding to a very small cyclic tree  $T_2$  with cyclic edge generator  $c_2$ , and let  $T_1$  be any other very small cyclic tree. Then there is a constant  $C = C(T_1, T_2)$  such that for any  $x \in F_k$  and  $n \geq 0$  the following hold:*

$$\ell_{T_1}(\delta_2^{\pm n}(x)) \geq \ell_{T_2}(x)[n\ell_{T_1}(c_2) - C] - \ell_{T_1}(x), \tag{2.1}$$

$$\ell_{T_1}(\delta_2^{\pm n}(x)) \leq \ell_{T_2}(x)[n\ell_{T_1}(c_2) + C] + \ell_{T_1}(x). \tag{2.2}$$

These bounds are shown in [12] to hold not only for cyclic subgroups, but for any finitely generated malnormal subgroup of  $F_k$ ; in particular any proper free factor of  $F_k$ .

We will also need the following notions from [12] for Section 5.

Suppose that  $T$  is a very small cyclic tree for an amalgamated free product  $F_k = A *_c B$ . After possibly interchanging  $A \leftrightarrow B$ , there is a basis  $\mathcal{T} = \mathcal{A} \cup \mathcal{B}$  for  $F_k$  such that  $c \in \mathcal{A}$ , and such that  $\mathcal{A}$  is a basis for  $A$  and  $\mathcal{B} \cup \{c\}$  is a basis for  $B$ . Such a basis is called a *basis relative to  $T$* . If  $x \in F_k$  and  $\ell_T(x) = 2m > 0$ , then  $x$  is conjugate to a cyclically reduced word of the form

$$x_1 c^{i_1} y_1 c^{j_1} \dots x_m c^{i_m} y_m c^{j_m}$$

where for  $s = 1, \dots, m$ , each  $y_s$  is a word in  $\mathcal{B}$ , each  $x_s$  a word in  $\mathcal{A}$ , such that both  $zx_s$  and  $x_s z$  are reduced for  $z = c, c^{-1}$ .

Now suppose that  $T$  is a very small cyclic tree for an HNN-extension  $F_k = A*_{\langle tc't^{-1}=c \rangle}$ . After possibly interchanging  $A \leftrightarrow tAt^{-1}$ , there is a basis  $\mathcal{A} \cup \{t_0\}$  for  $F_k$  such that  $t = t_0 a$  for some  $a \in A$ ,  $c \in \mathcal{A}$  and  $\mathcal{A} \cup \{t_0^{-1}ct_0\}$  is a basis for  $A$ . If  $x \in F_k$  and  $\ell_T(x) = m > 0$ , then  $x$  is conjugate to a cyclically reduced word of the form

$$x_1(c^{i_1}t_0)^{\epsilon_1}x_2(c^{i_2}t_0)^{\epsilon_2} \dots x_m(c^{i_m}t_0)^{\epsilon_m}$$

where for  $s = 1, \dots, m$ ,  $x_s$  is a word in  $\mathcal{A} \cup \{t_0^{-1}ct_0\}$ ,  $\epsilon_s \in \{\pm 1\}$ ; and if  $\epsilon_s = 1$ , then  $x_s z$  is a reduced word for  $z = c, c^{-1}$ ; and if  $\epsilon_r = -1$  then  $zx_{s+1}$  is a reduced word for  $z = c, c^{-1}$ .

In either of two above cases, we say that the specific word is *T-reduced*.

**2.4. Currents.** Measured geodesic currents for hyperbolic groups were first defined by Bonahon [8]. Recently, (measured geodesic) currents for free groups have seen much activity through the work of Kapovich and Lustig [21], [23], [24], [22]. We briefly introduce the parts of the theory needed for the sequel; see [21] for further details.

The group  $F_k$  is hyperbolic and hence has a boundary  $\partial F_k$ . We denote

$$\partial^2 F_k = \{(x_1, x_2) \in \partial F_k \times \partial F_k \mid x_1 \neq x_2\}.$$

This is naturally identified with the space of oriented geodesics in a Cayley tree for  $F_k$ . There is fixed-point free involution “flip” map  $\sigma: \partial^2 F_k \rightarrow \partial^2 F_k$  defined by  $\sigma(x_1, x_2) = (x_2, x_1)$  which corresponds to reversing the orientation on the geodesic.

A (*measured geodesic*) *current* on  $F_k$  is an  $F_k$ -invariant positive Radon measure on  $\partial^2 F_k / \sigma$ . The set  $\text{Curr}(F_k)$  is the set of all currents on  $F_k$ , topologized with the weak topology. There is an action of  $\mathbb{R}_{>0}$  on  $\text{Curr}(F_k) - \{0\}$ , and the quotient  $\mathbb{P}\text{Curr}(F_k)$  is a compact space. There is a continuous left action of  $\text{Out } F_k$  on  $\text{Curr}(F_k)$  and  $\mathbb{P}\text{Curr}(F_k)$  defined by  $\phi v(S) = \nu(\phi^{-1}(S))$ , where  $\phi \in \text{Out } F_k$ ,  $\nu \in \text{Curr}(F_k)$ , and where  $S$  is a measurable set of  $\partial^2 F_k / \sigma$ . There is a slight abuse of notation here as strictly speaking  $\phi^{-1}(S)$  is not well-defined. But for any two  $\Phi_0, \Phi_1 \in \text{Aut } F_k$  representing  $\phi \in \text{Out } F_k$ , there is an  $x \in F_k$  such that  $x\Phi_0^{-1}(S) = \Phi_1^{-1}(S)$  and hence  $\nu(x\Phi_0^{-1}(S)) = \nu(\Phi_1^{-1}(S))$  since  $\nu$  is  $F_k$ -invariant.

Given a basis  $\mathcal{A}$  of  $F_k$ , we have an identification between  $\partial^2 F_k / \sigma$  and unoriented geodesics in  $T_{\mathcal{A}}$ , the Cayley tree for  $\mathcal{A}$ . For a nontrivial  $g \in F_k$  (thought of as a vertex in  $T_{\mathcal{A}}$ ) and  $\nu \in \text{Curr}(F_k)$ , we define the *two-sided cylinder*

$\text{Cyl}_{\mathcal{A}}(g) = \{\text{unoriented geodesics in } T_{\mathcal{A}} \text{ containing the vertices } 1 \text{ and } g\} \subset \partial^2 F_k / \sigma$   
and denote

$$\langle g, \nu \rangle_{\mathcal{A}} = \nu(\text{Cyl}_{\mathcal{A}}(g)).$$

As the sets  $\bigcup_{h,g \in F_k} h\text{Cyl}_{\mathcal{A}}(g)$  form a basis for the topology of  $\partial^2 F_k/\sigma$ , and as  $\nu(h\text{Cyl}_{\mathcal{A}}(g)) = \nu(\text{Cyl}_{\mathcal{A}}(g))$ , a current  $\nu \in \text{Curr}(F_k)$  is determined by its values  $\langle g, \nu \rangle_{\mathcal{A}}$ .

Using these notions there is a useful normalization of a current  $\nu$  relative to the basis  $\mathcal{A}$ . Put

$$\omega_{\mathcal{A}}(\nu) = \sum_{x \in \mathcal{A}} \langle x, \nu \rangle_{\mathcal{A}}.$$

The following lemma provides a useful way to show convergence in  $\mathbb{P}\text{Curr}(F_k)$ :

**Lemma 2.6** ([21], Lemmas 2.11 and 3.5). *Let  $\mathcal{A}$  be a basis for  $F_k$ . Then*

$$\lim_{m \rightarrow \infty} [\nu_m] = [\nu]$$

*if and only if for every nontrivial  $g \in F_k$ ,*

$$\lim_{m \rightarrow \infty} \frac{\langle g, \nu_m \rangle_{\mathcal{A}}}{\omega_{\mathcal{A}}(\nu_m)} = \frac{\langle g, \nu \rangle_{\mathcal{A}}}{\omega_{\mathcal{A}}(\nu)}.$$

Particularly useful are the *counting currents*, defined as follows. Given a nontrivial  $h \in F_k$  that is not a proper power, define the current  $\eta_h$  by

$$\langle g, \eta_h \rangle_{\mathcal{A}} = \langle g^{\pm 1}, h \rangle_{\mathcal{A}}.$$

Here  $\langle g^{\pm 1}, h \rangle_{\mathcal{A}}$  is the number of *occurrences* of  $g$  or  $g^{-1}$  in the cyclic word determined by  $h$ ; specifically, this is the number of times either of the reduced words  $g$  or  $g^{-1}$  appear as a subword of the cyclic word determined by  $h$ . When  $h = f^m$  where  $m \geq 1$  and  $f$  is not a proper power, define  $\eta_h = m\eta_f$ . The current  $\eta_h$  only depends on the conjugacy class of  $h$ , and for  $\phi \in \text{Out } F_k$  we have  $\phi\eta_h = \eta_{\phi(h)}$ . Notice that for any nontrivial  $h \in F_k$  we have  $\omega_{\mathcal{A}}(\eta_h) = \ell_{\mathcal{A}}(h)$ . Although we will not explicitly use it, we remark that the set  $\{[\eta_h]\}_{h \in F_k - \{1\}}$  is dense in  $\mathbb{P}\text{Curr}(F_k)$ .

Similarly we define  $o(g^{\pm 1}, h)_{\mathcal{A}}$  as the number occurrences of  $g$  or  $g^{-1}$  in the word  $h$ ; specifically, this is the number of times the reduced words  $g$  or  $g^{-1}$  appear as a subword of the word  $h$ . A direct application of the Bounded Cancellation Lemma 2.1 gives the following.

**Lemma 2.7.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be bases for  $F_k$  and fix  $a \in \mathcal{A}$ . Then there exists a constant  $C \geq 0$  such that if  $w, w'$ , and  $ww'$  are all reduced words in  $\mathcal{B}$  and  $ww'$  is cyclically reduced in  $\mathcal{B}$ , then*

$$o(a^{\pm 1}, w)_{\mathcal{A}} + o(a^{\pm 1}, w')_{\mathcal{A}} \leq \langle a^{\pm 1}, ww' \rangle_{\mathcal{A}} + C.$$

*Proof.* Let  $B = \text{BCC}(\mathcal{B}, \mathcal{A})$  so that

$$o(a^{\pm 1}, w)_{\mathcal{A}} + o(a^{\pm 1}, w')_{\mathcal{A}} - 2B \leq \langle a^{\pm 1}, ww' \rangle_{\mathcal{A}}.$$



Since  $ww'$  is cyclically reduced with respect to  $\mathcal{B}$ , as a word in  $\mathcal{A}$  we have  $ww' = z x z^{-1}$  where  $|z|_{\mathcal{A}} \leq B$  and  $x$  is cyclically reduced in  $\mathcal{A}$ . Thus

$$o(a^{\pm 1}, ww')_{\mathcal{A}} \leq \langle a^{\pm 1}, ww' \rangle_{\mathcal{A}} + 2B.$$

Therefore, for  $C = 4B$ , the lemma holds. □

As in the Outer space setting, a hyperbolic fully irreducible element acts with North-South dynamics on  $\mathbb{P}\text{Curr}(F_k)$ . Here is a weak version of this statement that is sufficient for our needs.

**Theorem 2.8** ([28], cf. [5], Proposition 4.11). *Every hyperbolic fully irreducible element  $\phi \in \text{Out } F_k$  acts on  $\mathbb{P}\text{Curr}(F_k)$  with exactly two fixed points,  $[\mu_+]$  and  $[\mu_-]$ . Further, for any nontrivial  $h \in F_k$ ,*

$$\lim_{m \rightarrow \infty} [\phi^m \eta_h] = [\mu_+].$$

The currents  $[\mu_+]$  and  $[\mu_-]$  are called the *stable* and *unstable* currents of  $\phi$ , respectively. The stable and unstable currents of  $\phi^{-1}$  are  $[\mu_-]$  and  $[\mu_+]$ , respectively.

The existence of a continuous  $\text{Out } F_k$ -invariant *intersection form* is established by the following.

**Theorem 2.9** ([23], Theorem A). *There is a unique continuous map*

$$\langle \cdot, \cdot \rangle : \overline{cv}_k \times \text{Curr}(F_k) \rightarrow \mathbb{R}_{\geq 0}$$

such that

(1) *for any  $h \in F_n$  we have  $\langle T, \eta_h \rangle = \ell_T(h)$ .*

Further, this map is

(2) *Out  $F_k$ -invariant:  $\langle T\psi, \mu \rangle = \langle T, \psi\mu \rangle$ ;*

(2) *homogeneous with respect the first coordinate:  $\langle \lambda T, \mu \rangle = \lambda \langle T, \mu \rangle$  for  $\lambda > 0$ ;*

and

(4) *linear with respect to the second coordinate:  $\langle T, \lambda_1 \mu_1 + \lambda_2 \mu_2 \rangle = \lambda_1 \langle T, \mu_1 \rangle + \lambda_2 \langle T, \mu_2 \rangle$  for  $\lambda_1, \lambda_2 \geq 0$ .*

The actions of  $\text{Out } F_k$  on  $\overline{cv}_k$  and  $\text{Curr}(F_k)$  satisfy a type of “unique-ergodicity” with respect to this intersection form.

**Theorem 2.10** ([24], Theorem 1.3). *Let  $\phi \in \text{Out } F_k$  be a hyperbolic fully irreducible element with stable and unstable trees  $[T_+], [T_-] \in \overline{CV}_k$  and stable and unstable currents  $[\mu_+], [\mu_-] \in \mathbb{P}\text{Curr}(F_k)$ . The following statements hold.*

(1) *If  $\mu \in \text{Curr}(F_k) - \{0\}$ , then  $\langle T_{\pm}, \mu \rangle = 0$  if and only if  $[\mu] = [\mu_{\mp}]$ .*

(2) *If  $T \in \overline{cv}_k$ , then  $\langle T, \mu_{\pm} \rangle = 0$  if and only if  $[T] = [T_{\mp}]$ .*

The difference in signs  $\pm$  and  $\mp$  between the above and its version in [24] is due to our use of the right action of  $\text{Out } F_k$  on  $\overline{cv}_k$ .

**2.5. Bestvina–Feighn hyperbolic  $\text{Out}(F_k)$ -complex.** The final space we consider is given by the following theorem.

**Theorem 2.11** ([5], Main Theorem). *For any finite collection  $\phi_1, \dots, \phi_n$  of fully irreducible elements of  $\text{Out } F_k$  there is a connected  $\delta$ -hyperbolic graph  $\mathcal{X}$  equipped with an (isometric) action of  $\text{Out } F_k$  such that:*

- (1) *the stabilizer in  $\text{Out } F_k$  of a simplicial tree in  $\overline{CV}_k$  has bounded orbits;*
- (2) *the stabilizer in  $\text{Out } F_k$  of a proper free factor  $F \subset F_k$  has bounded orbits; and*
- (3)  *$\phi_1, \dots, \phi_n$  have nonzero translation lengths.*

The  $\delta$ -hyperbolicity of such a complex  $\mathcal{X}$  makes it comparable to the curve complex for the mapping class group, although its use is significantly restricted by its dependence on a finite set of fully irreducible elements. For our purposes the actual definition of  $\mathcal{X}$  is not necessary; we need only that non-fully irreducible elements of  $\text{Out } F_k$  act on  $\mathcal{X}$  with bounded orbits, and that the action of the elements  $\phi_1, \dots, \phi_n$  on  $\mathcal{X}$  have nonzero translation length and satisfy a property known as WPD (weak proper discontinuity). We refer the reader to [5], [6] for further details.

### 3. Producing hyperbolic automorphisms

In this section we show how to produce a hyperbolic outer automorphism with a specified action on  $H_1(F_k, \mathbb{Z})$ . This involves examining the dynamics of elements on  $\text{Curr}(F_k)$ . Using the “unique-ergodicity” and continuity of the intersection form  $\langle \cdot, \cdot \rangle$  we can mimic an argument due to Fathi ([17], Theorem 2.3) giving a construction of pseudo-Anosov homeomorphisms.

**Proposition 3.1.** *Let  $\phi \in \text{Out } F_k$  be a hyperbolic fully irreducible outer automorphism with stable and unstable currents  $[\mu_+]$  and  $[\mu_-]$  in  $\mathbb{P}\text{Curr}(F_k)$ . Suppose  $\psi \in \text{Out } F_k$  is such that  $[\psi\mu_+] \neq [\mu_-]$ . Then there is an  $M \geq 0$  such that for  $m \geq M$  the element  $\phi^m\psi$  is hyperbolic.*

*Proof.* Let  $\lambda_+$  and  $\lambda_-$  be the expansion factors for  $\phi$  and  $\phi^{-1}$  respectively, and let  $\lambda = \min\{\lambda_+, \lambda_-\} > 1$ . Also let  $T_+$  and  $T_-$  be representatives of the stable and unstable trees for  $\phi$  in  $\overline{cv}_k$ . Thus  $T_+\phi = \lambda_+T_+$  and  $T_-\phi^{-1} = \lambda_-T_-$ .

Hence for each  $m \geq 0$  and any  $\mu \in \text{Curr}(F_k)$  we have

$$\langle T_+, \phi^m\psi\mu \rangle = \langle T_+\phi^m, \psi\mu \rangle \geq \lambda^m \langle T_+, \psi\mu \rangle,$$

and

$$\langle T_-\psi, \psi^{-1}\phi^{-m}\mu \rangle = \langle T_-, \phi^{-m}\mu \rangle = \langle T_-\phi^{-m}, \mu \rangle \geq \lambda^m \langle T_-, \mu \rangle.$$

Now define  $\alpha(\mu) = \max\{\langle T_+, \mu \rangle, \langle T_-, \psi, \mu \rangle\}$ . Then

$$\alpha(\phi^m \psi \mu) \geq \langle T_+, \phi^m \psi \mu \rangle \geq \lambda^m \langle T_+, \psi \mu \rangle,$$

and

$$\alpha(\psi^{-1} \phi^{-m} \mu) \geq \langle T_-, \psi, \psi^{-1} \phi^{-m} \mu \rangle \geq \lambda^m \langle T_-, \mu \rangle.$$

Hence

$$\max\{\alpha(\phi^m \psi \mu), \alpha(\psi^{-1} \phi^{-m} \mu)\} \geq \lambda^m \beta(\mu),$$

where  $\beta(\mu) = \max\{\langle T_+, \psi \mu \rangle, \langle T_-, \mu \rangle\}$ . Now  $\beta(\mu) = 0$  if and only if both  $\langle T_+, \psi \mu \rangle$  and  $\langle T_-, \mu \rangle$  are equal to 0. Applying the “unique-ergodicity” (Theorem 2.10), we have that if  $\mu \neq 0$  then  $\langle T_-, \mu \rangle = 0$  if and only if  $[\mu] = [\mu_+]$ , and  $\langle T_+, \psi \mu_+ \rangle = 0$  if and only if  $[\psi \mu_+] = [\mu_-]$ . By assumption  $[\psi \mu_+] \neq [\mu_-]$ , and hence  $\beta(\mu)$  is strictly positive. Therefore  $\alpha(\mu)/\beta(\mu)$  defines a continuous function on  $\mathbb{P}\text{Curr}(F_k)$ . Since  $\mathbb{P}\text{Curr}(F_k)$  is compact, there is a constant  $K$  such that  $\alpha(\mu)/\beta(\mu) < K$  for all  $\mu \in \text{Curr}(F_k) - \{0\}$ , i.e.,  $K\beta(\mu) > \alpha(\mu)$ . For  $m$  such that  $\lambda^m \geq K$ , we obtain

$$\max\{\alpha(\phi^m \psi \mu), \alpha(\psi^{-1} \phi^{-m} \mu)\} > \alpha(\mu) \quad \text{for all } \mu \in \text{Curr}(F_k) - \{0\}.$$

It is now easy to see that  $\phi^m \psi$  acts on  $\text{Curr}(F_k) - \{0\}$  without a periodic orbit.

Notice that if  $\theta \in \text{Out } F_k$  has a periodic conjugacy class, say  $\theta^\ell$  fixes the conjugacy class of  $c$ , then  $\theta^\ell \eta_c = \eta_{\theta^\ell(c)} = \eta_c$ , and hence  $\theta$  acts on  $\text{Curr}(F_k) - \{0\}$  with a periodic orbit. Thus as  $\phi^m \psi$  acts on  $\text{Curr}(F_k) - \{0\}$  without a periodic orbit it does not have a periodic conjugacy class, i.e.,  $\phi^m \psi$  is hyperbolic.  $\square$

#### 4. Producing fully irreducible automorphisms

In this section we show how to produce a fully irreducible element of  $\text{Out } F_k$  with a specified action on  $H_1(F_k, \mathbb{Z})$ . This involves examining the dynamics of elements on the  $\delta$ -hyperbolic Bestvina–Feighn complex  $\mathcal{X}$  from Theorem 2.11. We begin with a theorem about the isometries of  $\delta$ -hyperbolic spaces. Even though the space we will ultimately consider has a right action, we will consider the more customary setting where the space has a left action; it is clear how to convert a right action into a left action.

We recall some basics about  $\delta$ -hyperbolic spaces needed for this section. Some references for this material are [1], [9], [20].

A geodesic metric space  $X$  is called  $\delta$ -hyperbolic if for any geodesic triangle in  $X$ , the  $\delta$ -neighborhood of the union of any two of the sides contains the third. There are various other equivalent notions. There is an *inner product* defined for points  $x, y \in X$  by

$$(x \cdot y)_w = \frac{1}{2}(d(x, w) + d(w, y) - d(x, y))$$

for a given basepoint  $w \in X$ . Associated to a  $\delta$ -hyperbolic space is a *boundary*  $\partial X$  which compactifies  $X$  as  $X \cup \partial X$  when  $X$  is locally compact. One definition of  $\partial X$  is as equivalence classes of sequences  $\{x_i\}$  with  $\lim_{i,j \rightarrow \infty} d(x_i, x_j) = \infty$  (the inner product is defined with respect to some basepoint), the equivalence relation is defined by  $\{x_i\} \sim \{y_i\}$  if  $\lim_{i \rightarrow \infty} d(x_i, y_i) = \infty$ . If  $f$  is an isometry of  $X$  with nonzero translation length (i.e.,  $\lim_{n \rightarrow \infty} \frac{1}{n} d(x, f^n(x)) > 0$  for all  $x \in X$ ), then the action of  $f$  extends to a continuous action on  $\partial X$  with exactly two fixed points. One fixed point is represented by the sequence  $\{f^n(x)\}$  for any  $x \in X$ ; the other is represented by  $\{f^{-n}(y)\}$  for any  $y \in X$ . These points are called the *attracting and repelling fixed points* of  $f$  respectively.

**Theorem 4.1.** *Suppose  $X$  is a  $\delta$ -hyperbolic space and  $f \in \text{Isom}(X)$  acts on  $X$  with nonzero translation length, with attracting and respectively repelling fixed points  $A_+$  and  $A_-$  in  $\partial X$ . If  $g \in \text{Isom}(X)$  acts on  $X$  such that  $gA_+ \neq A_-$ , then there is an  $M \geq 0$  such that for  $m \geq M$  the element  $f^m g$  acts on  $X$  with nonzero translation length.*

Before proving this theorem we need a lemma that allows us to locally build uniform quasi-geodesics. Recall that a  $(\lambda, \epsilon)$ -quasi-geodesic is a function  $\alpha : [a, b] \rightarrow X$  such that for all  $t, t' \in [a, b]$  we have

$$\frac{1}{\lambda} |t - t'| - \epsilon \leq d(\alpha(t), \alpha(t')) \leq \lambda |t - t'| + \epsilon.$$

We allow for the possibility that the domain of  $\alpha$  is  $\mathbb{R}$  or  $\mathbb{R}_{\geq 0}$ . A function  $\alpha : [a, b] \rightarrow X$  is an  $L$ -local  $(\lambda, \epsilon)$ -quasi-geodesic if for all  $a \leq a' \leq b' \leq b$  where  $b' - a' \leq L$ , the function  $\alpha|_{[a', b']}$  is a  $(\lambda, \epsilon)$ -quasi-geodesic. First we recall a standard fact about  $\delta$ -hyperbolic spaces.

**Lemma 4.2** ([9], Chapter III.H, Lemma 1.15). *Let  $X$  be a  $\delta$ -hyperbolic space, and let  $c_1 : [0, T_1] \rightarrow X$  and  $c_2 : [0, T_2] \rightarrow X$  be geodesics such that  $c_1(0) = c_2(0)$ . Let  $T = \max\{T_1, T_2\}$  and extend the shorter geodesic to  $[0, T]$  by the constant map. If  $K = d(c_1(T), c_2(T))$ , then  $d(c_1(t), c_2(t)) \leq 2(K + 2\delta)$  for all  $t \in [0, T]$ .*

The next lemma shows us that the sequence of points  $(f^m g)^n(x)$  defines a local quasi-geodesic with uniform constants.

**Lemma 4.3.** *Let  $X, f$  and  $g$  be as in Theorem 4.1. Fix  $x \in X$  and for  $m \geq 0$ , let  $\alpha_m$  be a geodesic connecting  $x$  to  $f^m g(x)$ . Then there is an  $\epsilon \geq 0$  such that for  $m \geq 0$  the concatenation of the geodesics  $\alpha_m \cdot f^m g(\alpha_m)$  is a  $(1, \epsilon)$ -quasi-geodesic.*

*Proof.* Let  $\beta_m = \alpha_m \cup f^m g(\alpha_m)$ ,  $d_m = d(x, f^m g(x))$  and consider the points  $g(x)$ ,  $f^{-m}(x)$  and  $g f^m g(x)$ . Notice  $f^{-m}(\beta_m)$  is a path from  $f^{-m}(x)$  to  $g f^m g(x)$  passing

through  $g(x)$ ; see Figure 1. As  $gA_+ \neq A_-$ , the inner product  $(f^{-m}(x), gf^m g(x))_{g(x)}$  stays bounded as  $m \rightarrow \infty$ . Hence there is a constant  $C \geq 0$  that does not depend on  $m$  such that

$$\begin{aligned} d(x, f^m g f^m g(x)) &= d(f^{-m}(x), g f^m g(x)) \\ &\geq d(f^{-m}(x), g(x)) + d(g(x), g f^m g(x)) - 2C \\ &= 2d_m - 2C. \end{aligned}$$

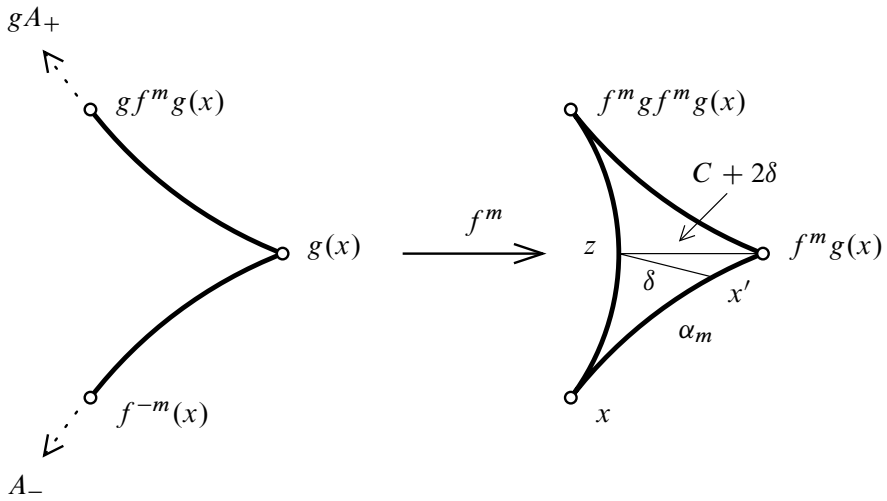


Figure 1. The geodesics in Lemma 4.3.

Fix a geodesic  $c$  from  $x$  to  $f^m g f^m g(x)$ , and let  $z$  be the midpoint on  $c$ . As  $X$  is  $\delta$ -hyperbolic, there is an  $x' \in \beta_m$  such that  $d(z, x') \leq \delta$ . Without loss of generality we can assume that  $x' \in \alpha_m$ . Thus

$$d(x', x) \geq d(x, z) - \delta \geq d_m - C - \delta,$$

and therefore

$$d(x', f^m g(x)) = d_m - d(x, x') \leq C + \delta,$$

from which we conclude  $d(z, f^m g(x)) \leq C + 2\delta$ . Let  $d'_m = d(x, f^m g f^m g(x))$  and define  $c_z: [0, d_m] \rightarrow X$  by  $c_z(t) = c(t)$  if  $0 \leq t \leq \frac{1}{2}d'_m$  and  $c_z(t) = z$  otherwise. Then by Lemma 4.2 we have for  $0 \leq t \leq d_m$  that  $d(\beta_m(t), c_z(t)) \leq 2(C + 4\delta)$ . Similarly define  $c'_z: [d_m, 2d_m] \rightarrow X$  by  $c'_z(t) = z$  if  $d_m \leq t \leq 2d_m - \frac{1}{2}d'_m$  and  $c'_z(t) = c(t + d'_m - 2d_m)$  otherwise. Then another application of Lemma 4.2 shows that for  $d_m \leq t \leq 2d_m$  we have  $d(\beta_m(t), c'_z(t)) \leq 2(C + 4\delta)$ . Notice that if  $0 \leq t \leq d_m \leq t' \leq 2d_m$  then

$$(t' - t) - 2C \leq d(c_z(t), c'_z(t')) \leq (t' - t)$$

as  $2d_m - d'_m \leq 2C$ . Therefore if  $0 \leq t \leq d_m \leq t' \leq 2d_m$  then

$$(t' - t) - (6C + 16\delta) \leq d(\beta_m(t), \beta_m(t')) \leq (t' - t).$$

The other cases ( $0 \leq t \leq t' \leq d_m$  or  $d_m \leq t \leq t' \leq 2d_m$ ) are clear since  $\alpha_m$  is a geodesic. □

Now to complete the proof of Theorem 4.1 we need the following theorem.

**Theorem 4.4** ([9], Chapter III.H, Theorems 1.7 and 1.13). *Let  $X$  be a  $\delta$ -hyperbolic space, and let  $\gamma : [a, b] \rightarrow X$  be an  $L$ -local  $(\lambda, \epsilon)$ -quasi-geodesic. Then there is an  $R = R(\delta, \lambda, \epsilon)$  such that if  $L > R$ , then for some  $\lambda' \geq 1$  and  $\epsilon' \geq 0$ , the path  $\gamma$  is a  $(\lambda', \epsilon')$ -quasi-geodesic.*

We can now give a proof of Theorem 4.1.

*Proof of Theorem 4.1.* Fix  $x \in X$ , and let  $\epsilon$  be given from Lemma 4.3 and let  $R = R(\delta, 1, \epsilon)$  be the constant from Theorem 4.4. As  $f$  has nonzero translation length, for  $m \geq 0$  we can let  $L_m = d(x, f^m g(x)) \geq d(g(x), f^m g(x)) - d(x, g(x)) \geq mt - d(x, g(x))$  for some  $t > 0$ . Let  $M$  be such that  $L_M > R$ . As in Lemma 4.3, let  $\alpha_m$  be a geodesic connecting  $x$  to  $f^m g(x)$ , and let  $\beta_m = \alpha_m \cdot f^m g(\alpha)$ . Then define a path  $\gamma : [0, \infty) \rightarrow X$  by

$$\gamma = \beta_m \bigcup_{f^m g(\alpha_m)} f^m g(\beta_m) \bigcup_{(f^m g)^2(\alpha_m)} (f^m g)^2(\beta_m) \dots$$

By Lemma 4.3,  $\gamma$  is an  $L_m$ -local  $(1, \epsilon)$ -quasi-geodesic and hence if  $m \geq M$  then  $\gamma$  is a  $(\lambda', \epsilon')$ -quasi-geodesic from some  $\lambda' \geq 1$  and  $\epsilon' \geq 0$  by Theorem 4.4. Therefore for any  $x' \in X$  and  $\ell \geq 0$  we have

$$\begin{aligned} d(x', (f^m g)^\ell(x')) &\geq d(x, (f^m g)^\ell(x)) - 2d(x', x) \\ &\geq \frac{1}{\lambda'} L_m \ell - \epsilon' - 2d(x', x) \end{aligned}$$

and hence  $f^m g$  has nonzero translation length. □

The fully irreducible analog of Proposition 3.1 follows easily from Theorems 2.11 and 4.1.

**Proposition 4.5.** *Let  $\phi \in \text{Out } F_k$  be a fully irreducible outer automorphism with stable and unstable trees  $[T_+]$  and  $[T_-]$  in  $\overline{CV}_k$ . Suppose  $\psi \in \text{Out } F_k$  is such that  $[T_+ \psi] \neq [T_-]$ . Then there is an  $M \geq 0$  such that  $m \geq M$  the element  $\phi^m \psi$  is fully irreducible.*

*Proof.* Let  $\mathcal{X}$  be the Bestvina–Feighn  $\delta$ -hyperbolic complex from Theorem 2.11 using  $\phi_1 = \phi$  and let  $A_+$  and  $A_-$  denote the attracting and repelling fixed points of  $\phi$  in  $\partial\mathcal{X}$ . What needs to be shown in order to apply Theorem 4.1 is that  $[T_+\psi] \neq [T_-]$  implies that  $A_+\psi \neq A_-$ . As the action of  $\phi$  on  $\mathcal{X}$  satisfies WPD, see Proposition 4.27 of [5], if  $A_+\psi = A_-$  then for some  $r, s > 0$  we have  $\psi\phi^r\psi^{-1} = \phi^{-s}$  ([6], Proposition 6). As the stable and unstable tree for positive powers of  $\phi$  are the same as for  $\phi$ , this would imply  $[T_+\psi] = [T_-]$ .

Now we can apply Theorem 4.1 to the pair  $\phi$  and  $\psi$  acting on  $\mathcal{X}$  to conclude that for large enough  $m$ , the element  $\phi^m\psi$  does not have a bounded orbit and hence by Theorem 2.11 is fully irreducible.  $\square$

We would like to thank Mladen Bestvina for suggesting the use of WPD in the above argument.

### 5. The stable current for a product of twists

In this section we examine the qualitative behavior of the stable and unstable currents associated to a product of Dehn twists. The main result is Corollary 5.4 which produces elements of  $\text{Out } F_k$  satisfying the hypotheses of Propositions 3.1 and 4.5. We begin with a simple lemma describing the change of a conjugacy class in  $F_k$  resulting from powers of a single twist.

**Lemma 5.1.** *Let  $T_1$  and  $T_2$  be very small cyclic trees with edge stabilizers  $c_1$  and  $c_2$  and associated Dehn twists  $\delta_1$  and  $\delta_2$ . Suppose  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are bases relative to  $T_1$  and  $T_2$  respectively such that  $c_2$  is cyclically reduced with respect to  $\mathcal{T}_1$  and  $C$  is the constant from Lemma 2.7 using these bases. Then for any  $x \in F_k$  and  $n \geq r > 0$  the following hold.*

$$\langle c_1^{\pm r}, \delta_1^n(x) \rangle_{\mathcal{T}_1} \geq (n - r + 1)\ell_{T_1}(x) - \langle c_1^{\pm 1}, x \rangle_{\mathcal{T}_1}, \tag{5.1}$$

$$\ell_{\mathcal{T}_1}(\delta_1^n(x)) \leq n\ell_{T_1}(x) + \ell_{\mathcal{T}_1}(x), \tag{5.2}$$

$$\ell_{\mathcal{T}_1}(\delta_1^n(x)) \geq n\ell_{T_1}(x) + \ell_{\mathcal{T}_1}(x) - \langle c_1^{\pm 1}, x \rangle_{\mathcal{T}_1}, \tag{5.3}$$

$$\langle c_1^{\pm 1}, \delta_2^{-n}(x) \rangle_{\mathcal{T}_1} \leq \ell_{T_2}(x)[n\langle c_1^{\pm 1}, c_2 \rangle_{\mathcal{T}_1} + 2C] + \langle c_1^{\pm 1}, x \rangle_{\mathcal{T}_1}, \tag{5.4}$$

$$\ell_{\mathcal{T}_1}(\delta_2^{-n}(x)) \leq \ell_{T_2}(x)[n\ell_{\mathcal{T}_1}(c_2) + 2C] + \ell_{\mathcal{T}_1}(x) \tag{5.5}$$

*Proof.* We begin by proving the first three inequalities. By replacing  $x$  by a conjugate we are free to assume that  $x$  is  $T_1$ -reduced as all of the quantities involved in the inequalities only depend on the conjugacy class of  $x$ . If  $T_1$  is dual to an amalgamated free product we have

$$x = x_1c_1^{i_1}y_1c_1^{j_1} \cdots x_m c_1^{i_m}y_m c_1^{j_m}.$$

Therefore

$$\delta_1^n(x) = x_1 c_1^{i_1+n} y_1 c_1^{j_1-n} \cdots x_m c_1^{i_m+n} y_m c_1^{j_m-n}$$

is a cyclically reduced word in  $\mathcal{T}_1$ . Hence by only counting the occurrences of  $c_1^{\pm r}$  that appear in the  $c_1^{i_s+n}$  and  $c_1^{j_s-n}$  we see that

$$\begin{aligned} \langle c_1^{\pm r}, \delta_1^n(x) \rangle_{\mathcal{T}_1} &\geq \sum_{s=1}^m (|i_s + n| - r + 1) + (|j_s - n| - r + 1) \\ &\geq 2m(n - r + 1) - \sum_{s=1}^m |i_s| + |j_s| \\ &\geq (n - r + 1) \ell_{T_1}(x) - \langle c_1^{\pm 1}, x \rangle_{\mathcal{T}_1}. \end{aligned}$$

A similar proof works if  $T_1$  is dual to an HNN-extension. This shows (5.1); the inequalities (5.2) and (5.3) follow similarly by looking at the given cyclically reduced expression for  $\delta_1^n(x)$ .

We now prove the last two inequalities. As before, by replacing  $x$  by a conjugate we are free to assume that  $x$  is  $T_2$ -reduced. If  $T_2$  is dual to an amalgamated free product we have

$$x = x_1 c_2^{i_1} y_1 c_2^{j_1} \cdots x_m c_2^{i_m} y_m c_2^{j_m}.$$

Therefore

$$\delta_2^{-n}(x) = x_1 c_2^{i_1-n} y_1 c_2^{j_1+n} \cdots x_m c_2^{i_m-n} y_m c_2^{j_m+n}$$

is a cyclically reduced word in  $\mathcal{T}_2$ . Hence by counting the number of occurrences of  $c_1^{\pm 1}$  in the various  $x_s, y_s, c_2^{i_s-n}$  and  $c_2^{j_s+n}$  we see that

$$\begin{aligned} \langle c_1^{\pm 1}, \delta_2^{-n}(x) \rangle_{\mathcal{T}_1} &\leq \ell_{T_2}(x) o(c_1^{\pm 1}, c_2^n)_{\mathcal{T}_1} \\ &\quad + \sum_{s=1}^m o(c_1^{\pm 1}, x_s)_{\mathcal{T}_1} + o(c_1^{\pm 1}, c_2^{i_s})_{\mathcal{T}_1} + o(c_1^{\pm 1}, y_s)_{\mathcal{T}_1} + o(c_1^{\pm 1}, c_2^{j_s})_{\mathcal{T}_1} \\ &\leq \ell_{T_2}(x) n \langle c_1^{\pm 1}, c_2 \rangle_{\mathcal{T}_1} + \langle c_1^{\pm 1}, x \rangle_{\mathcal{T}_1} + 4mC \\ &\leq \ell_{T_2}(x) [n \langle c_1^{\pm 1}, c_2 \rangle_{\mathcal{T}_1} + 2C] + \langle c_1^{\pm 1}, x \rangle_{\mathcal{T}_1}. \end{aligned}$$

A similar proof works if  $T_1$  is dual to an HNN-extension. This shows (5.4). The inequality (5.5) is just an application of the bounded cancellation lemma using the cyclically reduced expression for  $\delta_2^{-n}(x)$ . □

These estimates allow us to show our main technical result concerning the stable currents.



**Theorem 5.2.** *Let  $T_1$  and  $T_2$  be very small cyclic trees that fill, with edge stabilizers  $c_1$  and  $c_2$  and associated Dehn twist automorphisms  $\delta_1$  and  $\delta_2$ . Let  $N \geq 0$  be such that for  $n \geq N$ , we have that  $\delta_1^n \delta_2^{-n}$  is a hyperbolic fully irreducible outer automorphism with stable and unstable currents  $[\mu_+^n]$  and  $[\mu_-^n]$  in  $\mathbb{P}\text{Curr}(F_k)$ . Then*

$$\lim_{n \rightarrow \infty} [\mu_+^n] = [\eta_{c_1}] \text{ and } \lim_{n \rightarrow \infty} [\mu_-^n] = [\eta_{c_2}].$$

*Proof.* Let  $\mathcal{T}_1$  be a basis for  $F_k$  relative to  $T_1$ . Denote by  $\phi_n = \delta_1^n \delta_2^{-n}$ . Fix an element  $a \in \mathcal{T}_1$ , denote its conjugacy class by  $\alpha$ , and denote  $\phi_n^m(\alpha)$  by  $\alpha_n^m$ . Hence  $\phi_n^m \eta_\alpha = \eta_{\alpha_n^m}$ . As  $\ell_{T_2}(\alpha) > \ell_{T_1}(\alpha) = 0$ , Lemma 5.2 of [12] shows that since  $n$  is sufficiently large, for  $m \geq 0$  we have  $\ell_{T_2}(\alpha_n^m) \geq \ell_{T_1}(\alpha_n^m)$ . Let  $K > 0$  be such that for all  $m, n \geq 0$ ,

$$\ell_{\mathcal{T}_1}(\alpha_n^m) \leq K(\ell_{T_1}(\alpha_n^m) + \ell_{T_2}(\alpha_n^m)) \leq 2K\ell_{T_2}(\alpha_n^m).$$

Such a  $K$  exists by Theorem 1.4 in [24].

Then for each  $n \geq N$ , as  $\mu_+^n$  is the stable current for  $\phi_n$ , from the North-South dynamics of  $\phi_n$  on  $\mathbb{P}\text{Curr}(F_k)$  (Theorem 2.8), we have for any  $g \in F_k$  and  $\epsilon > 0$ , a constant  $M = M(n, g, \frac{\epsilon}{2})$  such that for  $m \geq M$  (Lemma 2.6),

$$\left| \frac{\langle g, \eta_{\alpha_n^m} \rangle_{\mathcal{T}_1}}{\omega_{\mathcal{T}_1}(\eta_{\alpha_n^m})} - \frac{\langle g, \mu_+^n \rangle_{\mathcal{T}_1}}{\omega_{\mathcal{T}_1}(\mu_+^n)} \right| < \frac{\epsilon}{2}. \tag{5.6}$$

We analyze how the current  $\eta_{\alpha_n^m}$  changes as  $m \rightarrow \infty$  in terms of  $n$ . Fix a basis  $\mathcal{T}_2$  that is relative to  $T_2$  such that  $c_2$  is cyclically reduced with respect to  $\mathcal{T}_1$ , and let  $C$  be larger than either constant  $C = C(T_1, T_2)$  from Theorem 2.5 or the constant  $C$  from Lemma 2.7, using the bases  $\mathcal{T}_1$  and  $\mathcal{T}_2$ . Applying (2.1), (5.1) and (5.4), for any  $m \geq 0$  and  $n \geq r > 0$  we have

$$\begin{aligned} & \langle c_1^{\pm r}, \delta_1^n \delta_2^{-n}(\alpha_n^m) \rangle_{\mathcal{T}_1} \\ & \geq (n - r + 1)\ell_{T_1}(\delta_2^{-n}(\alpha_n^m)) - \langle c_1^{\pm 1}, \delta_2^{-n}(\alpha_n^m) \rangle_{\mathcal{T}_1} \\ & \geq (n - r + 1) \left[ \ell_{T_2}(\alpha_n^m) [n\ell_{T_1}(c_2) - (C + 1)] \right. \\ & \quad \left. - \left[ \ell_{T_2}(\alpha_n^m) [n\langle c_1^{\pm 1}, c_2 \rangle_{\mathcal{T}_1} + 2C] + \langle c_1^{\pm 1}, \alpha_n^m \rangle_{\mathcal{T}_1} \right] \right] \\ & \geq n^2 \ell_{T_2}(\alpha_n^m) \ell_{T_1}(c_2) \\ & \quad - n \ell_{T_2}(\alpha_n^m) [(C + 1) + (r - 1)\ell_{T_1}(c_2) + \langle c_1^{\pm 1}, c_2 \rangle_{\mathcal{T}_1}] \\ & \quad - 2C \ell_{T_1}(\alpha_n^m) - \ell_{\mathcal{T}_1}(\alpha_n^m) \\ & \geq \ell_{T_2}(\alpha_n^m) \left[ n^2 \ell_{T_1}(c_2) \right. \\ & \quad \left. - n [(C + 1) + (r - 1)\ell_{T_1}(c_2) + \langle c_1^{\pm 1}, c_2 \rangle_{\mathcal{T}_1}] - (2C + 2K) \right] \\ & \geq \ell_{T_2}(\alpha_n^m) [n^2 \ell_{T_1}(c_2) - nC_1 - C_2] \end{aligned}$$

for some constants  $C_1 \geq 0$  and  $C_2 \geq 0$  that do not depend on  $m$ . Applying (2.2), (5.2) and (5.5) we also have for any  $m \geq 0$  and  $n > 0$  that

$$\begin{aligned} \ell_{\mathcal{T}_1}(\delta_1^n \delta_2^{-n}(\alpha_n^m)) &\leq n\ell_{T_1}(\delta_2^{-n}(\alpha_n^m)) + \ell_{\mathcal{T}_1}(\delta_2^{-n}(\alpha_n^m)) \\ &\leq n[\ell_{T_2}(\alpha_n^m)[n\ell_{T_1}(c_2) + (C + 1)]] \\ &\quad + \ell_{T_2}(\alpha_n^m)[n\ell_{\mathcal{T}_1}(c_2) + 2C] + \ell_{\mathcal{T}_1}(\alpha_n^m) \\ &\leq n^2\ell_{T_2}(x)\ell_{T_1}(c_2) \\ &\quad + n\ell_{T_2}(\alpha_n^m)[(C + 1) + \ell_{\mathcal{T}_1}(c_2)] + (2C + 2K)\ell_{T_1}(\alpha_n^m) \\ &\leq \ell_{T_2}(\alpha_n^m)[n^2\ell_{T_1}(c_2) + nC'_1 + C'_2] \end{aligned}$$

for some constants  $C'_1 \geq 0$  and  $C'_2 \geq 0$  that do not depend on  $m$ . Therefore, given  $r > 0$  there are constants  $\beta_1 \geq 0$  and  $\beta_2 \geq 0$  that do not depend on  $m$  such that for any  $n \geq r$ ,

$$\ell_{\mathcal{T}_1}(\delta_1^n \delta_2^{-n}(\alpha_n^m)) - \langle c_1^{\pm r}, \delta_1^n \delta_2^{-n}(\alpha_n^m) \rangle_{\mathcal{T}_1} \leq \ell_{T_2}(\alpha_n^m)[n\beta_1 + \beta_2]. \tag{5.7}$$

Now, applying (2.1) and (5.3), we have for any  $m \geq 0$  and  $n > 0$ ,

$$\begin{aligned} \ell_{\mathcal{T}_1}(\delta_1^n \delta_2^{-n}(\alpha_n^m)) &\geq n\ell_{T_1}(\delta_2^{-n}(\alpha_n^m)) + \ell_{\mathcal{T}_1}(\delta_2^{-n}(\alpha_n^m)) - \langle c_1^{\pm 1}, \delta_2^{-n}(\alpha_n^m) \rangle_{\mathcal{T}_1} \\ &\geq n[\ell_{T_2}(\alpha_n^m)[n\ell_{T_1}(c_2) - (C + 1)]] \\ &\quad - \ell_{T_2}(\alpha_n^m)[n\ell_{\mathcal{T}_1}(c_2) + 2C] - \langle c_1^{\pm 1}, \alpha_n^m \rangle_{\mathcal{T}_1} \\ &\geq n^2\ell_{T_2}(\alpha_n^m)\ell_{T_1}(c_2) \\ &\quad - n\ell_{T_2}(\alpha_n^m)[(C + 1) + \ell_{\mathcal{T}_1}(c_2)] - (2C + 2K)\ell_{T_2}(\alpha_n^m) \end{aligned}$$

Therefore, there are constants  $\gamma_1 \geq 0$ ,  $\gamma_2 \geq 0$  and  $\gamma_3 \geq 0$  that do not depend on  $m$  such that for  $n > 0$ ,

$$\ell_{\mathcal{T}_1}(\delta_1^n \delta_2^{-n}(\alpha_n^m)) \geq \ell_{T_2}(\alpha_n^m)[n^2\gamma_1 - n\gamma_2 - \gamma_3]. \tag{5.8}$$

As a first approximation, we will show that the currents  $\eta_{\alpha_n^m}$  converge to the correct value on  $\text{Cyl}_{\mathcal{T}_1}(c_1^r)$ . Notice  $\eta_{c_1}(\text{Cyl}_{\mathcal{T}_1}(c_1^r)) = 1$ . Suppose  $g = c_1^{\pm r}$  for some  $r > 0$ . Let  $\epsilon > 0$  and fix  $n \geq \max\{N, r\}$  large enough such that  $\epsilon(n^2\gamma_1 - n\gamma_2 - \gamma_3) >$

$2(n\beta_1 + \beta_2)$ . Now let  $m \geq M(n, g, \frac{\epsilon}{2})$ . Then

$$\begin{aligned} \left| \frac{\langle g, \eta_{c_1} \rangle_{\mathcal{T}_1}}{\omega_{\mathcal{T}_1}(\eta_{c_1})} - \frac{\langle g, \mu_+^n \rangle_{\mathcal{T}_1}}{\omega_{\mathcal{T}_1}(\mu_+^n)} \right| &\leq \left| \frac{\langle g, \eta_{c_1} \rangle_{\mathcal{T}_1}}{\omega_{\mathcal{T}_1}(\eta_{c_1})} - \frac{\langle g, \eta_{\alpha_n^{m+1}} \rangle_{\mathcal{T}_1}}{\omega_{\mathcal{T}_1}(\eta_{\alpha_n^{m+1}})} \right| \\ &\quad + \left| \frac{\langle g, \eta_{\alpha_n^{m+1}} \rangle_{\mathcal{T}_1}}{\omega_{\mathcal{T}_1}(\eta_{\alpha_n^{m+1}})} - \frac{\langle g, \mu_+^n \rangle_{\mathcal{T}_1}}{\omega_{\mathcal{T}_1}(\mu_+^n)} \right| \\ &< \left| 1 - \frac{\langle g, \eta_{\alpha_n^{m+1}} \rangle_{\mathcal{T}_1}}{\omega_{\mathcal{T}_1}(\eta_{\alpha_n^{m+1}})} \right| + \frac{\epsilon}{2} \\ &= \left| \frac{\ell_{\mathcal{T}_1}(\delta_1^n \delta_2^{-n}(\alpha_n^m)) - \langle c_1^{\pm r}, \delta_1^n \delta_2^{-n}(\alpha_n^m) \rangle_{\mathcal{T}_1}}{\ell_{\mathcal{T}_1}(\delta_1^n \delta_2^{-n}(\alpha_n^m))} \right| + \frac{\epsilon}{2} \\ &\leq \left| \frac{\ell_{T_2}(\alpha_n^m)[n\beta_1 + \beta_2]}{\ell_{T_2}(\alpha_n^m)[n^2\gamma_1 - n\gamma_2 - \gamma_3]} \right| + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{5.9}$$

Now suppose  $g \neq c_1^{\pm r}$  for any  $r > 0$ ; in this case  $\eta_{c_1}(\text{Cyl}_{\mathcal{T}_1}(g)) = 0$ . There is some  $a_0 \in \mathcal{T}_1 - \{c_1\}$  such that  $\langle a_0^{\pm 1}, g \rangle_{\mathcal{T}_1} > 0$ . Therefore for any  $m \geq 0$ ,

$$\langle a_0^{\pm 1}, \alpha_n^m \rangle_{\mathcal{T}_1} \langle a_0^{\pm 1}, g \rangle_{\mathcal{T}_1} \geq \langle g^{\pm 1}, \alpha_n^m \rangle_{\mathcal{T}_1} \tag{5.10}$$

as every occurrence of  $g^{\pm 1}$  in  $\alpha_n^m$  contains some occurrence of  $a_0^{\pm 1}$  in  $\alpha_n^m$  and such an occurrence can only be used  $\langle a_0^{\pm 1}, g \rangle_{\mathcal{T}_1}$  times. Since

$$\frac{1}{\ell_{\mathcal{T}_1}(\alpha_n^m)} \sum_{x \in \mathcal{T}_1 - \{c_1\}} \langle x^{\pm 1}, \alpha_n^m \rangle_{\mathcal{T}_1} = 1 - \frac{\langle c_1^{\pm 1}, \alpha_n^m \rangle_{\mathcal{T}_1}}{\ell_{\mathcal{T}_1}(\alpha_n^m)},$$

the computation in (5.9) combined with (5.10) shows that there is an  $n = n(g, \epsilon)$  such that  $2\langle g^{\pm 1}, \alpha_n^m \rangle_{\mathcal{T}_1} < \epsilon \ell_{\mathcal{T}_1}(\alpha_n^m)$  for  $m$  sufficiently large. Then for  $m \geq M(n, g, \frac{\epsilon}{2})$ ,

$$\begin{aligned} \left| \frac{\langle g, \eta_{c_1} \rangle_{\mathcal{T}_1}}{\omega_{\mathcal{T}_1}(\eta_{c_1})} - \frac{\langle g, \mu_+^n \rangle_{\mathcal{T}_1}}{\omega_{\mathcal{T}_1}(\mu_+^n)} \right| &\leq \left| \frac{\langle g, \eta_{c_1} \rangle_{\mathcal{T}_1}}{\omega_{\mathcal{T}_1}(\eta_{c_1})} - \frac{\langle g, \eta_{\alpha_n^m} \rangle_{\mathcal{T}_1}}{\omega_{\mathcal{T}_1}(\eta_{\alpha_n^m})} \right| \\ &\quad + \left| \frac{\langle g, \eta_{\alpha_n^m} \rangle_{\mathcal{T}_1}}{\omega_{\mathcal{T}_1}(\eta_{\alpha_n^m})} - \frac{\langle g, \mu_+^n \rangle_{\mathcal{T}_1}}{\omega_{\mathcal{T}_1}(\mu_+^n)} \right| \\ &< \left| \frac{\langle g, \eta_{\alpha_n^m} \rangle_{\mathcal{T}_1}}{\omega_{\mathcal{T}_1}(\eta_{\alpha_n^m})} \right| + \frac{\epsilon}{2} \\ &= \left| \frac{\langle g^{\pm 1}, \alpha_n^m \rangle_{\mathcal{T}_1}}{\ell_{\mathcal{T}_1}(\alpha_n^m)} \right| + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{5.11}$$

Putting together (5.9) and (5.11), we have that  $\lim_{n \rightarrow \infty} [\mu_+^n] = [\eta_{c_1}]$ . The same argument applied to  $\phi_n^{-1}$  shows that  $\lim_{n \rightarrow \infty} [\mu_-^n] = [\eta_{c_2}]$ .  $\square$

**Remark 5.3.** We remark that Theorem 5.2 is analogous to the surface setting. Given two simple closed curves  $\alpha, \beta \subset S_g$  that fill, the stable and unstable measured laminations  $[\Lambda_+^n]$  and  $[\Lambda_-^n]$  in the Thurston boundary of Teichmüller space associated to the pseudo-Anosov mapping classes  $\delta_\alpha^n \delta_\beta^{-n}$  converge to  $[\alpha]$  and  $[\beta]$  respectively. Here  $\delta_\alpha$  and  $\delta_\beta$  are the respective Dehn twist homeomorphisms about  $\alpha$  and  $\beta$ .

This raises a subtle point. To the hyperbolic fully irreducible outer automorphisms  $\delta_1^n \delta_2^{-n}$  in Theorem 5.2 are also associated the stable and unstable trees  $[T_+^n]$  and  $[T_-^n]$  in  $\overline{CV}_k$  (see Section 2.2). As  $\overline{CV}_k$  is compact, the associated sequences  $\{[T_+^n]\}$  and  $\{[T_-^n]\}$  have accumulation points. But in contrast with Theorem 5.2, it is not clear whether there is a single accumulation point for each respective sequence or how to characterize an accumulation point for either sequence. By Theorems 2.9, 2.10 and 5.2, the element  $c_2$  has a fixed point in any accumulation point of  $\{[T_+^n]\}$ , and similarly  $c_1$  has a fixed point in any accumulation point of  $\{[T_-^n]\}$ . However it is unlikely that this is a characterization of the accumulation points for the sequences  $\{[T_+^n]\}$  and  $\{[T_-^n]\}$ .

The following corollary is essential for our main theorem (Theorem 6.1).

**Corollary 5.4.** *Let  $T_1$  and  $T_2$  be very small cyclic trees that fill, with edge stabilizers  $c_1$  and  $c_2$  and associated Dehn twist automorphisms  $\delta_1$  and  $\delta_2$ . Let  $N \geq 0$  be such that for  $n \geq N$ , we have that  $\delta_1^n \delta_2^{-n}$  is a hyperbolic fully irreducible outer automorphism with stable and unstable currents  $[\mu_+^n]$  and  $[\mu_-^n]$  in  $\mathbb{P}\text{Curr}(F_k)$  and stable and unstable trees  $[T_+^n]$  and  $[T_-^n]$  in  $\overline{CV}_k$ . For  $\psi \in \text{Out } F_k$  such that the conjugacy class of  $\psi(c_1)$  is not equal to the conjugacy class of  $c_2$ , there is an  $N_1 \geq N$  such that for  $n \geq N_1$  we have  $[\psi \mu_+^n] \neq [\mu_-^n]$  and  $[T_+^n \psi] \neq [T_-^n]$ .*

*Proof.* As the conjugacy class of  $\psi(c_1)$  is not equal to the conjugacy class of  $c_2$  we have that  $[\psi \eta_{c_1}] \neq [\eta_{c_2}]$ . Fix disjoint open sets  $U_1$  and  $U_2$  of  $\mathbb{P}\text{Curr}(F_k)$  containing  $[\psi \eta_{c_1}]$  and  $[\eta_{c_2}]$  respectively. By Theorem 5.2, there is an  $N_1$  such that for  $n \geq N_1$  we have  $[\psi \mu_+^n] \in U_1$  and  $[\mu_-^n] \in U_2$ , and hence as  $U_1$  and  $U_2$  are disjoint,  $[\psi \mu_+^n] \neq [\mu_-^n]$ .

Additionally for  $n \geq N_1$  we have  $\langle T_+^n \psi, \mu_+^n \rangle = \langle T_+^n, \psi \mu_+^n \rangle > 0$  by Theorem 2.10, as  $[\psi \mu_+^n] \neq [\mu_-^n]$ . As  $\langle T_-^n, \mu_+^n \rangle = 0$ , this shows that  $[T_+^n \psi] \neq [T_-^n]$ .  $\square$

## 6. A hyperbolic fully irreducible automorphism for every matrix in $\text{GL}(k, \mathbb{Z})$

Our main theorem now follows easily.

**Theorem 6.1.** *Suppose  $k \geq 3$ . For any  $A \in \text{GL}(k, \mathbb{Z})$ , there is a hyperbolic fully irreducible outer automorphism  $\phi \in \text{Out } F_k$  such that  $\phi_* = A$ .*

*Proof.* Fix  $\psi \in \text{Out } F_k$  such that  $\psi_* = A$ . Let  $T$  be a very small cyclic tree dual to an amalgamated free product with edge stabilizer  $c_1$  (a primitive element of  $F_k$ ) and associated Dehn twist  $\delta_1$ . As is shown in [12], Remark 2.7, given any hyperbolic fully irreducible automorphism  $\theta \in \text{Out } F_k$ , the pair  $T$  and  $T\theta^\ell$  fill for sufficiently large  $\ell$ . The edge stabilizer for  $T\theta^\ell$  is  $\theta^{-\ell}(c_1)$ . Thus for large enough  $\ell$  we can assure that the very small cyclic trees  $T$  and  $T\theta^\ell$  fill and that the conjugacy class of  $\psi(c_1)$  is not equal to the conjugacy class of  $\theta^{-\ell}(c_1)$  (the edge stabilizer for  $T\theta^\ell$ ).

Let  $\delta_2$  be the associated Dehn twist for  $T\theta^\ell$ . By Theorem 2.4, Propositions 3.1 and 4.5 and Corollary 5.4, for large  $m$  and  $n$  the outer automorphism  $(\delta_1^n \delta_2^{-n})^m \psi$  is a hyperbolic fully irreducible element. Since both  $\delta_1$  and  $\delta_2$  act trivially on  $H_1(F_k, \mathbb{Z})$ , we have  $((\delta_1^n \delta_2^{-n})^m \psi)_* = \psi_* = A$ .  $\square$

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