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The $K(\pi, 1)$ conjecture for a class of Artin groups

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Abstract. Salvetti constructed a cellular space B_D for any Artin group A_D defined by a Coxeter graph D. We show that B_D is an Eilenberg–Mac Lane space if $B_{D'}$ is an Eilenberg–Mac Lane space for every subgraph D' of D involving no ∞-edges.

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1. Introduction

A *Coxeter matrix* is a symmetric $n \times n$ matrix whose entries $m(i, j)$ are either positive integers or the symbol ∞, with $m(i, j) = 1$ if and only if $i = j$. Such a matrix is represented by an n-vertex labelled graph D (called a *Coxeter graph*) with edge joining vertices i and j if and only if $m(i, j) \geq 3$; the edge is labelled by $m(i, j)$. The *Artin group* A_D is defined to be the group generated by the set of symbols $S = \{x_1, \dots, x_n\}$ subject to relations $(x_i x_j)_{m(i,j)} = (x_j x_i)_{m(i,j)}$ for all $i \neq j$, where $(xy)_m$ denotes the word $xyxyx\dots$ of length m. The *Coxeter group* W_D is the quotient of A_D obtained by imposing additional relations $x^2 = 1$ for $x \in S$.

For each Coxeter graph D there is an interesting finite CW-space B_D arising as a quotient of a union of certain convex polytopes (see Section 2 for precise details). It has fundamental group $\pi_1(B_D) = A_D$ and we have the following.

Conjecture 1. The space B_D is an Eilenberg–Mac Lane space $K(A_D, 1)$.

From work of Squier in the 1980s (published posthumously [16]) one can deduce that the conjecture holds whenever the Coxeter group W_D is finite. (Squier established a free $\mathbb{Z}A_D$-resolution R_*^D of $\mathbb{Z}$ having the same number of free generators in each degree as the cellular chain complex $C_*(\widetilde{B}_D)$. It is clear that R_*^D coincides with $C_*(\widetilde{B}_D)$ in degrees ≤ 2 and hence R_*^D is the cellular chain complex of the universal cover of some $K(A_D, 1)$. A detailed analysis suggests that R_*^D is in fact the cellular

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chain complex of $\widetilde{B}_D$.) Also, it follows immediately from a result of Appel and Schupp [1, Lemma 6] that the conjecture holds if, for every triple of generators $a, b, c \in S$, the three Artin relators $(ab)_k = (ba)_k$, $(bc)_l = (cb)_l$, $(ac)_m = (ca)_m$ are such that $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} \leq 1$. (In this case B_D is just the standard 2-dimensional CW-space associated to the presentation and Lemma 6 in [1] implies that any element of $\pi_2(B_D)$ would have to be represented by a non-positively curved piecewise euclidean 2-sphere.)

Given a Coxeter graph D we shall say that a subgraph D' is an ∞-*free* subgraph if (1) D' is a connected and full subgraph of D; (2) no edge of D' is labelled by ∞. (In a *full* subgraph an edge must be included if its two boundary vertices are present.) Our main result is obtained using a technique of D. E. Cohen [8] and is the following.

Theorem 2. *An Artin group A_D satisfies Conjecture 1 if $A_{D'}$ satisfies Conjecture 1 for every ∞-free subgraph D' in D.*

For Artin groups satisfying Conjecture 1 the cellular chains of the universal cover $\widetilde{B}_D$ yield an explicit small free $\mathbb{Z}A_D$-resolution from which cohomology calculations can be made. Section 4 gives such a cohomology calculation based on Theorem 2.

To place Theorem 2 in context we mention that there is an alternative statement of Conjecture 1. Every Coxeter group W_D acts canonically as a linear group generated by "reflections" on a real vector space V and properly discontinuously on an open cone $I \subset V$ called the *Tits cone*. Denote by A the set of reflecting hyperplanes of W_D and consider the following subspace of $\mathbb{C} \otimes V = V \oplus \mathbf{i}V$:

$$M(W_D) = I \oplus \mathbf{i}V \setminus \left(\bigcup_{H \in A} H \oplus \mathbf{i}H\right).$$

The group W_D acts freely and properly discontinuously on $M(W)$ and the quotient $N(W_D) = M(W_D)/W_D$ has fundamental group equal to A_D.

Conjecture 3. The space $N(W_D)$ is an Eilenberg–Mac Lane space $K(A_D, 1)$.

Conjecture 3 is known as the $K(\pi, 1)$-*conjecture* for Artin groups and is attributed to Arnold, Pham and Thom in [5]. It has been proved in many cases: Deligne [10] proved it for finite W_D; Hendriks [12] proved it for W_D of large type; Charney and Davis [5] proved it when W_D is 2-dimensional and when W_D is of FC type; Charney and Peifer [7] proved it for W_D of affine type $\tilde{A}_n$; Callegaro, Moroni and Salvetti [4] have recently proved it for W_D of affine type $\widetilde{B}_n$.

Salvetti [15] showed that the space B_D is homotopy equivalent to $N(W_D)$ for finite W_D. This homotopy equivalence was extended to arbitrary W_D by Charney and Davis [6]. Conjectures 1 and 3 are thus equivalent and so Theorem 2 can consequently be viewed as a generalisation of the solution to the $K(\pi, 1)$-conjecture for Artin groups of FC type provided in [5]. (Recall that A_D is said to be of *FC type* if $W_{D'}$ is finite for every ∞-free subgraph D' in D.)

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2. The space B_D

Let D be a Coxeter graph. Let $\bar{x}$ be the image in W_D of the generator $x \in S \subset A_D$ and set $\overline{S} = \{\bar{x} : x \in S\}$. We say that D is of *finite type* if the Coxeter group W_D is finite.

Assume for the moment that D is of finite type and let $n = |S|$. Then W_D can be realized as a group of orthogonal transformations of $\mathbb{R}^n$ with generators $\bar{x}$ equal to reflections [9]. Let A be the set of hyperplanes corresponding to all the reflections in W_D. For any point e in $\mathbb{R}^n \setminus A$ we denote by P_D the convex hull of the orbit of e under the action of W_D. The face lattice of the n-dimensional convex polytope P_D depends only on the graph D. (To see this, first note that the vertices of P_D are the points $w \cdot e$ for $w \in W_D$ and that there is an edge between $w \cdot e$ and $w' \cdot e$ if and only if $w^{-1}w' \in \overline{S}$. Thus the combinatorial type of the 1-skeleton of P_D does not depend on the choice of point e. Furthermore, each vertex of the n-dimensional polytope P_D is incident with precisely n edges; hence P_D is simple and the face lattice of the polytope is determined by the combinatorial type of the 1-skeleton [2].)

Label each edge in P_D by the generating reflection $\bar{x} = w^{-1}w' \in \overline{S}$ determined by the edge's boundary vertices $w \cdot e$, $w' \cdot e$. Define the *length* of an element g in W_D to be the shortest length of a word in the generators representing it. It is possible to orient each edge in P_D so that its initial vertex gv and final vertex $g'v$ are such that the length of g is less than the length of g'. With this edge orientation the 1-skeleton coincides with the Hasse diagram for the weak Bruhat order on W_D. Each k-face in P_D has a least vertex in the weak Bruhat order. Reading the edge labels along the boundary of any 2-face, starting at the least vertex and using edge orientations to determine exponents ± 1, yields a relator $(xy)_{m(i,j)}(yx)^{-1}_{m(i,j)}$ of the Artin group A_D. Furthermore, if F is any k-face of P_D, then $V_F = \{w \in W_D : w \cdot e \in F\}$ is a left coset of the *parabolic subgroup* $\langle T \rangle$ of W_D generated by some subset $T \subset \overline{S}$ of size $|T| = k$; this induces an isomorphism between the face lattice of P_D and the poset of cosets $\{w \cdot \langle T \rangle : T \subset \overline{S}, w \in W_D\}$ ordered by inclusion.

The above description of the polytope P_D is well known. (We note that many authors prefer to deal with the dual polytope: since P_D is simple the dual is simplicial.)

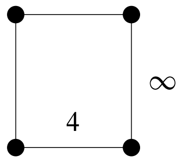
The space B_D is obtained from the polytope P_D by isometrically identifying any two cells with similarly labelled 1-skeleta. More precisely, the group W_D acts cellularly on P_D. If a k-face F is mapped to a k-face F' under the action of $w \in W_D$, then there is a unique $w_0 \in W_D$ which maps F to F' in such a way that the least vertex of F maps to the least vertex of F'; we identify $w_0 \cdot f$ with f for each point $f \in F$. Thus the face lattice of B_D is isomorphic to the poset of subsets of $\overline{S}$.

Suppose now that D is not of finite type. We define a subgraph D_i of D to be *maximal finite* if D_i is a full subgraph of D of finite type that is not contained in any larger subgraph of finite type. Let $D_1, \dots, D_k$ be the list of maximal finite subgraphs of D. We denote by $D_i \cap D_j$ the full subgraph of D with vertices common to D_i and

D_j. There is a canonical embedding of the polytope $P_{D_i \cap D_j}$ into the polytope P_{D_i}; such embeddings allow us to define P_D as the amalgamated sum of the polytopes $P_{D_1}, \dots, P_{D_k}$. The space B_D is the connected space obtained from P_D by isometrically identifying any two cells with similarly labelled 1-skeleta; the identification is the unique one which respects orientations of edges. The face lattice of the space B_D is isomorphic to the poset $S^f = \{T \subset \overline{S} \colon |\langle T \rangle| < \infty\}$ ordered by inclusion.

Note that if a Coxeter graph D with vertex set S is a disjoint union of two Coxeter graphs D', D'' with vertex sets S', S'' respectively, then there is a poset isomorphism $S^f = S'^f \times S''^f$. It is not difficult to see that this poset isomorphism extends to a CW-homeomorphism $B_D = B_{D'} \times B_{D''}$.

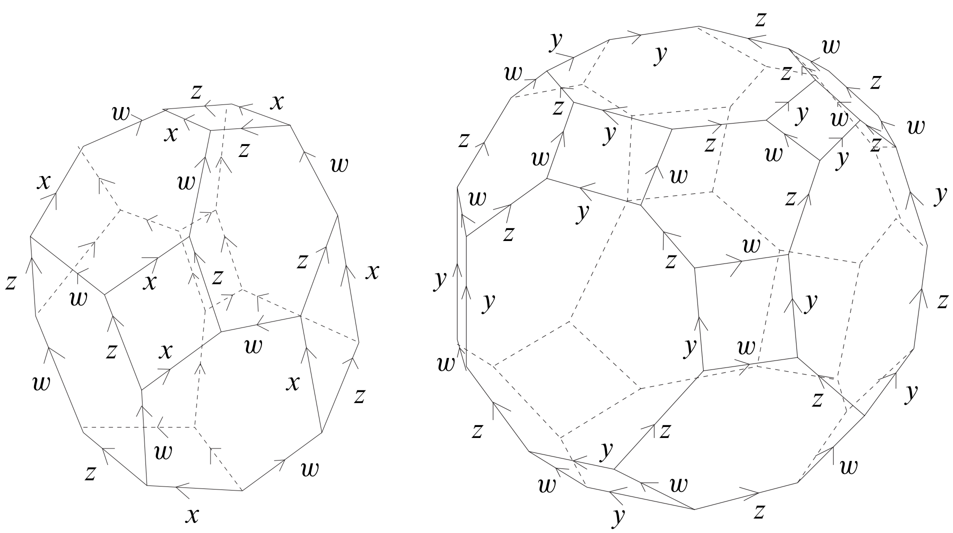
Example. Consider the graph



where edges whose label is not indicated are assumed to have edge label 3. Letting vertices correspond to generators w, x, y, z (starting at the top left corner and working clockwise) the associated Artin group is

$$A_D = \langle w, x, y, z : wxw = xwx,\ wy = yw,\\ wzw = zwz,\ xz = zx,\ yzyz = zyzy \rangle.$$

The 3-dimensional space B_D is obtained from the following two 3-dimensional polytopes by identifying similarly labelled faces and edges.



The space B_D contains four 1-cells, five 2-cells and two 3-cells. □

3. Proof of Theorem 2

Suppose that A_D satisfies the hypothesis of the Theorem 2. Let X_D denote the universal covering space of B_D. We shall use induction on the number of infinity edges in D and the number of connected components in D to show that X_D is contractible.

If there are no infinity edges and the graph D is connected then X_D is contractible by hypothesis.

If D is not connected then A_D is a direct product $A_D = A_{D'} \times A_{D''}$ of two non-trivial Artin groups $A_{D'}$ and $A_{D''}$ where the graph D is the disjoint union of D' and D''. The space B_D is the direct product $B_{D'} \times B_{D''}$. Thus X_D is contractible if and only if both $X_{D'}$ and $X_{D''}$ are contractible. Hence, by induction on the number of connected components in D, it suffices to prove the theorem in the case where the graph D is connected.

Suppose that the Coxeter graph D is connected. Suppose that there is an infinity edge in D whose endpoints correspond to the generators $a, b \in S = \{x_1, \ldots, x_n\}$. Let $A_{\hat{a}}$ be the subgroup of A_D generated by $S \setminus \{a\}$, and $A_{\hat{a},\hat{b}}$ the subgroup generated by $S \setminus \{a, b\}$. Let $D \setminus \{a\}$ denote the graph obtained from D by removing vertex a and all edges incident with a. Let $D \setminus \{a, b\}$ be the subgraph obtained by removing vertices a, b and all edges incident with them. There are clearly surjective homomorphisms $A_{D\setminus\{a\}} \to A_{\hat{a}}$ and $A_{D\setminus\{a,b\}} \to A_{\hat{a},\hat{b}}$. A result of H. van der Lek [13] (see also [14]) shows that these surjections are in fact isomorphisms. Note that each of the groups $A_{D\setminus\{a\}}$, $A_{D\setminus\{b\}}$, $A_{D\setminus\{a,b\}}$ is an Artin group satisfying the hypothesis of the theorem and with Coxeter graph involving fewer infinity edges than are in D.

Suppose that D has $n \geq 1$ infinity edges. As an inductive hypothesis assume that the theorem holds for all Artin groups satisfying its hypothesis and having Coxeter graph with fewer than n infinity edges. Thus we can assume that $B_{D\setminus\{a\}}$, $B_{D\setminus\{b\}}$, $B_{D\setminus\{a,b\}}$ are classifying spaces for the subgroups $A_{\hat{a}}$, $A_{\hat{b}}$, $A_{\hat{a},\hat{b}}$. Consider the homotopy pushout

$$\begin{array}{ccc} B_{D\setminus\{a,b\}} & \longrightarrow & B_{D\setminus\{a\}} \\ \downarrow & & \downarrow \\ B_{D\setminus\{b\}} & \longrightarrow & W. \end{array}$$

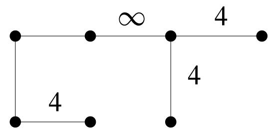
The space $W = B_{D\setminus\{a\}} \cup B_{D\setminus\{b\}}$ is precisely the space $W = B_D$. Now by a theorem of J. H. C. Whitehead (see for example [3], Chapter II-7) the space W is a classifying space. Hence its universal cover X_D is contractible. □

An argument similar to the above was used in [8] to study properties of graph products of groups. Also, a version of this proof for Artin groups of FC type can be found in [5] as a remark following Lemma 4.3.7.

4. An application

The cellular chain complex $C_*(X_D)$ has been implemented in the computational algebra package HAP [11]. In cases where the $K(\pi,1)$ conjecture is known to hold this chain complex is a free $\mathbb{Z}A_D$-resolution of $\mathbb{Z}$ and can be used to compute the cohomology of the Artin group A_D. The following was obtained in this way.

Proposition 4. *The Artin group A_D defined by the Coxeter graph*



has integral cohomology groups

$$H^0(A_D,\mathbb{Z})\cong\mathbb{Z},\quad H^1(A_D,\mathbb{Z})\cong\mathbb{Z}^5,\quad H^2(A_D,\mathbb{Z})\cong\mathbb{Z}^{11},$$
$$H^3(A_D,\mathbb{Z})\cong\mathbb{Z}_2\oplus\mathbb{Z}^{14},\quad H^4(A_D,\mathbb{Z})\cong\mathbb{Z}_2^2\oplus\mathbb{Z}^{12},\quad H^5(A_D,\mathbb{Z})\cong\mathbb{Z}_2\oplus\mathbb{Z}^6,$$
$$H^6(A_D,\mathbb{Z})\cong\mathbb{Z},\quad H^n(A_D,\mathbb{Z})=0\ (n\geq 7).$$

Proof. The graph D is such that for every ∞-free subgraph D' the Artin group $A_{D'}$ satisfies the $K(\pi,1)$ conjecture by results mentioned in Section 1. By Theorem 2 the group A_D itself satisfies the $K(\pi,1)$ conjecture. We can thus use the computer implementation of $C_*(X_D)$ in [11] to make the cohomology calculations. The space X_D is 6-dimensional in this example. □

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