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$\mathrm{SL}_n(\mathbb{Z}[t])$ is not FP_{n-1}

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Abstract. We prove the result from the title using the geometry of Euclidean buildings.

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1. Introduction

Little is known about the finiteness properties of $\mathrm{SL}_n(\mathbb{Z}[t])$ for arbitrary n.

In 1959 Nagao proved that if k is a field then $\mathrm{SL}_2(k[t])$ is a free product with amalgamation [Na]. It follows from his description that $\mathrm{SL}_2(\mathbb{Z}[t])$ and its abelianization are not finitely generated.

In 1977 Suslin proved that when $n \geq 3$, $\mathrm{SL}_n(\mathbb{Z}[t])$ is finitely generated by elementary matrices [Su]. It follows that $H_1(\mathrm{SL}_n(\mathbb{Z}[t]), \mathbb{Z})$ is trivial when $n \geq 3$.

More recent, Krstić and McCool proved in [Kr-Mc] that $\mathrm{SL}_3(\mathbb{Z}[t])$ is not finitely presented.

In this paper we provide a generalization of the results of Nagao and Krstić–McCool mentioned above for the groups $\mathrm{SL}_n(\mathbb{Z}[t])$.

Theorem 1. *If $n \geq 2$, then $\mathrm{SL}_n(\mathbb{Z}[t])$ is not of type FP_{n-1}.*

Recall that a group Γ is *of type FP_m* if there exists a projective resolution of $\mathbb{Z}$ as the trivial $\mathbb{Z}\Gamma$ module

$$P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

where each P_i is a finitely generated, projective $\mathbb{Z}\Gamma$ module.

In particular, Theorem 1 implies that there is no $K(\mathrm{SL}_n(\mathbb{Z}[t]), 1)$ with finite $(n-1)$-skeleton, where $K(G, 1)$ is the Eilenberg–Mac Lane space for G.

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1.1. Outline of the paper. The general outline of this paper is modelled on the proofs in [Bu-Wo 1] and [Bu-Wo 2], though some important modifications have to be made to carry out the proof in this setting.

As in [Bu-Wo 1] and [Bu-Wo 2], our approach is to apply Brown's filtration criterion [Br 1]. Here we will examine the action of $\mathrm{SL}_n(\mathbb{Z}[t])$ on the locally infinite Euclidean building for $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$. In Section 2 we will show that the infinite groups that arise as cell stabilizers for this action are of type FP_m for all m, which is a technical condition that is needed for our application of Brown's criterion.

In Section 3 we will demonstrate the existence of a family of diagonal matrices that will imply the existence of a "nice" isometrically embedded codimension 1 Euclidean space in the building for $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$. In [Bu-Wo 1] analogous families of diagonal matrices were constructed using some standard results from the theory of algebraic groups over locally compact fields. Because $\mathbb{Q}((t^{-1}))$ is not locally compact, our treatment in Section 3 is quite a bit more hands on.

Section 4 contains the main body of our proof. We use translates of portions of the codimension 1 Euclidean subspace found in Section 3 to construct spheres in the Euclidean building for $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ (also of codimension 1). These spheres will lie "near" an orbit of $\mathrm{SL}_n(\mathbb{Z}[t])$, but will be nonzero in the homology of cells "not as near" the same $\mathrm{SL}_n(\mathbb{Z}[t])$ orbit. Theorem 1 will then follow from Brown's criterion.

1.2. Background material. Our proof relies heavily on the geometry of the Euclidean and spherical buildings for $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$. A good source of information for the former topic is Chapter 6 of [Br 2]. For the latter, we recommend Chapter 5 of [Ti].

2. Stabilizers

Lemma 2. *If X is the Euclidean building for $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$, then the $\mathrm{SL}_n(\mathbb{Z}[t])$ stabilizers of cells in X are FP_m for all m.*

Proof. Let $x_0 \in X$ be the vertex stabilized by $\mathrm{SL}_n(\mathbb{Q}[[t^{-1}]])$. We denote a diagonal matrix in $\mathrm{GL}_n(\mathbb{Q}((t^{-1})))$ with entries $s_1, s_2, \dots, s_n \in \mathbb{Q}((t^{-1}))^\times$ by $D(s_1, s_2, \dots, s_n)$, and we let $\mathfrak{S} \subseteq X$ be the sector based at x_0 and containing vertices of the form $D(t^{m_1}, t^{m_2}, \dots, t^{m_n})x_0$ where each $m_i \in \mathbb{Z}$ and $m_1 \geq m_2 \geq \dots \geq m_n$.

The sector $\mathfrak{S}$ is a fundamental domain for the action of $\mathrm{SL}_n(\mathbb{Q}[t])$ on X (see [So]). In particular, for any vertex $z \in X$, there is some $h'_z \in \mathrm{SL}_n(\mathbb{Q}[t])$ and some integers $m_1 \geq m_2 \geq \dots \geq m_n$ with $z = h'_z D_z(t^{m_1}, t^{m_2}, \dots, t^{m_n})x_0$. We let $h_z = h'_z D_z(t^{m_1}, t^{m_2}, \dots, t^{m_n})$.

For any $N \in \mathbb{N}$, let W_N be the $(N+1)$-dimensional vector space

$$W_N = \{\, p(t) \in \mathbb{C}[t] \mid \deg\big(p(t)\big) \leq N \}$$

which is endowed with the obvious $\mathbb{Q}$–structure. If $N_1, \ldots, N_{n^2}$ in $\mathbb{N}$ are arbitrary then let

$$\boldsymbol{G}_{\{N_1,\ldots,N_{n^2}\}} = \left\{\boldsymbol{x} \in \prod_{i=1}^{n^2} W_{N_i} \mid \det(\boldsymbol{x}) = 1\right\}$$

where $\det(\boldsymbol{x})$ is a polynomial in the coordinates of $\boldsymbol{x}$. To be more precise this is obtained from the usual determinant function when one considers the usual $n \times n$ matrix presentation of $\boldsymbol{x}$, and calculates the determinant in $\mathrm{Mat}_n(\mathbb{C}[t])$.

For our choice of vertex $z \in X$ above, the stabilizer of z in $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ equals $h_z \mathrm{SL}_n(\mathbb{Q}[[t^{-1}]]) h_z^{-1}$. And with our fixed choice of h_z, there clearly exist some $N_i^z \in \mathbb{N}$ such that the stabilizer of the vertex z in $\mathrm{SL}_n(\mathbb{Q}[t])$ is $\boldsymbol{G}_{\{N_1^z,\ldots,N_{n^2}^z\}}(\mathbb{Q})$. Furthermore, conditions on N_i^z force a group structure on $\boldsymbol{G}_z = \boldsymbol{G}_{\{N_1^z,\ldots,N_{n^2}^z\}}$. Therefore, the stabilizer of z in $\mathrm{SL}_n(\mathbb{Q}[t])$ is the $\mathbb{Q}$–points of the affine $\mathbb{Q}$-group $\boldsymbol{G}_z$, and the stabilizer of z in $\mathrm{SL}_n(\mathbb{Z}[t])$ is $\boldsymbol{G}_z(\mathbb{Z})$.

The action of $\mathrm{SL}_n(\mathbb{Q}[t])$ on X is type preserving, so if $\sigma \subset \mathfrak{S}$ is a simplex with vertices $z_1, z_2, \ldots, z_m$, then the stabilizer of σ in $\mathrm{SL}_n(\mathbb{Z}[t])$ is simply

$$\left(\boldsymbol{G}_{z_1} \cap \cdots \cap \boldsymbol{G}_{z_m}\right)(\mathbb{Z}).$$

That is, the stabilizer of σ in $\mathrm{SL}_n(\mathbb{Z}[t])$ is an arithmetic group, and Borel–Serre proved that any such group is FP_m for all m [Bo-Se]. □

3. Polynomial points of tori

This section is devoted exclusively to a proof of the following

Proposition 3. *There is a group $A \leq \mathrm{SL}_n(\mathbb{Z}[t])$ such that the following holds:*

(i) $A \cong \mathbb{Z}^{n-1}$.

(ii) *There is some $g \in \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ such that gAg^{-1} is a group of diagonal matrices.*

(iii) *No nontrivial element of A fixes a point in the Euclidean building for $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$.*

The proof of this proposition is modelled on a classical approach to finding diagonalizable subgroups of $\mathrm{SL}_n(\mathbb{Z})$. The proof will take a few steps.

3.1. A polynomial over $\mathbb{Z}[t]$ with roots in $\mathbb{Q}((t^{-1}))$. Let $\{p_1, p_2, p_3, \ldots\} = \{2, 3, 5, \ldots\}$ be the sequence of prime numbers. Let $q_1 = 1$. For $2 \leq i \leq n$, let $q_i = p_{i-1} + 1$.

Let $f(x) \in \mathbb{Z}[t][x]$ be the polynomial given by

$$f(x) = \Big[\prod_{i=1}^{n}(x + q_i t)\Big] - 1.$$

It will be clear by the conclusion of our proof that $f(x)$ is irreducible over $\mathbb{Q}(t)$, but we will not need to use this directly.

Lemma 4. *There is some $\alpha \in \mathbb{Q}((t^{-1}))$ such that $f(\alpha) = 0$.*

Proof. We want to show that there are $c_i \in \mathbb{Q}$ such that if $\alpha = \sum_{i=0}^{\infty} c_i t^{1-in}$ then $f(\alpha) = 0$.

To begin let $c_0 = -1$. We will define the remaining c_i recursively. Define $c_{i,k}$ by $\alpha + q_k t = \sum_{i=0}^{\infty} c_{i,k} t^{1-in}$. Thus, $c_{i,k} = c_i$ when $i \geq 1$, each $c_{0,k}$ is contained in $\mathbb{Q}$, and $c_{0,1} = 0$.

That α is a root of f is equivalent to

$$1 = \prod_{k=1}^{n}(\alpha + q_k t) = \prod_{k=1}^{n}\Big(\sum_{i=0}^{\infty} c_{i,k} t^{1-in}\Big) = \sum_{i=0}^{\infty}\Big(\sum_{\sum_{k=1}^{n} i_k = i}\Big(\prod_{k=1}^{n} c_{i_k,k}\Big)\Big)t^{n(1-i)}.$$

Our task is to find c_m's so that the above is satisfied.

Note that for the above equation to hold we must have

$$0 \cdot t^n = \sum_{\sum_{k=1}^{n} i_k = 0}\Big(\prod_{k=1}^{n} c_{i_k,k}\Big)t^{n(1-0)}.$$

That is,

$$0 = \prod_{k=1}^{n} c_{0,k}$$

which is an equation we know is satisfied because $c_{0,1} = 0$. Now assume that we have determined $c_0, c_1, \ldots, c_{m-1} \in \mathbb{Q}$. We will find $c_m \in \mathbb{Q}$.

Notice that the first coefficient in our Laurent series expansion above which involves c_m is the coefficient for the t^{-nm} term. This follows from the fact that each i_k is nonnegative.

Since

$$\sum_{\sum_{k=1}^{n} i_k = m}\Big(\prod_{k=1}^{n} c_{i_k,k}\Big)$$

is the coefficient of the t^{-nm} term in the expansion of 1, we have

$$0 = \sum_{\sum_{k=1}^{n} i_k = m}\Big(\prod_{k=1}^{n} c_{i_k,k}\Big).$$

The above equation is linear over $\mathbb{Q}$ in the single variable c_m and the coefficient of c_m is nonzero. Indeed, $\sum_{k=1}^{n} i_k = m$, each $i_k \geq 0$, and $c_0, \dots, c_{m-1} \in \mathbb{Q}$ are assumed to be known quantities. Thus, $c_m \in \mathbb{Q}$. □

3.2. Matrices representing ring multiplication. By Lemma 4 we have that the field $\mathbb{Q}(t)(\alpha) \leq \mathbb{Q}((t^{-1}))$ is an extension of $\mathbb{Q}(t)$ of degree d where $d \leq n$. It follows that $\mathbb{Z}[t][\alpha]$ is a free $\mathbb{Z}[t]$-module of rank d with basis $\{1, \alpha, \alpha^2, \dots, \alpha^{d-1}\}$.

For any $y \in \mathbb{Z}[t][\alpha]$, the action of y on $\mathbb{Q}(t)(\alpha)$ by multiplication is a linear transformation that stabilizes $\mathbb{Z}[t][\alpha]$. Thus, we have a representation of $\mathbb{Z}[t][\alpha]$ into the ring of $d \times d$ matrices with entries in $\mathbb{Z}[t]$. We embed the ring of $d \times d$ matrices with entries in $\mathbb{Z}[t]$ into the upper left corner of the ring of $n \times n$ matrices with entries in $\mathbb{Z}[t]$.

By Lemma 4

$$\prod_{i=1}^{n} (\alpha + q_i t) = 1$$

so each of the following matrices are invertible:

$$\alpha + q_1 t, \ \alpha + q_2 t, \ \dots, \ \alpha + q_n t.$$

(We will be blurring the distinction between the elements of $\mathbb{Z}[t][\alpha]$ and the matrices that represent them.)

For $1 \leq i \leq n-1$, we let $a_i = \alpha + q_{i+1} t$. Since a_i is invertible, it is an element of $\mathrm{GL}_n(\mathbb{Z}[t])$, and hence has determinant ± 1. By replacing each a_i with its square, we may assume that $a_i \in \mathrm{SL}_n(\mathbb{Z}[t])$ for all i. We let $A = \langle a_1, \dots a_{n-1} \rangle$ so that A is clearly abelian as it is a representation of multiplication in an integral domain. This group A will satisfy Proposition 3.

3.3. *A* is free abelian on the a_i. To prove part (i) of Proposition 3 we have to show that if there are $m_i \in \mathbb{Z}$ with

$$\prod_{i=1}^{n-1} a_i^{m_i} = 1$$

then each $m_i = 0$. But the first nonzero term in the Laurent series expansion for α is $-t$, which implies that the first nonzero term in the Laurent series expansion for each a_i is $-t + q_{i+1} t = p_i t$. Hence, the first nonzero term of

$$\prod_{i=1}^{n-1} a_i^{m_i} = 1$$

is

$$\prod_{i=1}^{n-1} (p_i t)^{m_i} = t^0.$$

Thus

$$\prod_{i=1}^{n-1} p_i^{m_i} = 1$$

and it follows by the uniqueness of prime factorization that $m_i = 0$ for all i as desired.

Thus, part (i) of Proposition 3 is proved.

3.4. *A* is diagonalizable. Recall that α is a $d \times d$ matrix with entries in $\mathbb{Z}[t]$ where d is the degree of the minimal polynomial of α over $\mathbb{Q}(t)$. Let that minimal polynomial be $q(x)$. Because the characteristic of $\mathbb{Q}(t)$ equals 0, $q(x)$ has distinct roots in $\mathbb{Q}(t)(\alpha)$.

Let $Q(x)$ be the characteristic polynomial of the matrix α. The polynomial Q also has degree d and leading coefficient ± 1 with $Q(\alpha) = 0$. Therefore, $q = \pm Q$. Hence, Q has distinct roots in $\mathbb{Q}(t)(\alpha)$ which implies that α is diagonalizable over $\mathbb{Q}(t)(\alpha) \leq \mathbb{Q}((t^{-1}))$. That is to say that there is some $g \in \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ such that $g\alpha g^{-1}$ is diagonal.

Because every element of $\mathbb{Z}[t][\alpha]$ is a linear combination of powers of α, we have that $g(\mathbb{Z}[t][\alpha])g^{-1}$ is a set of diagonal matrices. In particular, we have proved part (ii) of Proposition 3.

3.5. *A* has trivial stabilizers. To prove part (iii) of Proposition 3 we begin with the following

Lemma 5. *If $\Gamma \leq \mathrm{SL}_n(\mathbb{Q}[t])$ is bounded under the valuation for $\mathbb{Q}((t^{-1}))$, then the eigenvalues for any $\gamma \in \Gamma$ lie in $\overline{\mathbb{Q}}$.*

Proof. We let X be the Euclidean building for $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$. By the Bruhat–Tits fixed point theorem, $\Gamma z = z$ for some $z \in X$.

Let $x_0 \in X$ be the vertex stabilized by $\mathrm{SL}_n(\mathbb{Q}[[t^{-1}]])$. We denote a diagonal matrix in $\mathrm{GL}_n(\mathbb{Q}((t^{-1})))$ with entries $s_1, s_2, \ldots, s_n \in \mathbb{Q}((t^{-1}))^\times$ by $D(s_1, s_2, \ldots, s_n)$, and we let $\mathfrak{S} \subseteq X$ be the sector based at x_0 and containing vertices of the form $D(t^{m_1}, t^{m_2}, \ldots, t^{m_n})x_0$ where each $m_i \in \mathbb{Z}$ and $m_1 \geq m_2 \geq \cdots \geq m_n$.

The sector $\mathfrak{S}$ is a fundamental domain for the action of $\mathrm{SL}_n(\mathbb{Q}[t])$ on X [So] which implies that there is some $h \in \mathrm{SL}_n(\mathbb{Q}[t])$ with $hz \in \mathfrak{S}$.

Clearly we have $(h\Gamma h^{-1})hz = hz$, and since eigenvalues of $h\Gamma h^{-1}$ are the same as those for Γ, we may assume that Γ fixes a vertex $z \in \mathfrak{S}$.

Fix $m_1, \ldots, m_n \in \mathbb{Z}$, $m_1 \geq \cdots \geq m_n \geq 0$, such that $z = D(t^{m_1}, \ldots, t^{m_n})x_0$. Without loss of generality, there is a partition of n – say $\{k_1, \ldots, k_\ell\}$ – such that

$$\{m_1, \ldots, m_n\} = \{q_1, \ldots, q_1,\ q_2, \ldots, q_2,\ \ldots,\ q_\ell, \ldots q_\ell\}$$

where each q_i occurs exactly k_i times and

$$q_1 > q_2 > \cdots > q_\ell.$$

We have that $D(t^{m_1}, \ldots, t^{m_n})^{-1}\Gamma D(t^{m_1}, \ldots, t^{m_n})x_0 = x_0$. That gives us, $D(t^{m_1}, \ldots, t^{m_n})^{-1}\Gamma D(t^{m_1}, \ldots, t^{m_n}) \subset \mathrm{SL}_n(\mathbb{Q}[[t^{-1}]])$. Furthermore, a trivial calculation of resulting valuation restrictions for the entries of

$$D(t^{m_1}, \ldots, t^{m_n})\mathrm{SL}_n(\mathbb{Q}[[t^{-1}]])D(t^{m_1}, \ldots, t^{m_n})^{-1}$$

shows that Γ is contained in a subgroup of $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ that is isomorphic to

$$\prod_{i=1}^{\ell} \mathrm{SL}_{k_i}(\mathbb{Q}) \ltimes U$$

where $U \leq \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ is a group of upper-triangular unipotent matrices.

The lemma is proved. □

Our proof of Proposition 3 will conclude by proving

Lemma 6. *No nontrivial element of A fixes a point in the Euclidean building for* $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$.

Proof. Suppose $a \in A$ fixes a point in the building. We will show that $a = 1$. Let $F(x) \in \mathbb{Z}[t][x]$ be the characteristic polynomial for $a \in \mathrm{SL}_n(\mathbb{Z}[t])$. Then

$$F(x) = \pm \prod_{i=1}^{n}(x - \beta_i)$$

where each $\beta_i \in \mathbb{Q}((t^{-1}))$ is an eigenvalue of a. By the previous lemma, each $\beta_i \in \overline{\mathbb{Q}}$. Hence, each $\beta_i \in \mathbb{Q} = \overline{\mathbb{Q}} \cap \mathbb{Q}((t^{-1}))$. It follows that $F(x) \in \mathbb{Z}[x]$ so that each β_i is an algebraic integer contained in $\mathbb{Q}$. We conclude that each β_i is contained in $\mathbb{Z}$.

Recall, that a has determinant 1, and that the determinant of a can be expressed as $\prod_{i=1}^{n} \beta_i$. Hence, each β_i is a unit in $\mathbb{Z}$, so each eigenvalue $\beta_i = \pm 1$. It follows – by the diagonalizability of a – that a is a finite order element of $A \cong \mathbb{Z}^{n-1}$. That is, $a = 1$. □

We have completed our proof of Proposition 3.

4. Body of the proof

Let $P \leq \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ be the subgroup where each of the first $n-1$ entries along the bottom row equal 0. Let $R_u(P) \leq P$ be the subgroup of elements that contain a $(n-1)\times(n-1)$ copy of the identity matrix in the upper left corner. Thus $R_u(P) \cong \mathbb{Q}((t^{-1}))^{n-1}$ with the operation of vector addition.

Let $L \leq P$ be the copy of $\mathrm{SL}_{n-1}(\mathbb{Q}((t^{-1})))$ in the upper left corner of $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$. We apply Proposition 3 to L (notice that the n in the proposition is now an $n-1$) to derive a subgroup $A \leq L$ that is isomorphic to $\mathbb{Z}^{n-2}$. By the same proposition, there is a matrix $g \in L$ such that gAg^{-1} is diagonal.

Let $b \in \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ be the diagonal matrix given in the notation from the proofs of Lemmas 2 and 5 as $D(t,t,\dots,t,t^{-(n-1)})$. Note that $b \in P$ commutes with L, and therefore, with A. Thus the Zariski closure of the group generated by b and A determines an apartment in X, namely $g^{-1}\mathcal{A}$ where $\mathcal{A}$ is the apartment corresponding to the diagonal subgroup of $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$.

4.1. Actions on $g^{-1}\mathcal{A}$. If $x_* \in g^{-1}\mathcal{A}$, then it follows from Proposition 3 that the convex hull of the orbit of x_* under A is an $(n-2)$-dimensional affine space that we will name V_{x_*}. Furthermore, the orbit Ax_* forms a lattice in the space V_{x_*}.

We let $g^{-1}\mathcal{A}(\infty)$ be the visual boundary of $g^{-1}\mathcal{A}$ in the Tits boundary of X. Recall that the Tits boundary of X is isomorphic to the spherical building for $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$. The definition of visual boundary used above is the standard definition from CAT(0) geometry.

The visual boundary of V_{x_*} is clearly an equatorial sphere in $g^{-1}\mathcal{A}(\infty)$. Precisely, we let P^- be the transpose of P. Then P and P^- are opposite vertices in $g^{-1}\mathcal{A}(\infty)$. It follows that there is a unique sphere in $g^{-1}\mathcal{A}(\infty)$ that is realized by all points equidistant to P and P^-. We call this sphere S_{P,P^-}.

Lemma 7. *The visual boundary of V_{x_*} equals S_{P,P^-}.*

Proof. Since $g \in P \cap P^-$, it suffices to prove that gV_{x_*} is the sphere in the boundary of $\mathcal{A}$ that is determined by the vertices P and P^-.

Note that gV_{x_*} is a finite Hausdorff distance from any orbit of a point in $\mathcal{A}$ under the action of the diagonal subgroup of L. The result follows by observing that the inverse transpose map on $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ stabilizes diagonal matrices while interchanging P and P^-. □

We let $R_1, R_2, \dots, R_{n-1}$ be the standard root subgroups of $R_u(P)$. Recall that associated to each R_i there is a closed geodesic hemisphere $H_i \subseteq \mathcal{A}(\infty)$ such that any nontrivial element of R_i fixes H_i pointwise and translates any point in the open hemisphere $\mathcal{A}(\infty) - H_i$ outside of $\mathcal{A}(\infty)$. Note that ∂H_i is a codimension 1 geodesic sphere in $\mathcal{A}(\infty)$.

We let $M \subseteq g^{-1}\mathcal{A}(\infty)$ be the union of chambers in $g^{-1}\mathcal{A}(\infty)$ that contain the vertex P. There is also an equivalent geometric description of M:

Lemma 8. *The union of chambers $M \subseteq g^{-1}\mathcal{A}(\infty)$ can be realized as an $(n-2)$-simplex. Furthermore,*

$$M = \bigcap_{i=1}^{n-1} g^{-1} H_i$$

and, when M is realized as a single simplex, each of the $n-1$ faces of M is contained in a unique equatorial sphere $g^{-1}\partial H_i = \partial g^{-1} H_i$.

Proof. Let $M' \subseteq \mathcal{A}(\infty)$ be the union of chambers in $\mathcal{A}(\infty)$ containing the vertex P. Since $M = g^{-1}M'$, it suffices to prove that M' is an $(n-2)$-simplex with $M' = \bigcap_{i=1}^{n-1} H_i$ and with each face of M' contained in a unique ∂H_i.

For any nonempty, proper subset $I \subseteq \{1, 2, \dots, n\}$, we let V_I be the $|I|$-dimensional vector subspace of $\mathbb{Q}((t^{-1}))^n$ spanned by the coordinates given by I, and we let P_I be the stabilizer of V_I in $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$. For example, $P = P_{\{1,2,\dots,n-1\}}$.

Recall that the vertices of $\mathcal{A}(\infty)$ are given by the parabolic groups P_I, that edges connect P_I and $P_{I'}$ exactly when $I \subseteq I'$ or $I' \subseteq I$, and that the remaining simplicial description of $\mathcal{A}(\infty)$ is given by the condition that $\mathcal{A}(\infty)$ is a flag complex.

We let $\mathcal{V}$ be the set of vertices in $\mathcal{A}(\infty)$ of the form P_J where $\emptyset \neq J \subseteq \{1, 2, \dots, n-1\}$. Note that M' is exactly the set of vertices $\mathcal{V}$ together with the simplices described by the incidence relations inherited from $\mathcal{A}(\infty)$. Thus, M' is easily seen to be isomorphic to a barycentric subdivision of an abstract $(n-2)$-simplex. Indeed, if $\overline{M}'$ is the abstract simplex on vertices $P_{\{1\}}, P_{\{2\}}, \dots, P_{\{n-1\}}$, then a simplex of dimension k in $\overline{M}'$ corresponds to a unique $P_J \in \mathcal{V}$ with $|J| = k+1$. So we have that M' can be topologically realized as an $(n-2)$-simplex.

Let F_i be a face of the simplex $\overline{M}'$. Then there is some $1 \leq i \leq n-1$ such that the set of vertices of F_i is exactly $\{P_{\{1\}}, P_{\{2\}}, \dots, P_{\{n-1\}}\} - P_{\{i\}}$.

Note that $R_i V_I = V_I$ exactly when $n \in I$ implies $i \in I$. It follows that R_i fixes M' pointwise, and thus $M' \subseteq H_i$ for all $1 \leq i \leq n-1$. Furthermore, if $P_I \in H_i$ for all $1 \leq i \leq n-1$, then $R_i P_I = P_I$ for all i so that $n \in I$ implies $i \in I$ for all $1 \leq i \leq n-1$. As I must be a proper subset of $\{1, 2, \dots, n\}$, we have $P_I \in \mathcal{V}$, so that $M' = \bigcap_{i=1}^{n-1} H_i$.

All that remains to be verified for this lemma is that $F_i \subseteq \partial H_i$. For this fact, recall that F_i is comprised of $(n-3)$-simplices in $\mathcal{A}(\infty)$ whose vertices are given by P_J where $J \subseteq \{1, 2, \dots, n-1\} - \{i\}$. Hence, if $\sigma \subseteq \mathcal{A}(\infty)$ is an $(n-3)$ simplex of $\mathcal{A}(\infty)$ with $\sigma \subseteq F_i$, then σ is a face of exactly 2 chambers in $\mathcal{A}(\infty)$: $\mathfrak{C}_P$ and $\mathfrak{C}_{P_{J'}}$ where $\mathfrak{C}_P$ contains P and thus $\mathfrak{C}_P \subseteq M'$, and $\mathfrak{C}_{P_{J'}}$ contains $P_{J'}$ where $J' = \{1, 2, \dots, n\} - \{i\}$ and thus $\mathfrak{C}_{P_{J'}} \not\subseteq M'$. Furthermore, $\sigma = \mathfrak{C}_P \cap \mathfrak{C}_{P_{J'}}$.

Since $R_i V_{J'} \neq V_{J'}$, it follows that $\mathfrak{C}_{P_{J'}}$ is not fixed by R_i. Since $\mathfrak{C}_{P_J}$ is fixed by R_i we have that $\sigma = \mathfrak{C}_P \cap \mathfrak{C}_{P_{J'}} \subseteq \partial H_i$. Therefore, $F_i \subseteq \partial H_i$. $\square$

For any vertex $y \in X$, we let $C_y \subseteq X$ be the union of sectors based at y and limiting to a chamber in M. Thus, C_y is a cone. Note also that because any chamber in $g^{-1}\mathcal{A}(\infty)$ has diameter less than $\pi/2$, it follows that $M \cap S_{P,P^-} = \emptyset$. Therefore, if we choose $x_*, y \in g^{-1}\mathcal{A}$ such that x_* is closer to P than y, then $C_y \subseteq g^{-1}\mathcal{A}$ and $V_{x_*} \cap C_y$ is a simplex of dimension $n-2$.

We will set on a fixed choice of y before x_*, and we will choose y to satisfy the below

Lemma 9. *There is some $y \in g^{-1}\mathcal{A}$ such that the $\mathbb{Q}[[t^{-1}]]$-points of $R_u(P)$ fix C_y pointwise.*

Proof. Let x_0 be the point in X stabilized by $\mathrm{SL}_n(\mathbb{Q}[[t^{-1}]])$. Recall that $R_u(P)M = M$ so that the $\mathbb{Q}[[t^{-1}]]$-points of $R_u(P)$ fix C_{x_0} pointwise.

Because $M \subseteq g^{-1}\mathcal{A}(\infty)$, there is a $y \in C_{x_0} \cap g^{-1}\mathcal{A}$. Any such y satisfies the lemma. □

Choose e such that with $x_* = e$ as above and with y as in Lemma 9, there exists a fundamental domain D_e for the action of A on V_e that is contained in C_y. The choice of e can be made by travelling arbitrarily far from y along a geodesic ray in $g^{-1}\mathcal{A}$ that limits to P.

By the choice of D_e we have that

$$AD_e = V_e$$

and that the $\mathbb{Q}([[t^{-1}]])$-points of $R_u(P)$ fix D_e.

4.2. The filtration.

We let

$$X_0 = \mathrm{SL}_n(\mathbb{Z}[t])D_e$$

and for any $i \in \mathbb{N}$ we choose an $\mathrm{SL}_n(\mathbb{Z}[t])$-invariant and cocompact space $X_i \subseteq X$ somewhat arbitrarily to satisfy the inclusions

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq \bigcup_{i=1}^{\infty} X_i = X.$$

In our present context, Brown's criterion takes on the following form [Br1].

Brown's Filtration Criterion. *By Lemma 2, the group $\mathrm{SL}_n(\mathbb{Z}[t])$ is not of type FP_{n-1} if for any $i \in \mathbb{N}$, there exists some class in the homology group $\widetilde{H}_{n-2}(X_0, \mathbb{Z})$ which is nonzero in $\widetilde{H}_{n-2}(X_i, \mathbb{Z})$.*

4.3. Translation to P moves away from filtration sets. The following is essentially Mahler's compactness criterion.

Lemma 10. *Given any $i \in \mathbb{N}$, there is some $k \in \mathbb{N}$ such that $b^k e \notin X_i$.*

Proof. The lemma follows from showing that the sequence

$$\{\mathrm{SL}_n(\mathbb{Z}[t])b^k e\}_k \subseteq \mathrm{SL}_n(\mathbb{Z}[t])\backslash X$$

is unbounded.

Since stabilizers of points in X are bounded subgroups of $\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$, the claim above follows from showing that the sequence

$$\{\mathrm{SL}_n(\mathbb{Z}[t])b^k\}_k \subseteq \mathrm{SL}_n(\mathbb{Z}[t])\backslash \mathrm{SL}_n(\mathbb{Q}((t^{-1})))$$

is unbounded.

But bounded sets in $\mathrm{SL}_n(\mathbb{Z}[t])\backslash\mathrm{SL}_n(\mathbb{Q}((t^{-1})))$ do not contain sequences of elements $\{\mathrm{SL}_n(\mathbb{Z}[t])g_\ell\}_\ell$ such that $1 \in \overline{g_\ell^{-1}(\mathrm{SL}_n(\mathbb{Z}[t]) - \{1\})g_\ell}$. And clearly b^k's contract some root groups to 1. Thus none of the sequences above is bounded. □

4.4. Applying Brown's criterion. As is described by Brown's criterion, we will prove Theorem 1 by fixing X_i and finding an $(n-2)$-cycle in X_0 that is nontrivial in the homology of X_i.

Recall that we denote the standard root subgroups of $R_u(P)$ by $R_1, \dots, R_{n-1}$. Each group $g^{-1}R_j g$ determines a family of parallel walls in $g^{-1}\mathcal{A}$. By Lemma 8, each face of the cone C_y is contained in a wall of one of these families.

Choose $r_j \in g^{-1}R_j g$ for all j such that $b^k e$ is contained in the wall determined by r_j where k is determined by i as in Lemma 10. In particular, $r_j b^k e = b^k e$.

The intersection of the fixed point sets in $g^{-1}\mathcal{A}$ of the elements $r_1, \dots, r_{n-1}$ determine a cone that we name Z. Note that Z is contained in – and is a finite Hausdorff distance from – the cone C_y.

Let $Z^- \subseteq g^{-1}\mathcal{A}$ be the closure of the set of points in $g^{-1}\mathcal{A}$ that are fixed by none of the r_j. The set Z^- is a cone based at $b^k e$, containing y, and asymptotically containing the vertex P^-.

As the walls of Z^- are parallel to those of Z – and hence of C_y, we have that $Z^- \cap V_e$ is an $(n-2)$-dimensional simplex. We will name this simplex σ.

The component of $Z^- - V_e$ that contains $b^k e$ is an $(n-1)$-simplex that has σ as a face. Call this $(n-1)$ simplex Y.

For any $\ell \in \mathbb{N}$, there are exactly 2^{n-1} possible subsets of the set $\{r_1^\ell, \dots, r_{n-1}^\ell\}$. For each such subset S_ℓ, we let

$$Y_{S_\ell} = \Big(\prod_{g \in S_\ell} g\Big)Y$$

and

$$\sigma_{S_\ell} = \Big(\prod_{g \in S_\ell} g \Big) \sigma.$$

Notice that the product of group elements in the equations above are well-defined regardless of the order of the multiplication since $R_u(P)$ is abelian. In the degenerate cases, $\prod_{g \in \emptyset} g = 1$, so $Y_\emptyset = Y$ and $\sigma_\emptyset = \sigma$.

For any $\ell \in \mathbb{N}$, we let $Y_\ell = \bigcup_{S_\ell} Y_{S_\ell}$. Because the wall in $g^{-1}\mathcal{A}$ determined by r_j^ℓ is the same as the wall determined by r_j, the space Y_ℓ is a closed ball containing $b^k e$ whose boundary sphere is $\bigcup_{S_\ell} \sigma_{S_\ell}$. Indeed the simplicial decomposition of Y_ℓ described above is isomorphic to the simplicial decomposition of the unit ball in $\mathbb{R}^{n-1}$ that is given by the $n-1$ hyperplanes defined by setting a coordinate equal to 0.

Let $\omega_\ell = \bigcup_{S_\ell} \sigma_{S_\ell}$. Thus $\omega_\ell = \partial Y_\ell$. Furthermore, the building X is $(n-1)$-dimensional and contractible, so any $(n-1)$-chain with boundary equal to ω_ℓ must contain Y_ℓ and thus $b^k e$. That is for all $\ell \in \mathbb{N}$

$$[\omega_\ell] \neq 0 \in \widetilde{H}_{n-2}(X - b^k e, \mathbb{Z})$$

If we can show that $\omega_\ell \subseteq X_0$ for some choice of ℓ, then we will have proved our main theorem by application of Brown's criterion since we would have

$$[\omega_\ell] \neq 0 \in \widetilde{H}_{n-2}(X_i, \mathbb{Z})$$

by Lemma 10.

Lemma 11. *There exists some $\ell \in \mathbb{N}$ such that $\omega_\ell \subseteq X_0$.*

Proof. For any $u \in R_u(P)$ there is a decomposition $u = u'u''$ where the entries of $u' \in R_u(P)$ are contained in $\mathbb{Q}[t]$ and the entries of $u'' \in R_u(P)$ are contained in $\mathbb{Q}[[t^{-1}]]$.

For any $a \in A$ and $u \in R_u(P)$ there is a power $\ell(a,u) \in \mathbb{N}$ such that

$$(a^{-1}u^{\ell(a,u)}a)' = ((a^{-1}ua)')^{\ell(a,u)} \in \mathrm{SL}_n(\mathbb{Z}[t]).$$

(For the above equality recall that $A \leq L$ normalizes $R_u(P)$ and the group operation on $R_u(P)$ is vector addition.)

There are only finitely many $a \in A$ such that $aD_e \cap \sigma \neq \emptyset$ (or equivalently, such that $aD_e \cap Z^- \neq \emptyset$). Call this finite set $\mathcal{D} \subseteq A$.

At this point we fix

$$\ell = \prod_{a \in \mathcal{D}} \prod_{i=1}^{n-1} \ell(a, r_i).$$

Thus,

$$\Big[a^{-1}\Big(\prod_{g\in S_\ell} g\Big)a\Big]' \in \mathrm{SL}_n(\mathbb{Z}[t])$$

for any $a \in \mathcal{D}$ and any $S_\ell \subseteq \{r_i^\ell\}_{i=1}^{n-1}$.

Because $\omega_\ell = \bigcup_{S_\ell} \sigma_{S_\ell}$ and $\sigma_{S_\ell} = (\prod_{g\in S_\ell} g)\sigma = (\prod_{g\in S_\ell} g)(AD_e \cap Z^-)$, we can finish our proof of this lemma by showing

$$\Big(\prod_{g\in S_\ell} g\Big)aD_e \subseteq X_0$$

for each $a \in \mathcal{D} \subseteq A \leq \mathrm{SL}_n(\mathbb{Z}[t])$ and each $S_\ell \subseteq \{r_i^\ell\}_{i=1}^{n-1}$. For this, recall that the $\mathbb{Q}[[t^{-1}]]$-points of $R_u(P)$ fix D_e and thus

$$\begin{aligned}\Big(\prod_{g\in S_\ell} g\Big)aD_e &= a[a^{-1}\Big(\prod_{g\in S_\ell} g\Big)a]D_e \\ &= a[a^{-1}\Big(\prod_{g\in S_\ell} g\Big)a]'[a^{-1}\Big(\prod_{g\in S_\ell} g\Big)a]''D_e \\ &= a[a^{-1}\Big(\prod_{g\in S_\ell} g\Big)a]'D_e \\ &\subseteq \mathrm{SL}_n(\mathbb{Z}[t])D_e \\ &= X_0\end{aligned}$$

□

References

[Bo-Se] A. Borel and J-P. Serre, Corners and arithmetic groups. *Comment. Math Helv* **48** (1973), 436–491. Zbl 0274.22011 MR 0387495

[Br 1] K. Brown, Finiteness properties of groups. *J. Pure Appl. Algebra* **44** (1987), 45–75. Zbl 0613.20033 MR 0885095

[Br 2] K. Brown, *Buildings*. Springer Monogr. Math., Springer-Verlag, New York 1998. Zbl 0922.20034 MR 1644630

[Bu-Wo 1] K.-U. Bux and K. Wortman, Finiteness properties of arithmetic groups over function fields. *Invent. Math.* **167** (2007), 355–378. Zbl 1126.20030 MR 2270455

[Bu-Wo 2] K.-U. Bux and K. Wortman, A geometric proof that $\mathrm{SL}_2(\mathbb{Z}[t,t^{-1}])$ is not finitely presented. *Algebr. Geom. Topol.* **6** (2006), 839–852. Zbl 1128.20037 MR R2240917

[Kr-Mc] S. Krstić and J. McCool, Presenting $\mathrm{GL}_n(k\langle T\rangle)$. *J. Pure Appl. Algebra* **141** (1999), 175–183. Zbl 0930.19001 MR 1706364

[Na] H. Nagao, On GL(2, $K[X]$). *J. Inst. Polytech. Osaka City Univ. Ser. A* **10** (1959), 117–121. Zbl 0092.02504 MR 0114866

[Su] A. A. Suslin, The structure of the special linear group over rings of polynomials. *Izv. Akad. Nauk SSSR Ser. Mat.* **41** (2) (1977), 235–252 (in Russian). Zbl 0354.13009 MR 0472792

[So] C. Soulé, Chevalley groups over polynomial rings. In *Homological group theory*, London Math. Soc. Lecture Note Ser. 36, Cambridge University Press, Cambridge 1979, 359–367. Zbl 0437.20036 MR 0564437

[Ti] J. Tits, *Buildings of spherical type and finite BN-pairs.* Lecture Notes in Math. 386. Springer-Verlag, Berlin 1974. Zbl 0295.20047 MR 0470099

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