Comment. Math. Helv. 84 (2009), 807–833 **Commentarii Mathematici Helvetici**

The structure of homotopy Lie algebras

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To J.-M. Lemaire for his 60th birthday

Abstract. In this paper we consider a graded Lie algebra, L, of finite depth m, and study the interplay between the depth of L and the growth of the integers $\dim L_i$. A subspace W in a graded vector space V is called full if for some integers d, N, q, $\dim V_k \leq d \sum_{i=k}^{k+q} \dim W_i$, $i \geq N$. We define an equivalence relation on the subspaces of V by $U \sim W$ if U and W are full in $U + W$. Two subspaces V, W in L are then called L-equivalent ($V \sim_L W$) if for all ideals $K \subset L$, $V \cap K \sim W \cap K$. Then our main result asserts that the set $\mathcal{L}$ of L-equivalence classes of ideals in L is a distributive lattice with at most 2^m elements. To establish this we show that for each ideal I there is a Lie subalgebra $E \subset L$ such that $E \cap I = 0$, $E \oplus I$ is full in L, and depth E + depth I $\leq$ depth L.

Mathematics Subject Classification (2000). 55P35, 55P62, 17B70.

Keywords. Homotopy Lie algebra, graded Lie algebra, ideal, radical of a Lie algebra, log index, growth.

1. Introduction

We work over a ground field $\mathbb{k}$ of characteristic $\neq 2$. A graded Lie algebra, L, is a graded vector space equipped with a Lie bracket $[\ ,\ ]\colon L \otimes L \to L$, satisfying

$$[x, y] + (-1)^{\deg x \cdot \deg y}[y, x] = 0$$

and

$$[x, [y, z]] = [[x, y], z] + (-1)^{\deg x \cdot \deg y}[y, [x, z]],$$

and $[x, [x, x]] = 0$, $x \in L_{\mathrm{odd}}$ if char $\mathbb{k} = 3$. (This condition is automatic if char $\mathbb{k}$ is not 3.)

As in the classical case, L has a universal enveloping algebra UL, and we define

$$\operatorname{depth} L = \text{least } m \text{ (or } \infty) \text{ such that } \operatorname{Ext}^m_{UL}(\mathbb{k}, UL) \neq 0.$$

Similarly, if M is an L-module, then

$$\text{grade}_L\, M = \text{ least } q \text{ (or } \infty) \text{ such that } \operatorname{Ext}^q_{UL}(M, UL) \neq 0.$$

The graded Lie algebra, L, is *connected* if $L = \{L_i\}_{i\geq 0}$ and of *finite type* if each $\dim L_i < \infty$; graded Lie algebras satisfying both condition are called cft graded Lie algebras.

Suppose now X is a simply connected CW complex of finite type. Then the rational homotopy Lie algebra, $L_X = \pi_*(\Omega X) \otimes \mathbb{Q}$ (with Lie bracket given by the Samelson product) is a cft graded Lie algebra. The motivation for the study of cft graded Lie algebras of finite depth is the following result.

Theorem ([1]). *If X is a simply connected CW complex of finite type, then*

$$\text{depth}\, L_X \leq \text{cat}_0\, X,$$

where $\text{cat}_0\, X$ *denotes the rational Lusternik–Schnirelmann category of X. In particular, if X is a finite CW complex, then* $\text{depth}\, L_X$ *is finite.*

For more details for all of the above, the reader is referred to [5].

An important question connected with the Lie algebra L_X is the behavior of the integers $\dim(L_X)_i$, since

$$\dim(L_X)_i = \text{rank}\, \pi_{i+1}(X).$$

In this regard, we have the following growth result.

Theorem ([9]). *Let X be a simply connected CW complex of finite type such that the sequence* $\dim H_k(X;\mathbb{Q})$ *grows at most exponentially. If* $\text{cat}_0\, X < \infty$*, then either* $\dim L_X < \infty$*, or else there is a positive integer d and a number $\alpha > 0$ such that given $\varepsilon > 0$,*

$$e^{(\alpha-\varepsilon)k} \leq \sum_{i=k}^{k+d} (\dim L_X)_i \leq e^{(\alpha+\varepsilon)k}, \quad k \geq K(\varepsilon).$$

Note that $e^{-\alpha}$ is just the radius of convergence of the power series $\sum \dim(L_X)_i z^i$.

In this paper we focus on the structure of cft graded Lie algebras of finite depth, with particular attention to the interplay between depth and growth of the integers $\dim L_i$, and to the structure of the ideals in L. Our aim is a classification theory for the ideals in a cft graded Lie algebra of finite depth, and in particular for the homotopy Lie algebras L_X of a space of finite Lusternik–Schnirelmann category. A crucial notion is that of full subspace.

Definition. A subspace W of a graded vector space $V = \{V_i\}_{i\geq 0}$ is *full* in V if for some fixed λ, q and N (all positive)

$$\dim V_k \leq \lambda \sum_{i=k}^{k+q} \dim W_i, \quad k \geq N.$$

An easy argument (Proposition 2.5) then shows that an equivalence relation on the subspaces of V is defined by

$$U \sim W \iff U \text{ and } W \text{ are full in } U + W.$$

Two subspaces V, W in a graded Lie algebra L are called *L-equivalent* ($V \sim_L W$) if for all ideals $K \subset L$, $V \cap K \sim W \cap K$. As we show in Section 5, the set $\mathcal{L}$ of L-equivalence classes $[I]$ of ideals $I \subset L$ is a distributive lattice under the operations $[I] \leq [J]$ if $I \cap J \sim_L I$, $[I] \vee [J] = [I + J]$ and $[I] \wedge [J] = [I \cap J]$. In such a lattice each maximal chain of strict inequalities $0 < [I(1)] < \cdots < [I(r)] = [I]$ has the same length r; the number r is the height $\mathrm{ht}[I]$ of $[I]$.

Now our main result (Theorem 5.7) reads as follows:

Theorem. *Let L be a cft graded Lie algebra of finite depth m and suppose* $\mathrm{ht}[L] = r$. *Then $r \leq m$. Moreover, the number ν_L of L-equivalence classes of ideals in L satisfies $\nu_L \leq 2^r$ and equality holds if and only if $L \sim_L I(1) \oplus \cdots \oplus I(r)$ where the $I(i)$ are ideals of height* 1.

The main step in the proof of this theorem is the following (Theorem 4.3).

Theorem. *Let I be an ideal in a cft graded Lie algebra L of finite depth. Then there is a Lie subalgebra $E \subset L$ such that,*

(ii) *$E \cap I = 0$ and $E \oplus I$ is full in L, and,*

(ii) $\mathrm{depth}\, E + \mathrm{depth}\, I = \mathrm{depth}(E \oplus I) \leq \mathrm{depth}\, L$.

Call an inclusion $W \subset V$ of graded vector spaces *strongly proper* if W is not full in V. Then the theorem above has the following consequence (Corollary to Theorem 4.3).

Proposition. *If I is a strongly proper ideal in a graded Lie algebra L, then* $\mathrm{depth}\, I <$ $\mathrm{depth}\, L$. *Thus the length of a sequence $I(1) \subset \cdots \subset I(r) \subset L$ of strongly proper inclusions of ideals has length at most* $\mathrm{depth}\, L$ ($r \leq \mathrm{depth}\, L$).

The proof of the theorem requires certain technology for the study of the relative size of graded vector spaces, which we set up in Section 2. Then in Section 3 we carry out a careful analysis of the relationship between depth L and $\text{grade}_L\, M$, showing that under certain hypotheses depth $L = \text{grade}_L\, M$ (Theorem 3.6). These hypotheses hold for the modules appearing in the Hochschild–Serre spectral sequence, which then constitute the main ingredient in the proof of the theorem.

The results in Sections 3 and 4 have a number of applications. First we note that upper and lower bounds on the rate of exponential growth of a graded vector space V are given by

$$\log\text{index}\, V = \limsup_k \frac{\log\dim V_k}{k}$$

and

$$\text{lower}\log\text{index}\, V = \lim_{q\to\infty}\liminf_k \frac{\log\sum_{i=k}^{k+q}\dim V_i}{k}.$$

In Section 5 we note that if W is full in V, then W and V have the same log index and the same lower log index. Thus the Lie subalgebra $E \oplus I$ in the theorem above has the same growth properties as L.

We then show that the sum, R, of the ideals $I \subset L$ with $\log\text{index}\, I < \log\text{index}\, L$ also satisfies $\log\text{index}\, R < \log\text{index}\, L$; thus R (called the *hyperradical* of L) has strictly lower depth. Define a sequence $R_r \subset R_{r-1} \subset \cdots \subset R_1 = R \subset L$ by defining R_i to be the hyperradical of R_{i-1}. Since each inclusion is strongly proper, it follows that $r \leq$ depth L; moreover, clearly for any ideal $I \subset L$,

$$\log\text{index}\, I = \log\text{index}\, R_i \quad \text{for some } i.$$

It follows that at most depth $L + 1$ numbers appear as the log index of an ideal I in L.

In Section 7 we show that in any cft graded Lie algebra of finite depth, either $\dim L_{\text{odd}}$ is finite or else for some d the integers $\sum_{j=k+1}^{k+d}\dim(L_{\text{odd}})_j$ grow faster than any polynomial.

Finally, the authors would like to thank the referee for the many helpful suggestions and comments.

2. Large and full subspaces

2.1. Definitions and characterization. Suppose $V = \{V_i\}_{i\geq 0}$ is a graded vector space of finite type, and let $\sigma = (\sigma_i)$ be a sequence of non-negative numbers.

Definition 2.1. A subspace $W \subset V$ is *σ-large* in V if for some fixed $q, \lambda, K \geq 0$,

$$\dim(V/W)_k \leq \lambda \sum_{i=k}^{k+q} \sigma_i, \quad k \geq K. \tag{1}$$

If Z is a graded vector space and W is $(\dim Z_i)$-large in V, we shall say simply that W is Z-large in V.

For instance $W \subset V$ has polynomial codimension if W is σ-large in V with $\sigma_i = i^m$ for some m.

Lemma 2.2. (i) *If $U \subset W$ is σ-large in W and if $W \subset V$ is σ-large in V, then U is σ-large in V.*

(ii) *The finite intersection of σ-large subspaces of V is also σ-large in V.*

(iii) *If $W \subset V$ is σ-large in V, then for each $r \geq 0$,*

$$\sum_{i=k}^{k+r} \dim(V/W)_i \leq \lambda(q+1) \sum_{i=k}^{k+r+q} \sigma_i, \quad k \geq K,$$

where q, λ, K are as in Definition 2.1.

Proof. (i) Choose λ, q, K so that Definition 2.1 is satisfied for both $U \subset W$ and $W \subset V$. Then

$$\dim(V/U)_k = \dim(W/U)_k + \dim(V/W)_k \leq \lambda \sum_{i=k}^{k+q} \sigma_i + \lambda \sum_{i=k}^{k+q} \sigma_i = 2\lambda \sum_{i=k}^{k+q} \sigma_i.$$

(ii) Suppose $W(1), \dots, W(r)$ are σ-large subspaces of V, and choose q, λ, K so that Definition 1 holds for each of the $W(j)$. The linear map $V \to V/W(1) \oplus \cdots \oplus V/W(r)$ factors to give an injection

$$V/W(1) \cap \cdots \cap W(r) \to V/W(1) \oplus \cdots \oplus V/W(r),$$

and so

$$\dim \left(\frac{V}{W(1) \cap \cdots \cap W(r)} \right)_k \leq \sum_{j=1}^{r} \dim \left(\frac{V}{W(j)} \right)_k \leq r\lambda \sum_{i=k}^{k+q} \sigma_i, \quad k \geq K.$$

(iii)

$$\sum_{i=k}^{k+r} \dim(V/W)_i \leq \sum_{i=k}^{k+r} \lambda \sum_{j=i}^{i+q} \sigma_j \leq \lambda(q+1) \sum_{i=k}^{k+r+q} \sigma_i. \qquad \square$$

Definition 2.3. A subspace $W \subset V$ is *full* in V if for some $q, \lambda, K \geq 0$,

$$\dim V_k \leq \lambda \sum_{i=k}^{k+q} \dim W_i, \quad k \geq K.$$

Lemma 2.4. *Suppose $U \subset W \subset V$.*

(i) *The following conditions are equivalent :*

- *W is full in V.*
- *W is W-large in V.*
- *The zero subspace is W-large in V.*

(ii) *If U is full in W and W is full in V, then U is full in V.*

(iii) *If W is S-large in V for some $S \subset V$, then $W + S$ is full in V.*

(iv) *If W is full in V and λ, q, K satisfy* $\dim V_k \leq \lambda \sum_{i=k}^{k+q} \dim W_i$, $k \geq K$ (*cf.* (i)), *then for any $r \geq 0$,*

$$\sum_{j=k}^{k+r} \dim V_j \leq \lambda(q+1) \sum_{j=k}^{k+r+q} \dim W_j, \quad k \geq K.$$

Proof. (i) The third condition simply states the definition of fullness, and trivially implies the second. If the second holds, then (for some λ, q, K)

$$\dim V_k = \dim W_k + \dim(V/W)_k \leq (\lambda + 1) \sum_{i=k}^{k+q} \dim W_i .$$

(ii) For suitable α, β, r, s, K,

$$\dim V_k \leq \alpha \sum_{i=k}^{k+r} \dim W_i \leq \alpha \sum_{i=k}^{k+r} \Big(\beta \sum_{j=i}^{i+s} \dim U_j\Big)$$
$$\leq \alpha\beta(r+1) \sum_{j=k}^{k+r+s} \dim U_j , \quad k \geq K.$$

(iii) For suitable λ, q, K and for $k \geq K$,

$$\dim V_k = \dim(V/W)_k + \dim W_k$$
$$\leq \lambda \sum_{i=k}^{k+q} \dim S_i + \dim W_k$$
$$\leq 2\lambda \sum_{i=k}^{k+q} \dim(S_i + W_i), \quad k \geq K,$$

because $\dim(S_k + W_k) \geq \frac{1}{2} (\dim S_k + \dim W_k)$.

(iv)

$$\sum_{i=k}^{k+r} \dim V_i \le \lambda \sum_{i=k}^{k+r} \sum_{j=i}^{i+q} \dim W_j = (q+1)\lambda \sum_{j=k}^{k+r+q} \dim W_j, \quad k \ge K. \qquad \square$$

Proposition 2.5. *An equivalence relation on the subspaces of V is defined by $U \sim W$ if and only if U and W are full in $U + W$.*

Proof. We have only to check transitivity. Suppose that U, W, Y are subspaces of V and $U \sim W$ and $W \sim Y$. The injection $W + Y \to U + W + Y$ induces a surjection

$$(W + Y)/W \to (U + W + Y)/(U + W).$$

Since W is full in $W + Y$ this implies that $U + W$ is full in $U + W + Y$. But U is full in $U + W$ and hence (Lemma 2.4 (ii)) U is full in $U + W + Y$. Therefore U is certainly full in $U + Y$. Similarly Y is full in $U + Y$ and so $U \sim Y$. $\square$

Definition 2.6. The equivalence relation above will be called *full equivalence* and will be denoted by $U \sim W$.

Proposition 2.7. *If $U_i \sim W_i$ are pairwise fully equivalent subspaces of V, then $U_1 + \cdots + U_r \sim W_1 + \cdots + W_r$.*

Proof. It is clearly sufficient to prove the proposition when $r = 2$; in this case we need show that $U_1 + U_2 \sim W_1 + U_2 \sim W_1 + W_2$. Thus we are reduced to show that $U_1 + W \sim U_2 + W$ if $U_1 \sim U_2$. By hypothesis, U_1 is full in $U_1 + U_2$. It follows from the obvious surjection $(U_1 + U_2)/U_1 \to (U_1 + U_2 + W)/(U_1 + W)$ that $U_1 + W$ is U_1-large in $U_1 + U_2 + W$. Thus it is certainly $(U_1 + W)$-large in $U_1 + U_2 + W$, and hence full in this space. Similarly $U_2 + W$ is full in $U_1 + U_2 + W$ and so $U_1 + W \sim U_2 + W$. $\square$

2.2. Log index and lower log index. Again suppose $V = \{V_i\}_{i \ge 0}$ is a graded vector space of finite type. The *log index* of V is the number given by

$$\log \operatorname{index} V = \limsup_k \frac{\log \dim V_k}{k};$$

it is the least number α such that for all $\varepsilon > 0$, there is a K such that $\dim V_k \le e^{(\alpha+\varepsilon)k}$, $k \ge K$. Thus it provides a sharp upper bound for exponential growth.

Note that if $\lambda = \log \operatorname{index} V < \infty$, then $e^{-\lambda}$ is the radius of convergence of the Hilbert series $\sum \dim V_k z^k$. One should also observe that if $\lambda > 0$, then the sum $\sum_{i=1}^{k} \dim V_i$ grows exponentially with k.

In the applications we shall use the following, seemingly more refined, measures.

Definition 2.8. The *upper* and *lower log indexes* of V are given, respectively, by

$$\text{upper log index } V = \lim_{q\to\infty} \limsup_k \frac{\log\big(\sum_{i=k}^{k+q} \dim V_i\big)}{k}$$

and

$$\text{lower log index } V = \lim_{q\to\infty} \liminf_k \frac{\log\big(\sum_{i=k}^{k+q} \dim V_i\big)}{k}.$$

Remark. The limits above exist because the sequences increase with q.

Lemma 2.9. (i) *For any q,*

$$\log\text{index } V = \limsup_k \frac{\log\big(\sum_{i=k}^{k+q} \dim V_i\big)}{k} = \text{upper log index } V.$$

(ii) *If L is a cft graded Lie algebra of finite depth then for some d,*

$$\liminf_k \frac{\log\big(\sum_{i=k}^{k+q} \dim L_i\big)}{k} = \text{lower log index } L, \quad q \geq d.$$

Proof. (i) This is straightforward.

(ii) By [9], Lemma 7, there is an integer d so that $Z = \{u \mid [u, L_{\leq d}] = 0\}$ is finite dimensional. Choose D so that $Z_{\geq D} = 0$.

Next, for any $s > d$ and $k > s + D$, write

$$\sum_{i=k}^{k+s} \dim L_i = e^{\gamma(k,s)k}.$$

Then for some $j \in [k-s, k]$, $\dim L_j \geq \frac{1}{s+1} e^{\gamma(k-s,s)(k-s)}$. Let $u_1, \ldots, u_p$ be a basis for $L_{\leq d}$ and note that, since $j \geq D$, for some λ we have $\dim[u_\lambda, L_j] \geq \frac{1}{p} \dim L_j$. Proceeding in this way yields an infinite sequence (u_{λ_ν}) such that

$$\dim[u_{\lambda_q}, [u_{\lambda_{q-1}}, [\ldots[u_{\lambda_1}, L_j]\ldots] \geq \left(\frac{1}{p}\right)^q \dim L_j \quad \text{for all } q.$$

But for some $q \leq s$, we have $\sum_{\nu=1}^{q} \deg u_{\lambda_\nu} + j \in [k, k+d]$. It follows that

$$\gamma(k,d) \geq (1 - s/k)\gamma(k-s,s) - \frac{Q(s)}{k},$$

for some $Q(s)$ independent of k. Letting $k \to \infty$, we see that $\liminf_k \gamma(k,d) = \liminf_k \gamma(k,s)$. Thus for $s \geq d$

$$\liminf_k \frac{\log\big(\sum_{i=k}^{k+d} \dim L_i\big)}{k} = \liminf_k \frac{\log\big(\sum_{i=k}^{k+s} \dim L_i\big)}{k},$$

and this is then obviously the lower log index of L. □

Remark. Lemma 2.9 shows that log index L and lower log index L give precise upper and lower bounds on the exponential growth of $\sum_{i=k}^{k+q} \dim L_i$.

Proposition 2.10. *Suppose U and W are fully equivalent subspaces of V. Then U and W have the same log index and the same lower log index.*

Proof. We need to show that if W is full in V then W and V have the same log index and lower log index. But then

$$\begin{aligned}\sup_{j\geq k} \frac{\log\dim V_j}{j} &\geq \sup_{j\geq k} \frac{\log\dim W_j}{j}\\ &\geq \sup_{j\geq k} \frac{\log\left(\frac{1}{q+1}\sum_{i=j}^{j+q} \dim W_i\right)}{j}\\ &\geq \sup_{j\geq k} \frac{\log\left(\frac{1}{q+1}\frac{1}{\lambda} \dim V_j\right)}{j}.\end{aligned}$$

Take limits as $k \to \infty$ to see that $\log\operatorname{index} V = \log\operatorname{index} W$.

On the other hand,

$$\begin{aligned}\sum_{i=k}^{k+r} \dim V_i &\leq \lambda(q+1) \sum_{i=k}^{k+r+q} \dim W_i \quad \text{(Lemma 2.4(iv))}\\ &\leq \lambda(q+1) \sum_{i=k}^{k+r+q} \dim V_i.\end{aligned}$$

Thus

$$\begin{aligned}\liminf_k \frac{\log\left(\sum_{i=k}^{k+r} \dim V_i\right)}{k} &\leq \liminf_k \left(\frac{\log\lambda(q+1)}{k} + \frac{\log\left(\sum_{i=k}^{k+r+q} \dim W_i\right)}{k}\right)\\ &\leq \liminf_k \left(\frac{\log\lambda(q+1)}{k} + \frac{\log\left(\sum_{i=k}^{k+r+q} \dim V_i\right)}{k}\right).\end{aligned}$$

Let $a_j = \frac{\log(\sum_{i=j}^{j+q+r} \dim W_i)}{j}$. Then

$$\inf_{j\geq k} a_j \leq \inf_{j\geq k}\left(\frac{\log\lambda(q+1)}{j} + a_j\right) \leq \frac{\log\lambda(q+1)}{k} + \inf_{j\geq k} a_j.$$

Taking limits as $k \to \infty$ gives

$$\liminf_k \left(\frac{\log\lambda(q+1)}{k} + \frac{\log\left(\sum_{i=k}^{k+q+r} \dim W_i\right)}{k}\right) = \liminf_k \frac{\log\left(\sum_{i=k}^{k+q+r} \dim W_i\right)}{k}.$$

Hence

$$\liminf_k \frac{\log \sum_{i=k}^{k+r} \dim V_j}{k} \leq \liminf_k \frac{\log \sum_{i=k}^{k+r+q} \dim W_i}{k}$$
$$\leq \liminf_k \frac{\log \sum_{i=k}^{k+r+q} \dim V_i}{k}.$$

Taking limits as $r \to \infty$ gives

$$\text{lower log index } V = \text{lower log index } W. \qquad \square$$

3. Growth and depth in a graded Lie algebra

Let L be a cft graded Lie algebra, let $\sigma = (\sigma_i)$ be a sequence of non-negative integers and let $M = \{M_i\}_{i\in\mathbb{Z}}$ be a $\mathbb{Z}$-graded L-module.

3.1. Thin modules

Definition 3.1. Given subspaces $V, W \subset M$, the *isotropy Lie subalgebra* L_V and the *co-isotropy Lie subalgebra* L^W are defined by

$$L_V = \{x \in L \mid x \cdot V = 0\} \quad \text{and} \quad L^W = \{x \in L \mid x \cdot M \subset W\}.$$

The L-module M is *σ-thin* if L_V and L^W are σ-large Lie subalgebras of L whenever $\dim V < \infty$ and $\operatorname{codim} W < \infty$.

Remark. If V and W are subspaces of a σ-thin L-module such that $\dim V < \infty$ and $\operatorname{codim} W < \infty$, then $E = L_V \cap L^W$ is a σ-large Lie subalgebra satisfying

$$E \cdot V = 0 \quad \text{and} \quad E \cdot M \subset W.$$

Lemma 3.2. *Let L be a cft graded Lie algebra and let $\sigma = (\sigma_i)_{i\geq 0}$ be a sequence of non-negative numbers. Then:*

(i) *The direct sum and the finite tensor product of σ-thin L-modules are σ-thin.*

(ii) *Any subquotient of a σ-thin L-module is σ-thin.*

(iii) *If M is a σ-thin L-module, then each $\wedge^q M$ is also σ-thin.*

(iv) *If M is a σ-thin L-module, then $M^\# = \operatorname{Hom}(M, \Bbbk)$ is also σ-thin.*

Proof. Elementary linear algebra suffices to prove the lemma, since a finite intersection of σ-large Lie subalgebra is σ-large. $\square$

Lemma 3.3. *Suppose L is a cft graded Lie algebra, $\sigma = (\sigma)_{i\ge 0}$ is a sequence of non-negative numbers, and $M = \{M_i\}_{i\ge 0}$ is an L-module concentrated in non-negative degrees. Then*

(i) *M is σ-thin if and only if L_V is σ-large in L, whenever V is a finite dimensional subspace of M.*

(ii) *The sum, N, of all the σ-thin submodules $N(\alpha) \subset M$ is itself σ-thin.*

(iii) *M is σ-thin if for some λ, q, K, dim $M_k \le \lambda \sum_{i=k}^{k+q} \sigma_i$, $k \ge K$.*

(iv) *M is σ-thin if and only if for some set $\{v_i\}$ of generators for M (as an L-module) each L_{v_i} is σ-large in L.*

Proof. (i) is immediate from the fact that $M = \{M_i\}_{i\ge 0}$.

(ii) Any finite dimensional subspace $V \subset N$ satisfies $V \subset N(\alpha_1)+\cdots+N(\alpha_r)$ for some finite subset $\alpha_1, \dots, \alpha_r$. Thus there are finite dimensional subspaces $V(\alpha_i) \subset N(\alpha_i)$ such that $V \subset V(\alpha_1) + \cdots + V(\alpha_r)$. Hence $L_V \supset \cap_i L_{V(\alpha_i)}$. Since the finite intersection of σ-large Lie subalgebras is σ-large, it follows that L_V is σ-large.

(iii) Let V be a finite dimensional subspace of M and $(x_i)_{1\le i\le N}$ be a basis of V. Then the action of L on the x_i induces a linear injection

$$(L/L_V)_k \to \oplus_{i=1}^N M_{k+\deg x_i}.$$

This implies that L_V is large in L.

(iv) We first show that if, for some $v \in V$, L_v is σ-large then $L_{a\cdot v}$ is σ-large for all $a \in UL$. In fact, because of (ii), it is sufficient to show this when $a = x_1 \cdots x_r$ ($x_i \in L$) and we proceed by induction on r.

Set $w = x_2 \cdots x_r \cdot v$ and let $S \subset L$ be the graded subspace of L defined by $S = \{y \in L \mid [y, x_1] \in L_w\}$. Since L_w is σ-large, by the induction hypothesis, we have for some λ, q, K that

$$\dim(L/L_w)_k \le \lambda \sum_{i=k}^{k+q} \sigma_i, \quad k \ge K,$$

and also

$$\dim(L/S)_k \le \dim(L/L_w)_{k+\deg x_1} \le \lambda \sum_{i=k+\deg x_1}^{k+\deg x_1+q} \sigma_i.$$

On the other hand, for $z \in L$ we have

$$z \cdot x_1 \cdots x_r \cdot v = z \cdot x_1 \cdot w = [z, x_1] \cdot w \pm x_1 \cdot z \cdot w$$

and so $L_{x_1\cdot w} \supset S \cap L_w$. Now the inequalities above yield

$$\dim(L/L_{x_1\cdot w})_k \le 2\lambda \sum_{i=k}^{k+\deg x_1+q} \sigma_i, \quad k \ge K.$$

Thus $L_{x\cdot w}$ is σ-large and the induction is closed.

Finally we have shown that if L_v is σ-large then $UL\cdot v$ is σ-thin, and so we may apply (ii) to complete the proof of (iv). $\square$

Lemma 3.4. *Let L be a cft graded Lie algebra, and let $\sigma = \{\sigma_i\}_{i\geq 0}$ be a sequence of non negative numbers.*

(i) *If E is a σ-large Lie subalgebra of L, then the L-module $UL\otimes_{UE}\Bbbk$ is σ-thin.*

(ii) *If L acts by derivations in a Lie algebra F, and if L_{w_α} is σ-large for a set $\{w_\alpha\}$ of generators for the Lie algebra F, then F is a σ-thin L-module.*

Proof. (i) The vector space $UL\otimes_{UE}\Bbbk$ is generated as an L-module by the single element $v = 1\otimes 1$. Since $L_v = E$, which is σ-large, (i) follows from Lemma 3.3 (iv).

(ii) Let W be the linear span of the w_α. Then $UL\cdot W$ is a σ-thin L-module by Lemma 3.3 (iv). The natural linear map $UL\cdot W\to F$ extends to an L-linear algebra surjection $T(UL\cdot W)\to UF$. But $T(UL\cdot W)$ is σ-thin by Lemma 3.2 (i), and hence F, as a subquotient of $T(UL\cdot W)$ is σ-thin by Lemma 3.2 (ii). $\square$

3.2. The Hochschild–Serre spectral sequences. The invariants $\mathrm{Ext}^*_{UL}(M,N)$ and $\mathrm{Tor}^{UL}_*(M,N)$ will play an important role in this paper, when L is a cft graded Lie algebra and M and N are L-modules.

Let $V = \{V_i\}_{i\geq 0}$ be a graded vector space of finite type. We denote by $V^\#$ the dual vector space, $V^\#_k = \mathrm{Hom}(V_{-k},\Bbbk)$, and by $\wedge V^\#$ the free graded commutative algebra on $V^\#$. Then $\wedge^q V^\#$ is the linear span of the products $f_1\cdots f_q$, $f_i\in V^\#$, and its dual $\Gamma V = (\wedge V^\#)^\#$ is the free divided powers algebra on V.

The graded vector spaces $\mathrm{Tor}^{UL}_*(M,N)$ and $\mathrm{Ext}^*_{UL}(M,N)$ may be computed as the homology of complexes respectively of the form $\Gamma^*(sL)\otimes_\Bbbk M\otimes_\Bbbk N$ and $\mathrm{Hom}_\Bbbk(\Gamma^*(sL)\otimes_\Bbbk M,N)$ with twisted differentials ([11]). (Here sL is the suspension of L; $(sL)_k = L_{k-1}$.) Now suppose $E\subset L$ is a Lie subalgebra and write $L = E\oplus S$. Then there is a first quadrant spectral sequence (the Hochschild–Serre spectral sequence), that converges from

$$E^1_{p,q} = \mathrm{Tor}^{UE}_q(\Gamma^p s(L/E)\otimes M,N) \quad\text{to}\quad \mathrm{Tor}^{UL}_{p+q}(M,N).$$

When E is an ideal then

$$E^2_{p,q} = \mathrm{Tor}^{UL/E}_p(\Bbbk,\mathrm{Tor}^{UE}_q(M,N)).$$

There is also a Hochschild–Serre spectral sequence for Ext,

$$\mathrm{Ext}^q_{UE}(\Gamma^p s(L/E)\otimes M,N)\Longrightarrow \mathrm{Ext}^{p+q}_{UL}(M,N).$$

For more details on the Hochschild–Serre spectral sequences, see [5] and [9].

Now we recall two results obtained in [9] and related to cft graded Lie algebras of finite depth, that we will use several times in the text.

Lemma 3.5 ([9], Lemma 4). *Suppose M and N are L-modules where L is a cft graded Lie algebra and each N_i is finite dimensional. If $\operatorname{Ext}^m_{UL}(M,N) \neq 0$ then for some finitely generated Lie subalgebra $E \subset L$ and for some finitely generated L-submodule $P \subset M$ the restrictions $\operatorname{Ext}^m_{UL}(M,N) \to \operatorname{Ext}^m_{UE}(M,N)$ and $\operatorname{Ext}^m_{UL}(M,N) \to \operatorname{Ext}^m_{UL}(P,N)$ are nonzero.*

Lemma 3.6 ([9], Lemma 6). *Let $E \subset L$ be a Lie subalgebra of a cft graded Lie algebra L. Suppose for some m, that the restriction map $\operatorname{Ext}^m_{UL}(\Bbbk, UL) \to \operatorname{Ext}^m_{UE}(\Bbbk, UL)$ is non-zero. Let Z be the centralizer of E in L. Then Z is finite dimensional.*

3.3. Minimal subalgebras

Definition 3.7. Let $\sigma = (\sigma_i)_{i\geq 0}$ be a sequence of non-negative numbers.

- A cft graded Lie algebra L is *σ-minimal with respect to an ideal I* if every σ-large Lie subalgebra E with $I \subset E \subset L$ satisfies depth $E \geq$ depth L.
- A cft graded Lie algebra L is *σ-minimal* if L is σ-minimal with respect to 0, i.e., if depth $E \geq$ depth L for all σ-large subalgebras E of L.
- If Z is any graded vector space and L is $(\dim Z_i)$-minimal (resp. $(\dim Z_i)$-minimal with respect to I), we shall say that L is *Z-minimal* (resp. *Z-minimal with respect to I*) .

Theorem 3.8. *Let $\sigma = (\sigma_i)_{i\geq 0}$ be a sequence of non-negative numbers and let I be an ideal in a cft graded Lie algebra L. If $M = \{M_i\}_{i\in\mathbb{Z}}$ is a σ-thin L-module satisfying $M \neq 0$ and $I \cdot M = 0$, and if L is σ-minimal with respect to I, then*

$$\operatorname{depth} L = \operatorname{grade}_L M.$$

We begin with two preliminary lemmas.

Lemma 3.9. *Let I be an ideal in a cft graded Lie algebra L, and let $\sigma = (\sigma_i)_{i\geq 0}$ be a sequence of non-negative numbers. If $M = \{M_i\}_{i\in\mathbb{Z}}$ is any σ-thin L-module for which $I \cdot M = 0$ and $M \neq 0$, then I extends to a σ-large Lie subalgebra $E \subset L$ such that*

$$\operatorname{depth} E \leq \operatorname{grade}_L M.$$

Proof. Let $m = \mathrm{grade}_L\, M$. Then for some finitely generated submodule $N \subset M$, the restriction $\mathrm{Ext}^m_{UL}(M, UL) \to \mathrm{Ext}^m_{UL}(N, UL)$ is non-zero. Denote by $v_1, \ldots, v_r$ a set of generators of N. Then the short exact sequence $0 \to UL \cdot v_1 \to N \to N/(UL \cdot v_1) \to 0$ induces an exact sequence $\mathrm{Ext}^m_{UL}(UL \cdot v_1, UL) \to \mathrm{Ext}^m_{UL}(N, UL) \to \mathrm{Ext}^m_{UL}(N/(UL \cdot v_1), UL)$. It follows that there exists a subquotient module of N, of the form $UL \cdot v$, for which $\mathrm{Ext}^m_{UL}(UL \cdot v, UL) \neq 0$. Moreover, as a subquotient of M, $UL \cdot v$ is σ-thin (Lemma 3.2 (ii)).

Consider the short exact sequence of L-modules

$$0 \to K \to UL \otimes_{UL_v} \Bbbk \to UL \cdot v \to 0.$$

Since $UL \cdot v$ is σ-thin, L_v is σ-large in L. Hence $UL \otimes_{UL_v} \Bbbk$ and K are also σ-thin (Lemma 3.2 (i) and Lemma 3.2 (ii) respectively). Note also that since $UL \cdot v$ is a subquotient of M, $I \cdot UL \cdot v = 0$. In particular, $I \subset L_v$ and since I is an ideal, it follows that $I \cdot (UL \otimes_{UL_v} \Bbbk) = 0$ and hence $I \cdot K = 0$.

On the other hand from the short exact sequence above, we deduce that either $\mathrm{Ext}^{m-1}_{UL}(K, UL) \neq 0$ or else $\mathrm{Ext}^m_{UL}(UL \otimes_{UL_v} \Bbbk, UL) \neq 0$. In the first case the lemma follows by induction on m. In the second one we use the standard isomorphism

$$\mathrm{Ext}^m_{UL}(UL \otimes_{UL_v} \Bbbk, UL) \cong \mathrm{Ext}^m_{UL_v}(\Bbbk, UL)$$

to conclude that depth $L_v \leq m$. Set $E = L_v$ in this case. □

Lemma 3.10. *Suppose $I \subset E$ with I and E respectively an ideal and a Lie subalgebra in a cft graded Lie algebra L. If L is σ-minimal with respect to I, and if E is σ-large in L, then* depth $L =$ depth E. *In particular, E is σ-minimal with respect to I.*

Proof. It follows from the Hochschild–Serre spectral sequence that

$$\mathrm{Tor}^{UE}_p(\Gamma^q sL/E, (UL)^\#) \implies \mathrm{Tor}^{UL}_{p+q}(\Bbbk, (UL)^\#)$$

that there exist p, q with $p + q =$ depth L, and such that

$$\mathrm{grade}_E\, \Gamma^q sL/E \leq p.$$

Since L/E is a σ-thin E-module and $I \cdot L/E = 0$, Lemma 3.9 gives a Lie subalgebra F, σ-large in E, with $I \subset F \subset E$, and satisfying

$$\mathrm{depth}\, F \leq \mathrm{grade}_E\, \Gamma^q sL/E.$$

Since L is σ-minimal with respect to I, depth $L \leq$ depth F; i.e., $p + q \leq p$. Thus $q = 0$ and depth $F \leq$ depth E. But L was σ-minimal with respect to I, so that depth $L \leq$ depth E. This gives depth $L =$ depth E. □

Proof of Theorem 3.8. By Lemma 3.9, L contains a σ-large Lie subalgebra F containing I, and such that $\operatorname{depth} F \leq \operatorname{grade}_L M$. Now take a Lie subalgebra E of F that is σ-minimal with respect to I. Then, $\operatorname{depth} E \leq \operatorname{depth} F$, and so $\operatorname{depth} E \leq \operatorname{grade}_L M$. Since $\operatorname{depth} E = \operatorname{depth} L$ (Lemma 3.8), it follows that $\operatorname{depth} L \leq \operatorname{grade}_L M$.

Next, let $M_+ = \{M_i\}_{i\geq 0}$ and set $N = M/M_+$; both M_+ and N are σ-thin L-modules. If $M_+ \neq 0$, we can find a short exact sequence of L-modules of the form

$$0 \to K \to M_+ \to \Bbbk x \to 0.$$

As observed at the start of the proof of the theorem (applied to K instead of M), $\operatorname{depth} L \leq \operatorname{grade}_L K$. Thus if $m = \operatorname{depth} L$ we have the exact sequence

$$0 \to \operatorname{Ext}^m_{UL}(\Bbbk x, UL) \to \operatorname{Ext}^m_{UL}(M_+, UL),$$

which implies that $\operatorname{grade}_L M_+ \leq \operatorname{depth} L$. It follows that $\operatorname{grade}_L M_+ = \operatorname{depth} L$ and so, if $N = 0$, the theorem is proved.

Next, suppose $N \neq 0$. Since N is concentrated in negative degrees, and since $(UL)^\#$ is also concentrated in negative degrees, it follows that $(N \otimes (UL)^\#)^\# = N^\# \otimes UL$ as L-modules with diagonal action.

On the other hand $\operatorname{Tor}^{UL}_*(N, (UL)^\#) = \operatorname{Tor}^{UL}_*(\Bbbk, N \otimes (UL)^\#)$, and dualizing gives $\operatorname{Ext}^*_{UL}(N, UL) = \operatorname{Ext}^*_{UL}(\Bbbk, N^\# \otimes UL))$. Since $N^\# \otimes UL$ is a free UL-module (diagonal action) this shows that $\operatorname{grade}_L N = \operatorname{depth} L$. Thus if $M_+ = 0$, the theorem is proved.

Finally, suppose that $M_+ \neq 0$ and $N \neq 0$. Since $\operatorname{depth} L = \operatorname{grade}_L M_+ = \operatorname{grade}_L N = m$, the short exact sequence

$$0 \to M_+ \to M \to N \to 0$$

and the consequent exact sequences

$$\operatorname{Ext}^i_{UL}(M, UL) \leftarrow \operatorname{Ext}^i_{UL}(M, UL) \leftarrow \operatorname{Ext}^i_{UL}(N, UL) \leftarrow 0, \quad i \leq m,$$

imply that $\operatorname{grade}_L M = m = \operatorname{depth} L$. □

4. Weak complements

Theorem 4.1. *Let E and I be respectively a Lie subalgebra and an ideal in a cft graded Lie algebra L, such that $E \cap I = 0$, and let $\sigma = (\sigma_i)_{i\geq 0}$ be a sequence of non-negative numbers.*

(i) *If E is σ-minimal and I is a σ-thin E-module* (*adjoint representation*), *then $E \oplus I$ is σ-minimal with respect to I, and*

$$\operatorname{depth}(E \oplus I) = \operatorname{depth} E + \operatorname{depth} I.$$

(ii) *If, moreover, $L/(E \oplus I)$ is a σ-thin E-module, then*

$$\operatorname{depth}(E \oplus I) \le \operatorname{depth} L.$$

Proof. (i) Use the inclusions $E, I \to (E \oplus I)$ and multiplication in $U(E \oplus I)$ to write $U(E \oplus I) = UI \otimes UE$. Then for $x \in E$, $a \in UI$, $b \in UE$, we have

$$x \cdot (a \otimes b) = (\operatorname{ad} x)a \otimes b + (-1)^{\deg a \deg x} a \otimes x \cdot b.$$

It follows that $\operatorname{Tor}^{UI}(\Bbbk, U(E \oplus I)^{\#}) = \operatorname{Tor}^{UI}(\Bbbk, (UI)^{\#}) \otimes (UE)^{\#}$ as E-modules. Thus the Hochschild–Serre spectral sequence converges from

$$E^2_{p,q} = \operatorname{Tor}^{UE}_p(\operatorname{Tor}^{UI}_q(\Bbbk, (UI)^{\#}), (UE)^{\#}) \quad \text{to} \quad \operatorname{Tor}^{U(E \oplus I)}_{p+q}(\Bbbk, (U(E \oplus I))^{\#}).$$

Now since I is a σ-thin E-module so is each $\Gamma^q sI \otimes (UI)^{\#}$, and hence so are the subquotients $\operatorname{Tor}^{UI}_q(\Bbbk, (UI)^{\#})$. By Theorem 3.6, either $\operatorname{Tor}^{UI}_q(\Bbbk, (UI)^{\#}) = 0$, or else $\operatorname{depth} E = \operatorname{grade}_E \operatorname{Tor}^{UI}_q(\Bbbk, (UI)^{\#})$. Hence $E^2_{p,q} = 0$ for $q < \operatorname{depth} I$ or for $p < \operatorname{depth} E$, and $E^2_{p,q} \neq 0$ when $q = \operatorname{depth} I$ and $p = \operatorname{depth} E$. A standard corner argument now shows that $\operatorname{depth}(E \oplus I) = \operatorname{depth} E + \operatorname{depth} I$.

Finally, we show that $E \oplus I$ is σ-minimal with respect to I. In fact let $F \subset E$ be any σ-large Lie subalgebra. Form the Hochschild–Serre spectral sequence

$$E^2_{p,q} = \operatorname{Tor}^{UF}_p(\operatorname{Tor}^{UI}_q(\Bbbk, (UI)^{\#}), (UF)^{\#}) \implies \operatorname{Tor}^{U(F \oplus I)}_{p+q}(\Bbbk, (U(F \oplus I))^{\#}).$$

We deduce that for some $q \ge \operatorname{depth} I$, $\operatorname{grade}_F \operatorname{Tor}^{UI}_q(\Bbbk, (UI)^{\#}) \le \operatorname{depth}(F \oplus I) - q$. But according to Lemma 3.9 there is a σ-large Lie subalgebra $E' \subset F$ such that $\operatorname{depth} E' \le \operatorname{grade}_F \operatorname{Tor}^{UI}_q(\Bbbk, (UI)^{\#})$. Thus

$$\begin{aligned} \operatorname{depth} E' &\le \operatorname{depth}(F \oplus I) - q \le \operatorname{depth}(F \oplus I) - \operatorname{depth} I \\ &\le \operatorname{depth}(E \oplus I) - \operatorname{depth} I = \operatorname{depth} E. \end{aligned}$$

Since E is σ-minimal these inequalities are equalities; in particular $\operatorname{depth}(F \oplus I) = \operatorname{depth}(E \oplus I)$ and $E \oplus I$ is σ-minimal with respect to I.

(ii) Consider the Hochschild–Serre spectral sequence converging from

$$E_1^{p,q} = \operatorname{Ext}^q_{U(E \oplus I)}(\Gamma^p sL/(E \oplus I), UL) \quad \text{to} \quad \operatorname{Ext}^{p+q}_{UL}(\Bbbk, UL).$$

Since $L/(E \oplus I)$ is a σ-thin E-module annihilated by I, it is also a σ-thin $E \oplus I$-module. Thus each $\Gamma^p sL/(E \oplus I)$ is a σ-thin $(E \oplus I)$-module annihilated by I. Thus, since $E \oplus I$ is σ-minimal with respect to I, Theorem 3.8 asserts that either $\Gamma^p sL/(E \oplus I) = 0$, or else

$$\operatorname{depth}(E \oplus I) = \operatorname{grade}_{E \oplus I}(\Gamma^p sL/(E \oplus I)).$$

Since $\operatorname{Ext}^q_{U(E \oplus I)}(\Gamma^p sL/(E \oplus I), UL) \neq 0$ for some $p + q = \operatorname{depth} L$, the theorem follows. □

Definition 4.2. Let I be an ideal in a cft graded Lie algebra of finite depth. A *weak complement* for I in L is a Lie subalgebra $E \subset L$ such that $E \cap I = 0$, $E \oplus I$ is full in L, and for some sequence $\sigma = (\sigma_i)_{i \geq 1}$ satisfying $0 \leq \sigma_i \leq \dim I_i$, $i \geq 1$: E is σ-minimal, and I and $L/(E \oplus I)$ are σ-thin E-modules.

Theorem 4.3. *Let I be an ideal in a cft graded Lie algebra of finite depth.*

(i) *There is an I-large Lie subalgebra $F \subset L$ such that $F \cap I = 0$. If E is any I-minimal, I-large Lie subalgebra of F then E is a weak complement for I in L.*

(ii) *If E is any weak complement for I in L, then*

$$\operatorname{depth} E + \operatorname{depth} I = \operatorname{depth}(E \oplus I) \leq \operatorname{depth} L.$$

Proof. (i). Since $\operatorname{depth} I < \infty$, there are elements $x_1, \dots, x_r \in I$ such that the Lie subalgebra , G, generated by the x_i satisfies $\operatorname{Ext}^*_{UI}(\Bbbk, UL) \to \operatorname{Ext}^*_{UG}(\Bbbk, UL)$ is non-zero (Lemma 3.5). This implies that $A = \{ y \in I \mid [y, x_i] = 0, 1 \leq i \leq r \}$ is a finite dimensional Lie subalgebra (Lemma 3.6). Choose n so that A is concentrated in degrees $< n$ and set

$$F = \{ y \in L_{\geq n} \mid [y, x_i] = 0, 1 \leq i \leq r \}.$$

Evidently $F \cap I = 0$.

On the other hand, F is the kernel of the linear map $L_{\geq n} \to I \oplus \cdots \oplus I$ given by $x \mapsto ([x, x_1], \dots, [x, x_r])$. Thus

$$\dim L_k / F_k \leq \sum_{i=1}^{r} \dim I_{k + \deg x_i}.$$

It follows that F is I-large in L, and so E is also I-large in L. Thus for some λ, q, N we have $\dim(L/E)_k \leq \lambda \sum_{i=k}^{k+q} \dim I_i$, $k \geq N$. It follows that $\dim L_k \leq (\lambda + 1) \sum_{i=k}^{k+q} \dim(E_i \oplus I_i)$, $k \geq N$ and so $E \oplus I$ is full in L. Finally, since E is I-large in L, Lemma 3.3 (iii) asserts that $L/(E \oplus I)$ is I-thin. □

Proposition 4.4. *Let J and K be ideals in a cft graded Lie algebra L of finite depth. Then there is a weak complement, E, for $J \cap K$ in K that is also a weak complement for J in $J + K$.*

Proof. By Theorem 4.3 (i) we may choose E to be $J \cap K$-minimal and such that $J \cap K$ and $K/(E \oplus J \cap K)$ are $(J \cap K)$-thin E-modules. Note that $E \cap J = (E \cap K) \cap J = 0$.

Set $\sigma_i = \dim(J \cap K)_i$ and note that because $[E, J] \subset [K, J] \subset J \cap K$ it follows that J is a σ-thin E-module. Moreover, $K/(E \oplus J \cap K)$ maps onto $(K + J)/(E \oplus J)$ and so $(K + J)/(E \oplus J)$ is also a σ-thin E-module. Finally, this surjection also shows that $E \oplus J$ is full in $K + J$ since $E \oplus J \cap K$ is full in K. □

Proposition 4.5. *Let $I \subset L$ be an ideal in a cft graded Lie algebra, and suppose that for some p, the restriction map*

$$\mathrm{Ext}^p_{UL}(\Bbbk, UL) \to \mathrm{Ext}^p_{UI}(\Bbbk, UL)$$

is non-zero. Then I is full in L.

Proof. Suppose $\alpha \in \mathrm{Ext}^p_{UL}(\Bbbk, UL)$ restricts to a non-zero element in $\mathrm{Ext}^p_{UI}(\Bbbk, UL)$. This in turn would restrict to a non-zero element in $\mathrm{Ext}^p_{UE}(\Bbbk, UL)$, where E is a finitely generated Lie subalgebra of I, see Proposition 3.1 in [3]. Let $x_1, \dots, x_r$ generate E. Then by [9], Lemma 6, the centralizer of E in L is finite dimensional. Therefore for n enough large, the map

$$L_n \to \oplus_{j=1}^r I_{n+\deg x_j}, \quad y \to \sum [y, x_i]$$

is injective. This gives the result. □

5. L-equivalence

It is immediate from Proposition 2.5 that an equivalence relation on the ideals of a cft graded Lie algebra, L, is defined by:

$$I \sim_L J \iff \text{for all ideals } K \subset L, I \cap K \sim J \cap K.$$

Definition and notation. The relation above will be called *L-equivalence* and the set of L-equivalence classes of ideals in L will be denoted by $\mathcal{L}$. If I is an ideal in L its L-equivalence class will be denoted by $[I]$. Finally, the number (possibly ∞) of L-equivalence classes of ideals will be denoted by ν_L, and for any subspace $V \subset L$ the number of L-equivalence classes represented by L-ideals contained in V will be denoted by $\nu_L(V)$.

Our next aim is to establish the following two results.

Proposition 5.1. *Let L be a cft graded Lie algebra. Then the structure of a distributive lattice in $\mathcal{L}$ is defined by*

$$[I] \le [J] \iff J \cap I \sim_L I, \quad [I] \vee [J] = [I + J]$$

and

$$[I] \wedge [J] = [I \cap J].$$

Proposition 5.2. *Let $J \subset I$ be ideals in a cft graded Lie algebra L. Then any maximal chain of strict inequalities in $\mathcal{L}$ of the form*

$$[J] < [I(1)] < \cdots < [I(r)] = [I]$$

has the same length. Moreover

$$r \leq \text{depth}\, I - \text{depth}\, J.$$

Definition. The length r in the chain above in Proposition 5.2 is called the *height* of $[I]$ over $[J]$. When $[J] = [0]$, r is called the *height* of $[I]$ and denoted by $\text{ht}[I]$.

Remark. Clearly the height of $[I]$ over $[J]$ is just $\text{ht}[I] - \text{ht}[J]$.

Before proving Proposition 5.1 we establish some preliminary lemmas.

Lemma 5.3. *Suppose I and J are ideals in a cft graded Lie algebra. Then,*

(i) $\text{depth}\, I \leq \text{depth}(J + I)$,

(ii) *if* $\text{depth}\, I = \text{depth}(J + I)$ *then $J \cap I$ is full in J.*

In particular, if $I \subset J$ and $\text{depth}\, I = \text{depth}\, J$, *then I is full in J.*

Proof. By Proposition 4.4 there is a weak complement, E, for $J \cap I$ in J that is also a weak complement for I in $I + J$. Thus

$$\text{depth}\, E + \text{depth}\, I = \text{depth}(E \oplus I) \leq \text{depth}(J + I).$$

It follows that $\text{depth}\, I \leq \text{depth}(J + I)$ and if equality holds then $\text{depth}\, E = 0$. This implies that E is finite dimensional ([1]). Since $E \oplus (J \cap I)$ is full in J it follows that $J \cap I$ is full in J. □

Lemma 5.4. *Let L be a cft graded Lie algebra of finite depth m. Then $[L, L]$ is full in L. In particular, if I and J are ideals in L then $[I, J]$ is full in $I \cap J$.*

Proof. Let E be a weak complement for $[L, L]$ in L. Since $[E, E] \subset E \cap [L, L]$, E is abelian. Since E has finite depth it is finite dimensional [1]. Now because $E \oplus [L, L]$ is full in L, $[L, L]$ is full in L. Finally, note that

$$[I \cap J, I \cap J] \subset [I, J] \subset I \cap J$$

to derive the last assertion. □

Lemma 5.5. *If I, J, K are ideals in L, then*

$$(I + J) \cap K \sim I \cap K + J \cap K.$$

Proof. $(I + J) \cap K \sim [I + J, K] = [I, K] + [J, K] \sim I \cap K + J \cap K$. □

Lemma 5.6. *Let I, J be ideals in a cft graded Lie algebra of finite depth. Then:*

(i) $I \sim_L J \iff I \sim_L (I+J) \sim_L J$.

(ii) $I \sim_L J \iff I \sim_L (I \cap J) \sim_L J$.

(iii) *If* $I \subset J$ *and* depth $I =$ depth J, *then* $I \sim_L J$.

(iv) *If* $I(i) \sim_L J(i)$ *are pairs of* L*-equivalent ideals in* L, *then*

$$I(1) + \cdots + I(r) \sim_L J(1) + \cdots + J(r).$$

(v) *For any ideal* K, $(I+J) \cap K \sim_L I \cap K + J \cap K$.

(vi) *If* $I \sim_L J$ *and* K *is any ideal in* L *then* $I \cap K \sim_L J \cap K$.

Proof. (i) We need only show that $I \sim_L J \implies I \sim_L (I+J)$. If K is any ideal in L, then by Lemma 5.5 and Proposition 2.7,

$$I \cap K \sim I \cap K + I \cap K \sim I \cap K + J \cap K \sim (I+J) \cap K.$$

Thus $I \sim_L I + J$.

(ii) We need only prove that $I \sim_L J \implies I \sim_L I \cap J$. Again let K be an ideal in L. Then

$$(I \cap J) \cap K = I \cap (J \cap K) \sim J \cap (J \cap K) = J \cap K.$$

Thus $I \cap J \sim_L J$.

(iii) Let K be an ideal in L. Since $I \subset I + (J \cap K) \subset J$ we have

$$\text{depth}\, I \leq \text{depth}(I + (J \cap K)) \leq \text{depth}\, J,$$

and so depth $I = \text{depth}(I + (J \cap K))$. It follows from Lemma 5.3 that $I \cap (J \cap K)$ is full in $J \cap K$. But $I \cap J = I$ and so $I \cap K$ is full in $J \cap K$. Thus $I \cap K \sim J \cap K$ for all K; i.e., $I \sim_L J$.

(iv) We need only show that if $I \sim_L J$ and H is an ideal in L, then $I + H \sim_L J + H$. But for any ideal K we have by Lemma 5.5 and Proposition 2.7

$$K \cap (I+H) \sim (K \cap I) + (K \cap H) \sim (K \cap J) + (K \cap H) \sim K \cap (J+H).$$

Thus $I + H \sim_L J + H$.

(v) For any ideal $H \subset L$ we have by Lemma 5.5

$$(I+J) \cap K \cap H \sim (I \cap K \cap H) + (J \cap K \cap H) \sim ((I \cap K) + (J \cap K)) \cap H.$$

Thus $(I+J) \cap K \sim_L I \cap K + J \cap K$.

(vi) For any L-ideal H, $(I \cap K) \cap H = I \cap (K \cap H) \sim J \cap K \cap H = (J \cap K) \cap H$. □

Proof of Proposition 5.1. If follows from Lemma 5.6 (vi) that the condition $I \cap J \sim_L I$ depends only on $[I]$ and $[J]$; thus the partial order is well defined. Clearly [0] and $[L]$ are initial and terminal elements. It follows from Lemma 5.6 (iv) and Lemma 5.6 (vi) that $[I] \vee [J]$ and $[I] \wedge [J]$ only depend on $[I]$ and $[J]$ and Lemma 5.6 (v) shows that the lattice is distributive. □

Proof of Proposition 5.2. The first assertion is a standard fact about distributive lattices. The second follows from Lemma 5.6 (iii), which asserts that if $[J] < [I]$ thendepth $J <$ depth I. □

Theorem 5.7. *Let L be a cft graded Lie algebra of finite depth m and height r. Then*

(i) $r \leq m$.

(ii) $\nu_L \leq 2^r$.

(iii) $\nu_L = 2^r$ *if and only if* $L \sim_L I(1) \oplus \cdots \oplus I(r)$ *where the* $I(i)$ *are infinite dimensional ideals. In this case* $I(i)$ *has height* 1.

(iv) *If* $\nu_L = 2^m$ *then* $\mathrm{ht}[L] = \mathrm{depth}\, L$ *and the* $I(i)$ *are infinite dimensional ideals of depth* 1.

For the proof of Theorem 5.7 we require one more lemma.

Lemma 5.8. *Let L be a cft graded Lie algebra.*

(i) *If* $I \subset J$ *are L-ideals then* $\nu_L(I) \leq \nu_L(J)$.

(ii) *If I and J are L-ideals and* $I \sim_L J$ *then* $\nu_L(I) = \nu_L(J)$. *In particular,* $\nu_l[I]$ *is well defined.*

(iii) *if I is the direct sum of L-ideals J and K* ($I = J \oplus K$), *then* $\nu_L(I) = \nu_L(J)\nu_L(K)$.

Proof. (i) The set of L-equivalence classes of L-ideals in I is clearly a subset of the L-equivalence classes of L-ideals in J. Thus $\nu_L(I) \leq \nu_L(J)$.

(ii) Since $I \sim_L (I \cap J)$ (Lemma 5.6) any L-ideal H contained in I satisfies $H = (H \cap I) \sim_L (H \cap I \cap J)$ (Lemma 5.6 (vi)). Thus the set of L-equivalence classes of L-ideals in I coincides with the set of L-equivalence classes of L-ideals in $I \cap J$, and so $\nu_L(I) = \nu_L(I \cap J) = \nu_L(J)$.

(iii) Any L-ideal H in I satisfies $H \sim_L (H \cap J) \oplus (H \cap K)$, and if G is another L-ideal in I such that $G \cap J \sim_L H \cap J$ and $G \cap K \sim_L H \cap K$ then $G \sim_L (G \cap J) \oplus (G \cap K) \sim_L (H \cap J) \oplus (H \cap K) \sim_L H$. It follows that $\nu_L(I) = \nu_L(J)\nu_L(K)$. □

Proof of Theorem 5.7. Proposition 5.2 asserts that $\mathrm{ht}[L] \leq \operatorname{depth} L = m < \infty$. This is statement (i).

Next let $0 < [J(1)] < \cdots < [J(r)] = [L]$ be a maximal chain of strict inclusions in $\mathcal{L}$, and let $\mathcal{L}(k)$ denote the subset of $\mathcal{L}$ of elements $[J] \leq [J(k)]$. Then, for any k, $1 \leq k \leq r$, let $[K] \in \mathcal{L}$ be an element of minimum height satisfying the two conditions:

$$[K] \leq [J(k)] \quad \text{and} \quad [K] \not\leq [J(k-1)].$$

We shall show that the map $\varphi(k)\colon \mathcal{L}(k-1) \times \mathbb{Z}_2 \to \mathcal{L}(k)$ given by

$$([J], 0) \mapsto [J] \quad \text{and} \quad ([J], 1) \mapsto [J] \vee [K]$$

is a surjection.

In fact, our conditions above imply that $[J(k-1)] \vee [K] = [J(k)]$. Thus for any $[J] \in \mathcal{L}(k)$ we have $[J] = ([J] \wedge [J(k-1)]) \vee ([J] \wedge [K])$. If $[J] \wedge [K] \not\leq [J(k-1)]$ then it too satisfies the conditions above and has height $\leq \mathrm{ht}[K]$. But $\mathrm{ht}[K]$ was a minimum; thus $[J] \wedge [K] = [K]$ in this case and it follows that $\varphi(k)$ is indeed a surjection. In particular, $\nu_L[J(k)] \leq 2\nu_L[J(k-1)]$ and so $\nu_L(L) \leq 2^r \nu_L(0) = 2^r$. This proves (ii).

For (iii), suppose first that $\nu_L = 2^r$. Reversing the argument above we see that $\nu_L[J(k)] = 2\nu_L[J(k-1)]$, each k, and so each $\varphi(k)$ is a bijection. But clearly

$$\varphi(k)([0], 1) = [K] = ([J(k-1)] \wedge [K]) \vee [K] = \varphi(k)([J(k-1)] \wedge [K], 1).$$

Thus $J(k-1) \cap K \sim 0$; i.e., it is finite dimensional and concentrated in degrees $< n$, some n. Now set $I(k) = J(k) \cap K_{\geq n}$. Then $I(k) \sim_L K_{\geq n} \sim_L K$ and so $[J(k)] = [J(k-1) \oplus I(k)]$. This also implies that $I(k)$ is not L-equivalent to zero; i.e., $\dim I(k)$ is infinite. Finally, by construction $[L] = [I(1) \oplus \cdots \oplus I(r)]$.

Conversely, suppose $L \sim_L I(1) \oplus \cdots \oplus I(r)$, where each $I(i)$ is an infinite dimensional ideal. Then $[0] < [I(i)]$, each i, and so $\nu_L[I(i)] \geq 2$. By Lemma 5.8 (iii), and part (ii), $2^r \geq \nu_L = \prod_{i=1}^r \nu_L[I(i)] \geq 2^r$. Thus these inequalities are equalities and $2^r = \nu_L$ and $2 = \nu_L[I(i)]$, $1 \leq i \leq r$. This implies that each $I(i)$ has height 1.

(iv) If $\nu_L = 2^m$ we must have $m = r$. Since the $I(i)$ are infinite dimensional, $\operatorname{depth} I(i) \geq 1$ and because $m = \operatorname{depth} L = \sum_{i=1}^m \operatorname{depth} I(i)$ (by Lemma 5.8) we have $\operatorname{depth} I(i) = 1$, each i. □

Corollary. *Let L be a cft graded Lie algebra and assume $L \sim_L I(1) \oplus \cdots \oplus I(r)$, where the $I(i)$ are infinite dimensional ideals of height 1. Then* $\mathrm{ht}[L] = r$ *and every element $[I] \in \mathcal{L}$ of height $s \geq 1$ is uniquely of the form $[I] = [I_{i_1}] \vee \ldots [I_{i_s}]$.*

Proof. It is a trivial consequence of the distributive law that

$$[I_{i_1}] \vee \cdots \vee [I_{i_s}] = [I_{j_1}] \vee \cdots \vee [I_{j_q}]$$

if and only if $s = q$ and $\{i_1, \dots, i_s\} = \{j_1, \dots, j_q\}$. Thus the elements of the form $[I_{i_1}] \vee \dots \vee [I_{i_s}]$, $1 \le s \le r$ are $2^r - 1$ distinct elements of $\mathcal{L}$, and so $\nu_L \ge 2^r$.

On the other hand, because each $I(i)$ has height 1, it is also an immediate consequence of the distributive law that $0 < [I(1)] < \dots < [I(1)] \vee \dots \vee [I(k)] < \dots < [L]$ is a chain of maximum length, so that $\mathrm{ht}[L] = r$ and $\nu_L \le 2^r$. Thus $\nu_L = 2^r$ and $\mathcal{L} = \{[0], [I_{i_1}] \vee \dots \vee [I_{i_s}]\}$. □

Remark. Theorem 5.7 and its corollary show that the cft graded Lie algebras L satisfying $\mathrm{ht}[L] = r$ and $\nu_L = 2^r$ are the analogues in this setting of the classical semi-simple Lie algebras. Note that this includes as a special case the cft graded Lie algebras L with depth $L = m$ and $\nu_L = 2^m$.

Proposition 5.9. *Let L be a cft graded Lie algebra. If*

$$2\mathrm{ht}[L] \le \mathrm{depth}\, L - 1$$

then L contains a free Lie algebra on two generators.

Lemma 5.10. *Suppose $J \subset I$ are ideals in a cft graded Lie algebra satisfying $[J] < [I]$ and* depth $J + 1 =$ depth I. *Then I contains an infinite dimensional Lie subalgebra of depth* 1.

Proof. Since $[J] < [I]$ there is an ideal $H \subset L$ such that $J \cap H$ is not full in $I \cap H$. Set $K = I \cap H$; then $J \cap K$ is not full in K. Thus a weak complement, E, for $J \cap K$ in K is infinite dimensional.

Next note that since $J \subset J + K \subset I$, either depth $J =$ depth$(J + K)$ or depth$(J+K) =$ depth I. The first equality would imply $J \sim_L (J+K)$ (Lemma 5.7) and thus (intersection with K) $J \cap K \sim_L K$, which is impossible because $J \cap K$ is not full in K. Thus

$$\mathrm{depth}(J + K) = \mathrm{depth}\, J + 1.$$

But since E may be chosen to also be a weak complement for J in $J + K$ (Proposition 4.4), Theorem 4.3 yields

$$\mathrm{depth}\, E + \mathrm{depth}\, J \le \mathrm{depth}(J + K) = \mathrm{depth}\, J + 1.$$

This gives depth $E \le 1$. But E is infinite dimensional and thus depth $E = 1$. □

Proof of Proposition 5.9. Let $0 < [I(1) < \dots < [I(r)] = [L]$ be a chain of strict inclusions in $\mathcal{L}$, with $r = \mathrm{ht}[L]$. We may assume $I(1) \subset \dots \subset I(r)$, and then it follows from Lemma 5.7 that $0 <$ depth $I(1) < \dots <$ depth $I(r)$. In view of our hypothesis either depth $I(1) = 1$ or for some i, depth $I(i + 1) =$ depth $I(i) + 1$. Lemma 5.8 then implies that $I(i+1)$ contains an infinite dimensional Lie subalgebra of depth 1. Finally according to [7] each infinite dimensional Lie subalgebra of depth 1 contains a free Lie algebra on two generators. □

6. The hyperradical

Recall that the *radical* of a cft graded Lie algebra L is the sum of its solvable ideals. In [1], Theorem C, it is shown that if depth $L < \infty$, then the radical of L is finite dimensional.

Definition 6.1. The *hyperradical* R of cft graded Lie algebra, L, is the sum of the ideals $I \subset L$ satisfying

$$\log \text{index}\, I < \log \text{index}\, L.$$

By convention, $R = \{0\}$ if there is no infinite dimensional ideal I of L with $\log \text{index}\, I < \log \text{index}\, L$. Clearly R is an ideal.

Theorem 6.2. *Let R be the hyperradical of an infinite dimensional cft graded Lie algebra L of finite depth, and let (x) denote the ideal in L generated by $x \in L$. Then*

(i) $x \in R$ *if and only if* $\log \text{index}(x) < \log \text{index}\, L$,

(ii) $\log \text{index}\, R < \log \text{index}\, L$, *and* depth $R <$ depth L.

Proof. (i) Suppose x is a finite sum $x = \sum_{i=1}^{p} x_i$ where x_i belongs to an ideal I_i with $\log \text{index}\, I_i < \log \text{index}\, L$. There is then an integer N and a non negative real number ε such that for $n \geq N$ and $i \leq p$, we have

$$\frac{\log \dim (I_i)_n}{n} \leq \log \text{index}\, L - \varepsilon.$$

If $I = I_1 + \cdots + I_p$, this implies that $\log \text{index}\, I < \log \text{index}\, L$. In particular, $\log \text{index}(x) < \log \text{index}\, L$.

(ii) By [9], Lemma 4, R contains a finitely generated Lie subalgebra E for which $\text{Ext}^*_{UR}(\Bbbk, UR) \to \text{Ext}^*_{UE}(\Bbbk, UR)$ is non-zero. Let $x_1, \ldots, x_r \in R$ generate E. If $I = (x_1) + \cdots + (x_r)$, it follows a fortiori that $\text{Ext}^*_{UR}(\Bbbk, UR) \to \text{Ext}^*_{UI}(\Bbbk, UR)$ is non-zero. Thus by Proposition 4.4, I is full in R. Now Proposition 2.10 and the argument in (i) above give $\log \text{index}\, R = \log \text{index}\, I < \log \text{index}\, L$. Thus R is not full in L, and so Lemma 4.6 shows that depth $R <$ depth L. □

Corollary 6.3. *Let L be an infinite dimensional cft graded Lie algebra of finite depth. For any $\lambda \geq 0$, let $J \subset L$ be the sum of all the ideals I satisfying* $\log \text{index}\, I \leq \lambda$. *Then* $\log \text{index}\, J \leq \lambda$.

Proof. If $\log \text{index}\, J > \lambda$, then J is its own hyperradical, which is impossible by Theorem 6.2 (ii) . □

Proposition 6.4. *Let L be a cft graded Lie algebra of finite depth. Then L contains a full Lie subalgebra whose hyperradical is zero.*

Proof. Let $E \subset L$ be a full Lie subalgebra of minimal depth, let R be the hyperradical of E, and let F be a weak complement for R in E. Since $R \oplus F$ is full in E and since $\log\text{index}\, R < \log\text{index}\, E$, it follows that F is full in E. Moreover, Theorem 4.2 asserts that $\operatorname{depth} F + \operatorname{depth} R \leq \operatorname{depth} E$.

But our hypothesis on E yields that $\operatorname{depth} F \geq \operatorname{depth} E$, and it follows that $\operatorname{depth} R = 0$; i.e., R is finite dimensional and concentrated in odd degrees. Choose n so that $R_{\geq n} = 0$. Then $E_{\geq n}$ is a full Lie subalgebra of E and, since it is an ideal, $\operatorname{depth} E_{\geq n} \leq \operatorname{depth} E$; thus $E_{\geq n}$ also has minimal depth among its full Lie subalgebra s. Thus if $S \subset E_{\geq n}$ is its hyperradical, S is also finite dimensional and concentrated in odd degrees.

The ideal I in E generated by S is the image of the linear map $UE \otimes_{UE_{\geq n}} S \to UE$, and hence has polynomial growth. Since $\operatorname{depth} I < \infty$ this implies ([6]) that $\dim I < \infty$; i.e., $I \subset R$. Thus $S \subset R_{\geq n} = 0$ and so the hyperradical of $E_{\geq n}$ is zero. □

Proposition 6.5. *Let L be an infinite dimensional cft graded Lie algebra of finite depth m. Then at most m pairs (α, β) can satisfy*

$$\alpha = \log\text{index}\, I \quad \textit{and} \quad \beta = \log\text{index}\, I$$

for some ideal I.

Proof. Suppose $I_1, \dots, I_r$ are ideals with respective log indices and lower log indices ordered by lexicographic order $(\alpha_1, \beta_1) < \cdots < (\alpha_r, \beta_r)$. Then we can replace the sequence of ideals by the following sequence with the same sequence of log indices $I_1 \subset I_1 + I_2 \subset \cdots \subset I_1 + \cdots + I_r$. Since the (α_i, β_i) are distinct, no $I_1 + \cdots I_j$ is full in $I_1 + \cdots + I_{j+1}$. Therefore, by Lemma 4.6, $r \leq m$. □

Example 6.6. Let X be the space

$$S^3_a \vee S^3_b \vee S^5_z \cup_{[a,z]} e^8 \cup_{[a,[a,z]]} e^{10} \cup_{[b,[a,z]]} e^{10}.$$

Then L_X has depth 2 and the lattice $\mathcal{L}$ has exactly three elements.

The Sullivan minimal model of X is quasi-isomorphic to the differential graded algebra $(A, d) = (\wedge(x, y, z, t)/(xy, tz), d)$ where $\deg x = \deg y = 3$, $\deg z = 5$, $\deg z = 7$, $dx = dy = dz = 0$, $d(t) = yz$. The algebra (A, d) is a semifree $(\wedge(x, y)/(xy), 0)$-module ([5]). This gives a rational fibration

$$F = S^5 \vee S^7 \to X \to B = S^3 \vee S^3.$$

The ideal L_F has not the same log index as L_X, and so is neither L-equivalent to L_X or to 0. The exact sequence $0 \to L_F \to L_X \to L_B \to 0$ implies at once that

$$[0] < [L_F] < [L_X]$$

are the only elements of $\mathcal{L}$. In particular L_F is the hyperradical of L_X.

7. The odd and even part of a graded Lie algebra

Theorem 7.1. *Let L be a cft graded Lie algebra of finite depth.*

(i) *Either L_{odd} is contained in a finite dimensional ideal of L, or else for some d the integers $\sum_{j=k+1}^{k+d} \dim(L_{\text{odd}})_j$ grow faster than any polynomial in k.*

(ii) *The Lie subalgebra L_{even} is full in L.*

Proof. Let I be the Lie subalgebra generated by L_{odd}; I is clearly an ideal in L and hence has finite depth. Choose $x_1, \dots, x_n$ of odd degrees $e_1 \le \dots \le e_n$ that generate a Lie subalgebra F for which $\text{Ext}_*^{UI}(\Bbbk, UI) \to \text{Ext}_*^{UF}(\Bbbk, UI)$ is non-zero. The centralizer of the x_i in I is therefore finite dimensional, which implies that for some N the linear map $x \mapsto ([x, x_1], \dots, [x, x_n])$ is an injection $I_k \to I_{k+e_1} \oplus \dots \oplus I_{k+e_n}$, $k \ge N$. Since the e_i are odd, it follows that, for $k \ge N$,

$$\dim(I_{\text{odd}})_k \le \sum_{j=k+e_1}^{k+e_n} \dim(I_{\text{even}})_j \quad \text{and} \quad \dim(I_{\text{even}})_k \le \sum_{j=k+e_1}^{k+e_n} \dim(I_{\text{odd}})_j,$$

which implies that both I_{odd} and I_{even} are full in I.

Now suppose I is infinite dimensional. Then according to [6] for some d the integers $\sum_{j=k}^{k+d} \dim I_j$ grow faster than any polynomial in k. Since $\dim I_{2j} \le \sum_{i=2j+e_1}^{2j+e_n} \dim(I_{\text{odd}})_i$, it follows that $(d+2)\sum_{j=k}^{k+d+e_n} \dim(I_{\text{odd}})_j$ grow faster than any polynomial in k. And, of course, $I_{\text{odd}} = L_{\text{odd}}$.

Finally, let E be a weak complement for I in L. Then $E \subset L_{\text{even}}$ and $E \oplus I$ is full in L. Since I_{even} is full in I it follows that $E \oplus I_{\text{even}}$ is full in L and so L_{even} is full in L. □

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Received December 10, 2007

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