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A product formula for valuations on manifolds with applications to the integral geometry of the quaternionic line

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Abstract. The Alesker–Poincaré pairing for smooth valuations on manifolds is expressed in terms of the Rumin differential operator acting on the cosphere-bundle. It is shown that the derivation operator, the signature operator and the Laplace operator acting on smooth valuations are formally self-adjoint with respect to this pairing. As an application, the product structure of the space of SU(2)- and translation invariant valuations on the quaternionic line is described. The principal kinematic formula on the quaternionic line $\mathbb{H}$ is stated and proved.

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1. Smooth valuations on manifolds

Let M be a smooth manifold of dimension n. For simplicity, we suppose that M is oriented, although the whole theory works in the non-oriented case as well. Following Alesker, we set $\mathcal{P}(M)$ to be the set of compact submanifolds with corners.

Definition 1.1. A valuation on M is a real valued map μ on $\mathcal{P}(M)$ which is additive in the following sense: whenever $X, Y, X \cap Y$ and $X \cup Y$ belong to $\mathcal{P}(M)$, then

$$\mu(X \cup Y) + \mu(X \cap Y) = \mu(X) + \mu(Y).$$

A set $X \in \mathcal{P}(M)$ admits a conormal cycle $\operatorname{cnc}(X)$, which is a compactly supported Legendrian cycle on the cosphere bundle S^*M. Sometimes it will be convenient to think of S^*M as the set of pairs (p, P) with $p \in M$ and $P \subset T_pM$ an oriented hyperplane, at other places it is better to think of it as the set of pairs $(p, [\xi])$ where $p \in M$ and $\xi \in T_p^*M \setminus \{0\}$ and the brackets denote the equivalence class for the relation $\xi_1 \sim \xi_2 \iff \xi_1 = \lambda \xi_2, \lambda > 0$.

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A valuation μ on M is called *smooth* if there exist an $(n-1)$-form $\omega \in \Omega^{n-1}(S^*M)$ and an n-form $\phi \in \Omega^n(M)$ such that

$$\mu(X) = \operatorname{cnc}(X)(\omega) + \int_X \phi, \quad X \in \mathcal{P}(X). \tag{1}$$

If μ can be expressed in the form (1), we say that μ *is represented* by (ω, ϕ). The space of smooth valuations on M is denoted by $\mathcal{V}^\infty(M)$. It is a Fréchet space (see [6], Section 3.2 for the definition of the topology). If $M = V$ is a vector space, the subspace of translation invariant smooth valuations will be denoted by $\mathrm{Val}^{\mathrm{sm}}(V)$.

Let N be another oriented n-dimensional smooth manifold and $\rho\colon N \to M$ an orientation preserving immersion. Then ρ induces a map $\tilde\rho\colon S^*N \to S^*M$, sending (p, P) to $(\rho(p), T_p\rho(P))$. It clearly satisfies $\pi \circ \tilde\rho = \rho \circ \pi$.

The valuation $\rho^*\mu$ on N such that

$$\rho^*\mu(X) := \mu(\rho(X)), \quad X \in \mathcal{P}(N)$$

is again smooth. If μ is represented by (ω, ϕ), then $\rho^*\mu$ is represented by $(\tilde\rho^*\omega, \rho^*\phi)$. This follows from the fact that $\operatorname{cnc}(\rho(X)) = \tilde\rho_* \operatorname{cnc}(X)$. Note also that $\widetilde{\rho^{-1}} = (\tilde\rho)^{-1}$ if ρ is a diffeomorphism.

We will use some results of [11], which we would like to recall. The cosphere bundle S^*M is a contact manifold of dimension $2n-1$ with a global contact form α (α is not unique, but this will play no role here). The projection from S^*M to M will be denoted by π, it induces a linear map π_* (fiber integration) on the level of forms.

Given an $(n-1)$-form ω on S^*M, there exists a unique vertical form $\alpha \wedge \xi$ such that $d(\omega + \alpha \wedge \xi)$ is vertical (i.e. a multiple of α). The Rumin differential operator D is defined as $D\omega := d(\omega + \alpha \wedge \xi)$ [18]. The following theorem was proved in [11].

Theorem 1.2. *Let $\omega \in \Omega^{n-1}(S^*M)$, $\phi \in \Omega^n(M)$ and define the smooth valuation μ by* (1). *Then $\mu = 0$ if and only if*

(1) $D\omega + \pi^*\phi = 0$, *and*

(2) $\pi_*\omega = 0$ *for all* $p \in M$.

Moreover, if $D\omega + \pi^*\phi = 0$, then μ is a multiple of the Euler characteristic χ.

The support of a smooth valuation μ is defined as

$$\operatorname{spt}\mu := M \setminus \{p \in M : \text{there exists } p \in U \subset M \text{ open such that } \mu|_U = 0\}.$$

The subspace of $\mathcal{V}^\infty(M)$ consisting of compactly supported valuations will be denoted by $\mathcal{V}_c^\infty(M)$.

Let $\int\colon \mathcal{V}_c^\infty(M) \to \mathbb{R}$ denote the integration functional [7]. If μ has compact support, then $\int \mu := \mu(X)$, where $X \in \mathcal{P}(M)$ is an n-dimensional manifold with boundary containing spt μ in its interior. It is clear that, if μ is represented by (ω, ϕ) with compact supports, then

$$\int \mu = \int_M \phi = [\phi] \in H_c^n(M) = \mathbb{R}.$$

Before stating our main theorem we have to recall two other constructions of Alesker.

The first one is the Euler–Verdier involution $\sigma\colon \mathcal{V}^\infty(M) \to \mathcal{V}^\infty(M)$ [6]. Let $s\colon S^*M \to S^*M$ be the natural involution on S^*M, sending (p, P) to $(p, \overline{P})$, where $\overline{P}$ is the hyperplane P with the reversed orientation. If a valuation $\mu \in \mathcal{V}^\infty(M)$ is represented by the pair (ω, ϕ), then $\sigma\mu$ is defined as the valuation which is represented by the pair $((-1)^n s^*\omega, (-1)^n\phi)$.

The second construction is the Alesker–Fu product [9], which is a bilinear map

$$\mathcal{V}^\infty(M) \times \mathcal{V}^\infty(M) \to \mathcal{V}^\infty(M), \quad (\mu_1, \mu_2) \mapsto \mu_1 \cdot \mu_2.$$

We refer to [9] for its construction. It is characterized by the following properties:

(1) "·" is continuous and linear in both variables;

(2) if $\rho\colon N \to M$ is a diffeomorphism and $\mu_1, \mu_2 \in \mathcal{V}^\infty(M)$, then

$$\rho^*(\mu_1 \cdot \mu_2) = \rho^*\mu_1 \cdot \rho^*\mu_2;$$

(3) if m_1, m_2 are smooth measures on an n-dimensional vector space V, $A_1, A_2 \in \mathcal{K}(V)$ convex bodies with strictly convex smooth boundary and if $\mu_i \in \mathcal{V}^\infty(V)$, $i = 1, 2$ is defined by

$$\mu_i(K) = m_i(K + A_i), \quad K \in \mathcal{K}(V), \tag{2}$$

then

$$\mu_1 \cdot \mu_2(K) = m_1 \times m_2(\Delta(K) + A_1 \times A_2),$$

where $\Delta\colon V \to V \times V$ is the diagonal embedding.

Our first main theorem is the following relation between Alesker–Fu product, integration functional, Euler–Verdier involution and Rumin differential.

Theorem 1.3. *Let $\mu_1 \in \mathcal{V}^\infty(M)$ be represented by (ω_1, ϕ_2); let $\mu_2 \in \mathcal{V}_c^\infty(M)$ be represented by (ω_2, ϕ_2). Then*

$$\int \mu_1 \cdot \sigma\mu_2 = (-1)^n \int_{S^*M} \omega_1 \wedge (D\omega_2 + \pi^*\phi_2) + \int_M \phi_1 \wedge \pi_*\omega_2. \tag{3}$$

Let us call the pairing

$$\begin{aligned} \mathcal{V}^\infty(M) \times \mathcal{V}_c^\infty(M) &\to \mathbb{R}, \\ (\mu_1,\mu_2) &\mapsto \int \mu_1 \cdot \mu_2 =: \langle \mu_1, \mu_2\rangle \end{aligned} \tag{4}$$

the *Alesker–Poincaré pairing*. Note that Theorem 1.3 is equivalent to

$$\langle \mu_1,\mu_2\rangle = \int_{S^*M} \omega_1 \wedge s^*(D\omega_2 + \pi^*\phi_2) + \int_M \phi_1 \wedge \pi_*\omega_2. \tag{5}$$

From Theorem 1.3 and from the fact that the Poincaré pairings on M and S^*M are perfect, we get the following corollary (which was first proved by Alesker).

Corollary 1.4 ([7], Theorem 6.1.1). *The Alesker–Poincaré pairing* (4) *is a perfect pairing.*

Some more operators on $\mathcal{V}^\infty(M)$ were introduced in [11]. For this, we suppose that M is a Riemannian manifold. Then S^*M admits an induced metric, the *Sasaki metric* [20].

The first operator is the derivation operator Λ (which was denoted by $\mathfrak{L}$ in [11]). The metric on S^*M provides a canonical choice of α, namely $\alpha|_{(p,[\xi])} := \frac{1}{\|\xi\|}\pi^*\xi$ for all $(p,[\xi]) \in S^*M$. Let T be the Reeb vector field on S^*M (i.e. $\alpha(T) = 1$ and $\mathcal{L}_T\alpha = 0$).

If the smooth valuation μ is represented by (ω,ϕ), then $\Lambda\mu$ is by definition the valuation which is represented by $(\mathcal{L}_T\omega + i_T\pi^*\phi, 0)$.

Let us recall the definitions of the signature operator $\mathcal{S}$ and the Laplace operator Δ. Let $*$ be the Hodge star acting on $\Omega^*(S^*M)$. Let $\mu \in \mathcal{V}^\infty(M)$ be represented by (ω,ϕ). Then $\mathcal{S}\mu$ is defined as the valuation which is represented by $(*(D\omega + \pi^*\phi), 0)$.

The Laplace operator Δ is defined as $\Delta := (-1)^n\mathcal{S}^2$.

Our second main theorem shows that these operators fit well into Alesker's theory. In fact, they are formally self-adjoint with respect to the Alesker–Poincaré pairing.

Theorem 1.5. *For valuations $\mu_1 \in \mathcal{V}^\infty(M)$ and $\mu_2 \in \mathcal{V}_c^\infty(M)$, the following equations hold:*

$$\langle \Lambda\mu_1,\mu_2\rangle = \langle \mu_1,\Lambda\mu_2\rangle, \tag{6}$$

$$\langle \mathcal{S}\mu_1,\mu_2\rangle = \langle \mu_1,\mathcal{S}\mu_2\rangle, \tag{7}$$

$$\langle \Delta\mu_1,\mu_2\rangle = \langle \mu_1,\Delta\mu_2\rangle. \tag{8}$$

We will apply these theorems in the study of the integral geometry of $\mathrm{SU}(2)$. This group acts on the quaternionic line $\mathbb{H}$. In this setting, it is more natural to work with the space $\mathcal{K}(\mathbb{H})$ of convex sets instead of manifolds with corners. By Proposition 2.6. of [8], there is no loss of generality in doing so.

It was shown by Alesker [3] that the space of SU(2)-invariant and translation invariant valuations on the quaternionic line $\mathbb{H}$ is of dimension 10. For each purely complex number u of norm 1, let I_u be the complex structure given by multiplication from the right with u and $\mathbb{CP}_u^1$ the corresponding Grassmannian of complex lines (with its unique SU(2)-invariant Haar measure). Alesker defined a valuation Z_u by

$$Z_u(K) := \int_{\mathbb{CP}_u^1} \operatorname{vol}(\pi_L(K))dL, \quad K \in \mathcal{K}(\mathbb{H}). \tag{9}$$

He showed that $Z_i, Z_j, Z_k, Z_{\frac{i+j}{\sqrt{2}}}, Z_{\frac{i+j}{\sqrt{2}}}, Z_{\frac{i+j}{\sqrt{2}}}$, together with Euler characteristic χ, the volume vol and the intrinsic volumes $\operatorname{vol}_1, \operatorname{vol}_3$ form a basis of $\operatorname{Val}^{\mathrm{SU}(2)}$. Following a suggestion of Fu, we will state the kinematic formula using a more symmetric choice. Noting that $Z_u = Z_{-u}$ for all $u \in S^2$, the 12 vertices $\pm u_i$, $i = 1, \dots, 6$ of an icosahedron on S^2 define 6 valuations Z_{u_i}, $i = 1, \dots, 6$.

We endow SU(2) with its Haar measure and the semidirect product $\overline{\mathrm{SU}(2)} = \mathrm{SU}(2) \ltimes \mathbb{H}$ with the product measure. Let vol_k denote the k-dimensional intrinsic volume [15].

Theorem 1.6 (Principal kinematic formula for SU(2)). *Let $K, L \in \mathcal{K}(\mathbb{H})$. Then*

$$\begin{aligned}\int_{\overline{\mathrm{SU}(2)}} \chi(K \cap \bar{g}L)d\bar{g} = \chi(K)\operatorname{vol}(L) + \frac{4}{3\pi}\operatorname{vol}_1(K)\operatorname{vol}_3(L) \\ + \frac{17}{4}\sum_{i=1}^{6} Z_{u_i}(K)Z_{u_i}(L) - \frac{3}{4}\sum_{1\le i\neq j\le 6} Z_{u_i}(K)Z_{u_j}(L) \\ + \frac{4}{3\pi}\operatorname{vol}_3(K)\operatorname{vol}_1(L) + \operatorname{vol}(K)\chi(L).\end{aligned}$$

This theorem implies and generalizes the Poincaré formulas of Tasaki [19], (which contained an error in some constant) as we will explain in the last section.

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2. The Alesker–Poincaré pairing in terms of forms

In order to prove Theorem 1.3 and Theorem 1.4, we will need three lemmas which are of independent interest.

Lemma 2.1 (Partition of unity for valuations, [7], Proposition 6.2.1). *Let* $M = \bigcup_i U_i$ *be a locally finite open cover of* M*. Then there exist valuations* $\mu_i \in \mathcal{V}^\infty(M)$ *such that* $\operatorname{spt} \mu_i \subset U_i$ *and*

$$\sum_i \mu_i = \chi.$$

Proof. Let $1 = \sum_i f_i$ a partition of unity subordinate to $M = \bigcup_i U_i$. We represent χ by (ω, ϕ) and let μ_i be the valuation represented by $(\pi^* f_i \wedge \omega, f_i \phi)$. □

By inspecting the proof of Theorem 1.2 (which uses a local variational argument), one gets the following lemma.

Lemma 2.2. *Let* $\omega \in \Omega^{n-1}(S^*M)$*,* $\phi \in \Omega^n(M)$ *and define the smooth valuation* μ *by* (1)*. Then*

$$\operatorname{spt} D\omega + \pi^*\phi \subset \pi^{-1}(\operatorname{spt} \mu) \quad \textit{and} \quad \operatorname{spt} \pi_*\omega \subset \operatorname{spt} \mu.$$

Lemma 2.3. *Let* $\mu \in \mathcal{V}^\infty(M)$ *be compactly supported. Then* μ *can be represented by a pair* $(\omega, \phi) \in \Omega^{n-1}(S^*M) \times \Omega^n(M)$ *of compactly supported forms.*

Proof. We suppose M is non-compact (otherwise the statement is trivial). Let μ be represented by a pair (ω', ϕ'). Then $D\omega' + \pi^*\phi'$ is compactly supported. Since $H_c^n(S^*M) = \mathbb{R}$, there exists a compactly supported form $\phi \in \Omega_c^n(M)$ such that

$$[D\omega' + \pi^*\phi'] = [\pi^*\phi] \in H_c^n(S^*M).$$

In other words, there is a compactly supported form $\omega \in H_c^{n-1}(S^*M)$ such that $d\omega = D\omega = D\omega' + \pi^*\phi' - \pi^*\phi$. By Theorem 1.2, the pair (ω, ϕ) represents μ up to a multiple of χ. Since the valuation represented by (ω, ϕ) and the valuation μ are both compactly supported, whereas χ is not, they have to be the same. □

Proof of Theorem 1.3. Note first that the right hand side of (3) is well defined: since μ_2 is compactly supported, the same holds true for $D\omega_2 + \pi^*\phi_2$ and $\pi_*\omega_2$ by Lemma 2.2.

Next, both sides of (3) are linear in μ_1 and μ_2. Using Lemma 2.1, we may therefore assume that the supports of μ_1 and μ_2 are contained in the support of a coordinate chart. Since the Alesker–Fu product, the Euler–Verdier involution and the integration functional are natural with respect to diffeomorphisms, it suffices to prove (3) in the case where $M = V$ is a real vector space of dimension n.

Let us first suppose that μ_1 and $\sigma\mu_2$ are of the type (2). We thus have $\mu_1(K) = m_1(K + A_1)$ and $\sigma\mu_2(K) = m_2(K + A_2)$ with smooth measures m_1, m_2 and smooth convex bodies A_1, A_2 with strictly convex boundary.

The left hand side of (3) is given by

$$\begin{aligned}\int \mu_1 \cdot \sigma\mu_2 &= m_1 \times m_2(\Delta(V) + A_1 \times A_2)\\ &= \int_V m_1((\Delta V + A_1 \times A_2) \cap V \times \{x\})\, dm_2(x)\\ &= \int_V m_1(x - A_2 + A_1)\, dm_2(x)\\ &= \int_V \mu_1(x - A_2)\, dm_2(x)\\ &= \int_V \operatorname{cnc}(x - A_2)(\omega_1)\, dm_2(x) + \int_V \left(\int_{x-A_2} \phi_1\right) dm_2(x).\end{aligned} \tag{10}$$

Let $A \in \mathcal{K}(V)$ be smooth with strictly convex boundary. Its support function is defined by

$$\begin{aligned} h_A\colon V^* &\to \mathbb{R},\\ \xi &\mapsto \sup_{x \in A} \xi(x).\end{aligned}$$

Note that h_A is homogeneous of degree 1 and that $h_{-A}(\xi) = h_A(-\xi)$.

Define the map $G_A\colon S^*V \to S^*V, (x, [\xi]) \mapsto (x + d_\xi h_A, [\xi])$ (since h_A is homogeneous of degree 1, $d_\xi h_A \in V^{**} = V$ only depends on $[\xi]$). G_A is an orientation preserving diffeomorphism of S^*V.

It is easy to show ([10], [12]) that for $X \in \mathcal{K}(V)$

$$\operatorname{cnc}(X + A) = (G_A)_* \operatorname{cnc}(X). \tag{11}$$

We next compute that for all $(x, [\xi]) \in S^*V$

$$\begin{aligned} G_A \circ s(x, [\xi]) = G_A(x, [-\xi]) &= (x + d_{-\xi} h_A, [-\xi]) = (x - d_\xi h_{-A}, [-\xi])\\ &= s(x - d_\xi h_{-A}, [\xi]) = s \circ G_{-A}^{-1}(x, [\xi]).\end{aligned} \tag{12}$$

Let $\kappa_2 \in \Omega^n(V)$ be the form representing the measure m_2. The first term in (10) is equal to

$$\begin{aligned}\int_V \operatorname{cnc}(x - A_2)(\omega_1) dm_2(x) &= \int_V \operatorname{cnc}(\{x\})(G_{-A_2}^* \omega_1)\, dm_2(x)\\ &= \int_V \pi_*(G_{-A_2}^* \omega_1) \wedge \kappa_2\\ &= \int_{S^*V} G_{-A_2}^* \omega_1 \wedge \pi^* \kappa_2\\ &= \int_{S^*V} \omega_1 \wedge (G_{-A_2}^{-1})^* \pi^* \kappa_2.\end{aligned} \tag{13}$$

By (11) we have $(-1)^n Ds^*\omega_2 + (-1)^n \pi^*\phi_2 = G_A^*\pi^*\kappa_2$. Applying s^* to both sides and using (12), we get

$$(-1)^n (D\omega_2 + \pi^*\phi_2) = s^* G_{A_2}^* \pi^* \kappa_2 = (G_{-A_2}^{-1})^* \pi^* \kappa_2.$$

Hence (13) equals $(-1)^n \int_{S^*V} \omega_1 \wedge (D\omega_2 + \pi^*\phi_2)$, which is the first term in (3).

By Fubini's theorem, the second term in (10) equals

$$\int_V \left(\int_{x-A_2} \phi_1 \right) dm_2(x) = \int_V m_2(y + A_2)\phi_1(y) = \int_V \sigma\mu_2(\{y\})\phi_1(y).$$

For $y \in V$, we have $s_* \operatorname{cnc}(\{y\}) = (-1)^n \operatorname{cnc}(\{y\})$, since the antipodal map on S^{n-1} is orientation preserving precisely if n is even. Hence

$$\sigma\mu_2(\{y\}) = \pi_*\omega_2(y).$$

The second term in (10) thus equals $\int_V \phi_1 \wedge \pi_*\omega_2$, which corresponds to the second term in (3).

This finishes the proof in the case where μ_1 and $\sigma\mu_2$ are of type (2). By linearity of both sides, (3) holds true for linear combinations of such valuations. Given arbitrary $\mu_1 \in \mathcal{V}^\infty(M)$ and $\mu_2 \in \mathcal{V}_c^\infty(M)$, we find sequences $\mu_1^j \in \mathcal{V}^\infty(M)$ and $\mu_2^j \in \mathcal{V}_c^\infty(M)$ such that $\mu_1^j \to \mu_1$ and $\mu_2^j \to \mu_2$ and such that μ_1^j and $\sigma\mu_2^j$ are linear combinations of valuations of type (2) (compare [5] and [6]).

By definition of the topology on $\mathcal{V}^\infty(M)$ (see Section 3.2 of [6]) and the open mapping theorem, there are sequences (ω_1^j, ϕ_1^j) and (ω_2^j, ϕ_2^j) representing μ_1^j, μ_2^j and converging to (ω_1, ϕ_1), (ω_2, ϕ_2) in the C^∞-topology. By what we have proved,

$$\int \mu_1^j \cdot \sigma\mu_2^j = (-1)^n \int_{S^*M} \omega_1^j \wedge (D\omega_2^j + \pi^*\phi_2^j) + \int_M \phi_1^j \wedge \pi_*\omega_2^j$$

for all j. Letting j tend to infinity, Equation (3) follows. □

3. Self-adjointness of natural operators

Proof of Theorem 1.5. Note first the following equation:

$$\langle \sigma\mu_1, \mu_2 \rangle = (-1)^n \langle \mu_1, \sigma\mu_2 \rangle. \tag{14}$$

This equation is immediate from (5) and the fact that $s\colon S^*M \to S^*M$ preserves orientation if and only if n is even.

Let μ_i be represented by (ω_i, ϕ_i). By Lemma 2.3 we may suppose that ω_2 and ϕ_2 are compactly supported.

$\Lambda\mu_i$ is represented by $\xi_i := i_T(D\omega_i + \pi^*\phi_i)$. Since $D\omega_i + \pi^*\phi_i = \alpha \wedge \xi_i$, we get

$$\begin{aligned}\langle \Lambda\mu_1, \sigma\mu_2\rangle &= (-1)^n \int_{S^*M} \xi_1 \wedge (D\omega_2 + \pi^*\phi_2)\\ &= (-1)^n \int_{S^*M} \xi_1 \wedge \alpha \wedge \xi_2\\ &= -\int_{S^*M} \xi_2 \wedge (D\omega_1 + \pi^*\phi_1)\\ &= (-1)^{n+1} \int_{S^*M} \omega_1 \wedge D\xi_2 - \int_M \phi_1 \wedge \pi_*\xi_2\\ &= -\langle \mu_1, \sigma\Lambda\mu_2\rangle.\end{aligned} \tag{15}$$

Since D and s^* commute and since $i_T \circ s^* = -s^* \circ i_T$, it is easily checked that

$$\Lambda \circ \sigma = -\sigma \circ \Lambda. \tag{16}$$

Now (6) follows from (15) and (16).

Let us next prove (7) ((8) is an immediate consequence).

By Lemma 2.3 we may suppose that ω_2 and ϕ_2 have compact support. Then

$$\begin{aligned}\langle \mu_1, \sigma\mathcal{S}\mu_2\rangle &= (-1)^n \int_{S^*M} \omega_1 \wedge D * (D\omega_2 + \pi^*\phi_2)\\ &\qquad + \int_M \phi_1 \wedge \pi_* * (D\omega_2 + \pi^*\phi_2)\\ &= \int_{S^*M} (D\omega_1 + \pi^*\phi_1) \wedge *(D\omega_2 + \pi^*\phi_2)\\ &= \int_{S^*M} *(D\omega_1 + \pi^*\phi_1) \wedge (D\omega_2 + \pi^*\phi_2)\\ &= \int_{S^*M} (-1)^n s^* * (D\omega_1 + \pi^*\phi_1) \wedge s^*(D\omega_2 + \pi^*\phi_2)\\ &= \langle \sigma\mathcal{S}\mu_1, \mu_2\rangle.\end{aligned} \tag{17}$$

Since s changes the orientation of S^*M by $(-1)^n$, we get $s^* \circ * = (-1)^n * \circ s^*$ on $\Omega^*(S^*V)$. It follows that $\sigma \circ \mathcal{S} = (-1)^n \mathcal{S} \circ \sigma$. Therefore (7) follows from (14) and (17). □

Alesker defined the space $\mathcal{V}^{-\infty}(M)$ of *generalized valuations* on M by

$$\mathcal{V}^{-\infty}(M) := \left(\mathcal{V}_c^\infty(M)\right)^*,$$

where the star means the topological dual. This space is endowed with the weak topology. By the perfectness of the Alesker–Poincaré pairing, there is a natural dense embedding $\mathcal{V}^\infty(M) \hookrightarrow \mathcal{V}^{-\infty}(M)$.

Corollary 3.1. *Let M be a Riemannian manifold. Each of the operators $\Lambda, \mathcal{S}, \Delta$ acting on $\mathcal{V}^\infty(M)$ admits a unique continuous extension to $\mathcal{V}^{-\infty}(M)$.*

Proof. Uniqueness of the extension is clear, since $\mathcal{V}^\infty(M)$ is dense in $\mathcal{V}^{-\infty}(M)$. We let Λ act on $\mathcal{V}^{-\infty}$ by $\Lambda\xi(\mu) := \xi(\Lambda\mu)$. By Theorem 1.5, this is consistent with the embedding of $\mathcal{V}^\infty(M)$ into $\mathcal{V}^{-\infty}(M)$ and we are done. The cases of $\mathcal{S}$ and Δ are similar. □

4. The translation invariant case

From now on, V will denote an oriented n-dimensional real vector space. We will consider valuations on the space $\mathcal{K}(V)$ of compact convex sets (i.e. convex valuations).

A convex valuation μ on V is called translation invariant, if $\mu(x+K) = \mu(K)$ for all $K \in \mathcal{K}(V)$ and all $x \in V$.

A translation invariant convex valuation μ is said to be of degree k if $\mu(tK) = t^k\mu(K)$ for $t > 0$ and $K \in \mathcal{K}(V)$. By $\mathrm{Val}_k(V)$ we denote the space of translation invariant convex valuations of degree k. A valuation μ is even if $\mu(-K) = \mu(K)$ and odd if $\mu(-K) = -\mu(K)$, the corresponding spaces will be denoted by a superscript $+$ or $-$.

In [16] it is shown that the space of translation invariant valuations can be written as a direct sum

$$\mathrm{Val}(V) = \bigoplus_{k=0}^{n} \mathrm{Val}_k(V).$$

Each space $\mathrm{Val}_k(V)$ splits further as $\mathrm{Val}_k(V) = \mathrm{Val}_k^+(V) \oplus \mathrm{Val}_k^-(V)$.

The spaces $\mathrm{Val}_0(V)$ and $\mathrm{Val}_n(V)$ are both 1-dimensional (generated by χ and a Lebesgue measure respectively). For $\mu \in \mathrm{Val}(V)$, we denote by μ_n its component of degree n.

Let us prove the following version of Theorem 1.3 in the translation invariant case.

Theorem 4.1. *Let $\mu_1, \mu_2 \in \mathrm{Val}^{\mathrm{sm}}(V)$ be represented by translation invariant forms (ω_1, ϕ_1), (ω_2, ϕ_2) respectively. Then $(\mu_1 \cdot \sigma\mu_2)_n$ is represented by the n-form*

$$(-1)^n \pi_*(\omega_1 \wedge (D\omega_2 + \pi^*\phi_2)) + \phi_1 \wedge \pi_*\omega_2 \in \Omega^n(V).$$

Proof. The proof is similar to that of Theorem 1.3. Fix a Euclidean metric on V. For $R > 0$, let B_R denote the ball of radius R, centered at the origin. Let us suppose that

$\mu_1(K) = \mathrm{vol}(K + A_1)$ and $\sigma\mu_2(K) = \mathrm{vol}(K + A_2)$ for all $K \in \mathcal{K}(V)$. Then

$$\begin{aligned}\mu_1 \cdot \sigma\mu_2(B_R) &= \mathrm{vol}_{2n}(\Delta(B_R) + A_1 \times A_2)\\ &= \int_{B_R} \mathrm{vol}(x - A_2 + A_1)dx + o(R^n)\\ &= \int_{B_R} \mu_1(x - A_2) + o(R^n)\\ &= \int_{B_R} \mathrm{cnc}(x - A_2)(\omega_1)dx + \int_{B_R}\int_{x-A_2} \phi_1 dx + o(R^n).\end{aligned}$$

The first term is given by

$$\begin{aligned}\int_{B_R} \mathrm{cnc}(x - A_2)(\omega_1)dx &= \int_{B_R} \pi_*(G^*_{-A_2}\omega_1)dx + o(R^n)\\ &= \int_{B_R\times S^*(V)} G^*_{-A_2}\omega_1 \wedge \pi^*(dx) + o(R^n)\\ &= \int_{G_{-A_2}(B_R\times S^*(V))} \omega_1 \wedge (G^{-1}_{-A_2})^*\pi^* dx_2 + o(R^n)\\ &= (-1)^n \int_{B_R\times S^*(V)} \omega_1 \wedge (D\omega_2 + \pi^*\phi_2) + o(R^n)\\ &= (-1)^n \int_{B_R} \pi_*(\omega_1 \wedge (D\omega_2 + \pi^*\phi_2)) + o(R^n).\end{aligned}$$

The second term yields

$$\begin{aligned}\int_{B_R}\int_{x-A_2} \phi_1 dx &= \int_V \mathrm{vol}((y + A_2)\cap B_R)\phi_1(y)\\ &= \int_{B_R} \mathrm{vol}(y + A_2)\phi_1(y) + o(R^n)\\ &= \int_{B_R} \mu_2(\{y\})\phi_1(y) + o(R^n)\\ &= \int_{B_R} \phi_1 \wedge \pi_*\omega_2 + o(R^n).\end{aligned}$$

Therefore we obtain

$$\begin{aligned}(\mu_1 \cdot \sigma\mu_2)_n &= \lim_{R\to\infty} \frac{1}{R^n}\mu_1 \cdot \sigma\mu_2(B_R)\\ &= \lim_{R\to\infty} \frac{1}{R^n}\int_{B_R} (-1)^n\pi_*(\omega_1 \wedge (D\omega_2 + \pi^*\phi_2)) + \phi_1 \wedge \pi_*\omega_2.\end{aligned}$$

This finishes the proof of Theorem 4.1 in the case where μ_1 and $\sigma\mu_2$ are of type $K \mapsto \mathrm{vol}(K+A)$. Using linearity of both sides, it also hold for linear combinations of such valuations. Since they are dense in $\mathrm{Val}(V)$ (by Alesker's solution of McMullen's conjecture [1]), Theorem 4.1 is true in general. □

Let us next suppose that V is endowed with a Euclidean product. We can identify $\mathrm{Val}_n(V)$ with $\mathbb{R}$ by sending vol to 1. We get a symmetric bilinear form (called Alesker pairing)

$$\begin{aligned} \mathrm{Val}^{\mathrm{sm}}(V) \times \mathrm{Val}^{\mathrm{sm}}(V) &\to \mathbb{R}, \\ (\mu_1, \mu_2) &\mapsto \langle \mu_1, \mu_2 \rangle := (\mu_1 \cdot \mu_2)_n. \end{aligned}$$

Corollary 4.2. *For valuations $\mu_1, \mu_2 \in \mathrm{Val}^{\mathrm{sm}}(V)$, the following equations hold:*

$$\begin{aligned} \langle \Lambda \mu_1, \mu_2 \rangle &= \langle \mu_1, \Lambda \mu_2 \rangle, \\ \langle \mathcal{S} \mu_1, \mu_2 \rangle &= \langle \mu_1, \mathcal{S} \mu_2 \rangle, \\ \langle \Delta \mu_1, \mu_2 \rangle &= \langle \mu_1, \Delta \mu_2 \rangle. \end{aligned}$$

Proof. Analogous to the proof of Theorem 1.5. □

5. Kinematic formulas and Poincaré formulas

5.1. Kinematic formulas. In this section, we suppose that G is a subgroup of $\mathrm{O}(V)$ acting transitively on the unit sphere. By a result of Alesker [3], the space of translation invariant and G-invariant valuations Val^G is a finite-dimensional vector space.

Let $\phi_1, \dots, \phi_N$ a basis of Val^G. Suppose we have a kinematic formula

$$\int_{\bar{G}} \chi(K \cap \bar{g} L) d\bar{g} = \sum_{i,j=1}^{N} c_{i,j} \phi_i(K) \phi_j(L).$$

Here and in the following, G is endowed with its Haar measure and $\bar{G} := G \ltimes V$ with the product measure.

Set

$$k_G(\chi) := \sum_{i,j=1}^{N} c_{i,j} \phi_i \otimes \phi_j \in \mathrm{Val}^G \otimes \mathrm{Val}^G = \mathbf{Hom}(\mathrm{Val}^G, \mathrm{Val}^{G*}).$$

The Alesker pairing induces a bijective map

$$\mathbf{PD} \in \mathbf{Hom}(\mathrm{Val}^G, \mathrm{Val}^{G*}).$$

Fu [13] showed that these two maps are inverse to each other:

$$k_G(\chi) = \mathbf{PD}^{-1}. \tag{18}$$

For further use, we give another interpretation of (18). Let G be as above. The scalar product on the finite-dimensional space Val^G induces a scalar product on Val^{G*} such that **PD** is an isometry.

Given $K \in \mathcal{K}(V)$, let $\mu_K \in \mathrm{Val}^{G*}$ be defined by

$$\mu_K(\mu) = \mu(K), \quad \mu \in \mathrm{Val}^G.$$

Proposition 5.1 (Principal kinematic formula). *Let G be a subgroup of $\mathrm{O}(V)$ acting transitively on the unit sphere. Then for $K, L \in \mathcal{K}(V)$*

$$\int_{\bar{G}} \chi(K \cap \bar{g}L)\, d\bar{g} = \langle \mu_K, \mu_L \rangle.$$

Proof. Let $\phi_1, \dots, \phi_N$ be a basis of Val^G. Set $g_{ij} := \langle \phi_i, \phi_j \rangle$, $i, j = 1, \dots, N$. Let us denote by $(g^{ij})_{i,j=1\dots,N}$ the inverse matrix. Then

$$\int_{\bar{G}} \chi(K \cap \bar{g}L) d\bar{g} = \sum_{i,j} g^{ij} \phi_i(K)\phi_j(L) = \sum_{i,j} g^{ij} \mu_K(\phi_i)\mu_L(\phi_j) = \langle \mu_K, \mu_L \rangle. \quad \square$$

5.2. Klain functions. Let us suppose additionally that $-1 \in G$, which implies that $\mathrm{Val}^G \subset \mathrm{Val}^+$.

For $0 \leq k \leq n$, the action of G on V induces an action on the Grassmannian $\mathrm{Gr}_k(V)$. We set $\mathcal{P}_k := \mathrm{Gr}_k(V)/G$ for the quotient space. Given $u \in \mathcal{P}_k$, the space of k-planes contained in u admits a unique G-invariant probability measure and we define $Z_u \in \mathrm{Val}^G$ by

$$Z_u(K) := \int_{L \in u} \mathrm{vol}(\pi_L K) dL, \quad K \in \mathcal{K}(V).$$

Recall that the Klain function of an even, translation invariant valuation μ of degree k on a Euclidean vector space V is the function $\mathrm{Kl}_\mu \colon \mathrm{Gr}_k(V) \to \mathbb{R}$ such that the restriction of μ to $L \in \mathrm{Gr}_k(V)$ is given by $\mathrm{Kl}_\mu(L)$ times the Lebesgue measure. An even, translation invariant valuation is uniquely determined by its Klain function [14]. If M is a compact k-dimensional submanifold (possibly with boundaries or corners), then

$$\mu(M) = \int_M \mathrm{Kl}_\mu(T_p M) dp.$$

Alesker proved the existence of a duality operator (or Fourier transform) $\mathbb{F}$ on $\mathrm{Val}^{+,\mathrm{sm}}$ such that $\mathrm{Kl}_{\mathbb{F}\mu} = \mathrm{Kl}_\mu \circ \perp$ for all $\mu \in \mathrm{Val}^{+,\mathrm{sm}}$. $\mathbb{F}$ is formally self-adjoint with respect to the Alesker pairing.

Proposition 5.2. *Let $u, v \in \mathcal{P}_k$ and $L \in v$. Then*

$$\mathrm{Kl}_{Z_u}(L) = \langle \mathbb{F} Z_u, Z_v \rangle. \tag{19}$$

Proof. Immediate from Lemma 2.2. of [12]. □

Lemma 5.3. *There are finitely many elements $u_1, \dots, u_N$ such that $Z_{u_i}, i = 1, \dots, N$ is a basis of* Val_k^G.

Proof. Let $\phi_1, \dots, \phi_N$ be a basis of Val_k^G. Let m_i be the push-forward of a Crofton measure for ϕ_i on $\mathrm{Gr}_k(V)$ under the projection $\mathrm{Gr}_k(V) \to \mathcal{P}_k$.

By G-invariance of ϕ_i, we get

$$\phi_i(K) = \int_{\mathcal{P}_k} \int_{L \in u} \mathrm{vol}(\pi_L K)\, dL\, dm_i(u).$$

Now, for sufficiently close approximations of the m_i by discrete measures $\sum_{j=1}^{k_i} c_{i,j} \delta_{u_{i,j}}$ with $u_{i,j} \in \mathcal{P}_k, c_{i,j} \in \mathbb{R}$, the valuations $\sum_j c_{i,j} Z_{u_{i,j}}$ form a basis of Val_k^G. Hence $\{Z_{u_{i,j}}, i = 1, \dots, N, j = 1, \dots, k_i\}$ is a finite generating set of Val_k^G, from which we can extract a finite basis. □

5.3. Poincaré formulas. Poincaré formulas for G are special cases of the principal kinematic formula for G, when K and L are replaced by smooth compact submanifolds M_1 and M_2 (possibly with boundary) of complementary dimension (note that $M_1, M_2 \in \mathcal{P}(V)$, so there is no problem in evaluating a valuation in M_1 and M_2). Then the right hand side of the principal kinematic formula is the "average number" of intersections of M_1 and $\bar{g} M_2$.

Proposition 5.4 (General Poincaré formula). *Let M_1, M_2 be smooth compact submanifolds, possibly with boundaries, of complementary dimensions k and $n - k$. Then*

$$\int_{\bar{G}} \#(M_1 \cap \bar{g} M_2)\, d\bar{g} = \int_{M_1 \times M_2} \alpha(T_p M_1, T_q M_2)\, dp\, dq$$

with

$$\begin{aligned} \alpha \colon \mathcal{P}_k \times \mathcal{P}_{n-k} &\to \mathbb{R}, \\ (u, v) &\mapsto \langle Z_u, Z_v \rangle. \end{aligned}$$

Proof. Let $u_1, \dots, u_N$ be such that Z_{u_i}, $i = 1, \dots, N$ is a basis of Val_k^G. Let $v_1, \dots, v_N$ be such that $Z_{v_j}, j = 1, \dots, N$ is a basis of Val_{n-k}^G (note that the dimensions of these two spaces agree by Theorem 1.2.2 in [2]). Setting $g_{ij} := \langle Z_{u_i}, Z_{v_j} \rangle$

and (g^{ij}) for the inverse matrix, the principal kinematic formula implies that for all M_1 and M_2 as above

$$\begin{aligned}\int_{\bar{G}} \#(M_1 \cap \bar{g} M_2) d\bar{g} &= \sum_{i,j} g^{ij} Z_{u_i}(M_1) Z_{v_j}(M_2) \\ &= \int_{M_1 \times M_2} \sum_{i,j} g^{i,j} \operatorname{Kl}_{Z_{u_i}}(T_p M_1) \operatorname{Kl}_{Z_{v_j}}(T_q M_2)\, dp dq.\end{aligned}$$

This shows that

$$\alpha(u,v) = \sum_{i,j} g^{ij} \operatorname{Kl}_{Z_{u_i}}(u) \operatorname{Kl}_{Z_{v_j}}(v) = \langle Z_u, Z_v \rangle;$$

where the last equation follows from (19) and the self-adjointness of $\mathbb{F}$. □

6. Kinematic formulas for SU(2)

We apply the results of the preceding section to the special case $G = \mathrm{SU}(2)$ acting on the quaternionic line

$$\mathbb{H} = \{x_1 + x_2 i + x_3 j + x_4 k : (x_1, x_2, x_3, x_4) \in \mathbb{R}^4\}.$$

Since this action is transitive on the unit sphere, $\mathrm{Val}^{\mathrm{SU}(2)}$ is finite dimensional and $\mathrm{Val}_k^{\mathrm{SU}(2)}$ is one-dimensional except for the case $k = 2$. The quotient space $\mathcal{P}_2 := \mathrm{Gr}_2(\mathbb{H})/\mathrm{SU}(2)$ is the two-dimensional projective space $\mathbb{RP}^2 = S^2/\{\pm 1\}$ ([19]). Following Tasaki, we denote by $(\omega_1(L) : \omega_2(L) : \omega_3(L)) \in \mathbb{RP}^2$ the class of $L \in \mathrm{Gr}_2(\mathbb{H})$.

A canonical representative in the preimage of $(a : b : c) \in \mathbb{RP}^2$ is given by the 2-plane spanned by 1 and $ai + bj + ck$.

If $u = (a, b, c) \in S^2$, then the planes in u are the complex lines for the complex structure I_u which is defined by multiplication by u from the right on $\mathbb{H}$. We will therefore write $\mathbb{CP}_u$ instead of u. Note that $\mathbb{CP}_u = \mathbb{CP}_{-u}$.

The following SU(2)-invariant and translation invariant valuations of degree 2 were introduced by Alesker [3].

Definition 6.1. Given $u \in \mathbb{RP}^2$, set

$$Z_u(K) := \int_{\mathbb{CP}_u} \operatorname{vol}(\pi_L(K))\, dL, \quad K \in \mathcal{K}(\mathbb{H}).$$

Proof of Theorem 1.6. From Lemma 5.3 we infer that there is a finite number of points $u_1, \dots, u_N \in \mathbb{RP}^2$ such that Z_{u_i}, $i = 1, \dots, N$ form a basis of $\mathrm{Val}_2^{\mathrm{SU}(2)}$. Alesker showed that $N = 6$ and that $Z_i, Z_j, Z_k, Z_{\frac{i+j}{\sqrt{2}}}, Z_{\frac{i+k}{\sqrt{2}}}, Z_{\frac{j+k}{\sqrt{2}}}$ is such a basis.

Our aim is to compute the product $\langle Z_u, Z_v \rangle$ for $u, v \in \mathbb{RP}^2$. We will achieve it by first expressing each Z_u as a smooth valuation represented by some 3-form $\omega_u \in \Omega^3(S^*\mathbb{H})$ and then applying Theorem 4.1.

Since the metric induces a diffeomorphism between $S^*\mathbb{H}$ and $S\mathbb{H}$, we may as well work with the latter space. The image of the conormal cycle of a compact convex set K under this diffeomorphism is the normal cycle $\mathrm{nc}(K)$.

Let us introduce several differential forms on $S\mathbb{H}$, depending on the choice of the complex structure I_u. We follow the notation of [17].

Let α, β, γ be 1-forms on $S\mathbb{H}$ which, at a point $(x, v) \in S\mathbb{H}$, equal

$$\begin{aligned} \alpha(w) &= \langle v, d\pi(w) \rangle, & w &\in T_{(x,v)} S\mathbb{H}, \\ \beta_u(w) &= \langle v, I_u d\pi(w) \rangle, & w &\in T_{(x,v)} S\mathbb{H}, \\ \gamma_u(w) &= \langle v, I_u d\pi_2(w) \rangle, & w &\in T_{(x,v)} S\mathbb{H}. \end{aligned}$$

Note that α is the canonical 1-form (in particular independent of u), whereas β_u and γ_u depend on u.

Let Ω be the pull-back of the symplectic form on $(\mathbb{H}, I_u)$ to $S\mathbb{H}$, i.e.

$$\Omega_u(w_1, w_2) := \langle d\pi(w_1), I_u d\pi(w_2) \rangle, \quad w_1, w_2 \in T_{(x,v)} S\mathbb{H}.$$

Claim. Z_u is represented by the 3-form

$$\omega_u := \frac{1}{8\pi} \beta_u \wedge d\beta_u + \frac{1}{4\pi} \gamma_u \wedge \Omega_u.$$

Since ω_u is U(2)- and translation invariant and has bidegree $(2, 1)$ (with respect to the product decomposition $S\mathbb{H} = \mathbb{H} \times S(\mathbb{H})$), it represents some U(2)-invariant, translation invariant valuation μ_u of degree 2. Here U(2) is the unitary group for the complex structure I_u.

Now the space of valuations with these properties is of dimension 2 [2]. It is thus enough to show that the valuation Z_u and the valuation μ_u agree on the unit ball B as well as on a complex disk D_u.

It is clear that $Z_u(B) = \omega_2 = \pi$. It was shown by Fu (compare Equation (37) in [13],) that $Z_u(D_u) = \frac{\pi}{2}$.

By [11], the derivation of a smooth translation invariant valuation μ on a finite-dimensional Euclidean vector space is given by

$$\Lambda \mu(K) = \frac{d}{dt}\Big|_{t=0} \mu(K + tB).$$

It follows that, if μ is of degree k, then $\Lambda\mu(B) = k\mu(B)$.

It is easily checked that $\mathcal{L}_T\beta = \gamma$, $\mathcal{L}_T\gamma = 0$ and $\mathcal{L}_T^2\Omega = d\gamma$, so that

$$\mathcal{L}_T^2\omega_u = \frac{1}{2\pi}\gamma \wedge d\gamma.$$

Note that $\gamma \wedge d\gamma$ is twice the volume form on S^3, hence $\Lambda^2\mu_u = 2\pi\chi$. It follows that

$$\mu_u(B) = \frac{1}{2}\Lambda^2\mu_u(B) = \pi.$$

The restriction of β_u to the normal cycle of the complex disc D_u clearly vanishes. γ_u is the length element of the fibers of $\pi\colon \operatorname{nc}(D_u) \to D_u$ (which are circles), whereas Ω_u is the (pull-back of) the volume form on D_u. It follows that $\omega_u(D_u) = \frac{\pi}{2}$. The claim is proved.

Next, the Rumin operator is easily computed as

$$D\omega_u = d\Big(\omega_u + \frac{1}{8\pi}\alpha \wedge \beta_u \wedge \gamma_u - \frac{1}{8\pi}\alpha \wedge \Omega_u\Big) = \frac{1}{2\pi}\alpha \wedge \beta_u \wedge d\gamma_u.$$

From Theorem 4.1 we infer that $\mu_u \cdot \mu_v$ is represented by the 4-form

$$\frac{1}{16\pi^2}\pi_*((\beta_u \wedge d\beta_u + 2\gamma_u \wedge \Omega_u) \wedge \alpha \wedge \beta_v \wedge d\gamma_v) \in \Omega^4(\mathbb{H}).$$

If $u = (a : b : c)$ and $v = (\tilde{a} : \tilde{b} : \tilde{c})$, then

$$\begin{aligned}
&(\beta_u \wedge d\beta_u + 2\gamma_u \wedge \Omega_u) \wedge \alpha \wedge \beta_v \wedge d\gamma_v \\
&\quad = 2\big((a\tilde{b} - \tilde{a}b)^2 + (a\tilde{c} - \tilde{a}c)^2 + (b\tilde{c} - \tilde{b}c)^2 + 2(a\tilde{a} + b\tilde{b} + c\tilde{c})^2\big)d\operatorname{vol}_{S\mathbb{H}} \\
&\quad = 2(1 + (a\tilde{a} + b\tilde{b} + c\tilde{c})^2)d\operatorname{vol}_{S\mathbb{H}}.
\end{aligned}$$

It follows that

$$\langle Z_u, Z_v\rangle = \frac{1}{4}\big(1 + (a\tilde{a} + b\tilde{b} + c\tilde{c})^2\big). \tag{20}$$

Let $\pm u_i$, $i = 1, \dots, 6$ be the 12 vertices of an icosahedron I on S^2. They induce 6 valuations Z_{u_i}, $i = 1, \dots, 6$. Since the edge length a of I satisfies $\cos a = \frac{\sqrt{5}}{5}$, (20) implies that

$$\langle Z_{u_i}, Z_{u_j}\rangle = \begin{cases} \frac{1}{2}, & i = j, \\ \frac{3}{10}, & i \neq j. \end{cases} \tag{21}$$

Theorem 1.6 follows easily from (21) and (18). □

The general Poincaré formula (Proposition 5.4) implies the following (corrected version of the) Poincaré formula on the quaternionic line.

Corollary 6.2 (Poincaré formula for $\mathrm{SU}(2)$, [19]). *Let $M_1, M_2 \subset \mathbb{H}$ be compact smooth 2-dimensional submanifolds. Then*

$$\int_{\overline{\mathrm{SU}(2)}} \#(M_1 \cap \bar{g} M_2) d\bar{g} = \frac{1}{4} \int_{M_1 \times M_2} (1 + A(T_pM_1, T_qM_2))\, dp dq$$

with

$$A(T_pM_1, T_qM_2) = (\omega_1(T_pM_1)\omega_1(T_qM_2) + \omega_2(T_pM_1)\omega_2(T_qM_2) + \omega_3(T_pM_1)\omega_3(T_qM_2))^2.$$

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