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Erratum to "The Huber theorem for non-compact conformally flat manifolds"

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Abstract. An argument in our paper *The Huber theorem for non-compact conformally flat manifolds* [Comment. Math. Helv. 77 (2002), 192–220] was not justified. Using recent work by G. Tian and J. Viaclovsky, we show that our result holds true.

In [4] we consider a complete conformally flat Riemannian manifold (M^n, g) which satisfies the Sobolev inequality :

$$\mu_n(M,g)\left(\int_M u^{\frac{2n}{n-2}}\right)^{1-2/n} \leqslant \int_M |du|^2 \quad \text{for all } u \in C_0^\infty(M), \tag{1}$$

and whose Ricci tensor is in $L^{\frac{n}{2}}$. On page 208 we then assert that *"diameter is controlled from above and volume growth is controlled from below* (*from the Sobolev inequality*) *on each annulus* $M_{kr} - M_{k^{-1}r}$. *We can then infer from Anderson–Cheeger harmonic radius' theory that the rescaled annuli* $(M_{kr_i} - M_{k^{-1}r_i}, r_i^{-2}g)$ *are covered by a finite* (*and uniformly bounded*) *number of balls of uniformly bounded size where the metric coefficients are* $C^{1,\alpha}$*-close to the euclidean metric."* In fact, the trivial extrinsic diameter bound is not enough to ensure Anderson–Cheeger compactness (one needs an intrinsic diameter control) and this argument needs to be justified. This is what we intend to do below.

First observe that the needed intrinsic diameter control can be replaced by an upper bound on the volume growth of geodesic balls. Recently, G. Tian and J. Viaclovsky [5] investigated an issue closely related to ours, and proved the following result:

Theorem ([5]). *Let* (X^n, g) *be a complete noncompact Riemannian manifold of dimension* $n \geq 3$. *If there exists a constant* $C_1 > 0$ *such that* $\mathrm{vol}(B(q,s)) \geq C_1 s^n$, *for any* $q \in X$, $s \geq 0$, *if furthermore* $\sup_{S(r)} |K_g| = o(r^{-2})$ *as* $r \to \infty$, *where* $S(r)$ *is the sphere of radius* r *centered at a basepoint* p, *and if* $b_1(X) < \infty$, *then there exists a constant* C_2 *so that*

$$\mathrm{vol}(B(p,s)) \leq C_2 s^n \quad \textit{for any } s \geq 0. \tag{2}$$

Using this one can show that our argument remains true. Indeed, the Sobolev inequality implies that the manifold has an euclidean lower bound on the volume growth of geodesic balls. Moreover, Sobolev and $L^{\frac{n}{2}}$-integrability of the Ricci curvature imply that the space of L^2 harmonic 1-forms

$$\mathcal{H}^1(M) = \{h \in L^2(T^*M),\ d\alpha = 0,\ d^*\alpha = 0\}$$

and the first cohomology group with compact support $H^1_c(M)$ have finite dimensions [2], [3]. In particular M has a finite number of ends. As it is proved in [4] that $\sup_{S(r)} |K_g| = o(r^{-2})$ in our setting, one can apply Tian–Viaclovsky's theorem if one has finiteness of the first Betti number.

However the assumption on the first Betti number is only used in their paper to insure that the manifold has a finite number of ends (which we already have) and *a finite number of bad connected components of annuli*, as defined below. This is then used to prove the upper bound on the volume growth. A bad component of an annulus is defined as follows: if p is a point in a complete (connected) Riemannian manifold (M, g), let $B(r)$ be the geodesic ball of radius r centered at p and, for $R > r$, let the annulus $A(r, R)$ be the closure of $B(R) - B(r)$. Let (r_k) be an unbounded increasing sequence of positive real numbers and note $A_k = A(r_k, r_{k+1})$.

Definition ([5]). A connected component $\mathcal{C}$ of A_k is said to be bad if $S(r_k) \cap \mathcal{C}$ is disconnected. If $S(r_k) \cap \mathcal{C}$ is connected, we say that $\mathcal{C}$ is good.

We now state:

Claim. *The Sobolev inequality and $L^{\frac{n}{2}}$-integrability of the Ricci curvature imply that the number of bad connected components of any sequence of annuli is finite.*

This will follow from the following

Lemma. *If the image of $H^1_c(M)$ in $H^1(M)$ is zero (for instance if $H^1_c(M)$, $H^1(M)$ or $\mathcal{H}^1(M)$ is zero) then all connected components of A_k are good.*

If the dimension of the image of $H^1_c(M)$ in $H^1(M)$ is finite (for instance if one of the spaces $H^1_c(M)$, $H^1(M)$ or $\mathcal{H}^1(M)$ has finite dimension), then there are only a finite number of bad connected components.

Proof. To prove the first part, let $\mathcal{C}$ be a bad connected component of A_k, then $S(r_k) \cap \mathcal{C}$ has at least two connected components let S_1 be one of these connected components and S_2 be the union of the remaining other components ; choose $p_1 \in S_1$ and $p_2 \in S_2$. By definition there is a continuous curve c_1 in $\mathcal{C}$ from p_2 to p_1. And because $B(r_k)$ is connected there is also a continuous curve c_2 in $B(r_k)$ from p_1 to p_2. Let $\gamma_{\mathcal{C}}$ be the loop $c_2 \# c_1$. There also exist a smooth function f on $\mathcal{C}$ such that the support of f is a neighborhood of $S_1 \subset \mathcal{C}$, such that f is constant near $S(r_k) \cap \mathcal{C}$ and such that $f = 1$ on S_1. Then the 1-form $\alpha_{\mathcal{C}} = df$ has clearly an extension as a smooth closed 1-form on M with support in a small thickening of S_1 in $\mathcal{C}$. It is clear that

$$\int_{\gamma_{\mathcal{C}}} \alpha_{\mathcal{C}} = 1.$$

Hence $\alpha_{\mathcal{C}}$ defines a non zero class in the first cohomology group with compact support $H^1_c(M)$ and also in the first cohomology group $H^1(M)$. Now, M. Anderson ([1]) has noticed that the space $\operatorname{Im}\left(H^1_c(M) \to H^1(M)\right)$ always injects in the space of L^2 harmonic 1-forms $\mathcal{H}^1(M)$ and this yields the expected result.

To prove the second part of the lemma, let $\mathcal{C}_1, \dots, \mathcal{C}_k$ be different bad connected components in a sequence (A_i). We can assume that for a non decreasing sequence (j_i) one has $\mathcal{C}_i \subset A_{r_{j_i}}$. Consider now the loops $\gamma_i = \gamma_{\mathcal{C}_i}$ and the 1-forms $\alpha_i = \alpha_{\mathcal{C}_i}$. Then it is clear that

$$\int_{\gamma_i} \alpha_i = 1, \quad \int_{\gamma_i} \alpha_j = 0 \ \text{ if } i < j.$$

Hence $k \leq \dim\left[\operatorname{Im}\left(H^1_c(M) \to H^1(M)\right)\right]$. □

The proof of the claim is then done since our assumptions imply that the space of L^2 harmonic 1-forms is finite dimensional, as already noticed.

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