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Non-existence of n-dimensional T-embedded discs in $\mathbb{R}^{2n}$

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Abstract. We prove non-existence of C^2-smooth embeddings of n-dimensional discs to $\mathbb{R}^{2n}$ such that the tangent spaces at distinct points are pairwise disjoint.

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A number of recent papers concerned various non-degeneracy conditions on embedding and immersions of smooth manifolds in affine and projective spaces defined in terms of mutual positions of the tangent spaces at distinct points, see [1], [2], [3], [4], [7], [8], [9], [10], [11]. Following Ghomi [1], a C^1-embedded manifold $M^n \subset \mathbb{R}^N$ is called T-embedded if the tangent spaces to M at distinct points do not intersect. For example, the cubic curve (x, x^2, x^3) is a T-embedding of $\mathbb{R}$ to $\mathbb{R}^3$, and the direct product of such curves gives a T-embedding of $\mathbb{R}^n$ to $\mathbb{R}^{3n}$.

A T-embedding $M^n \to \mathbb{R}^N$ induces a topological embedding of the tangent bundle $TM \to \mathbb{R}^N$, hence $N \geq 2n$. One of the results in [1] is that no closed manifold M^n admits T-embeddings to $\mathbb{R}^{2n}$. In this note we extends this result as follows (note that we assume more differentiability than Ghomi).

Theorem 1. *There exist no C^2-smooth T-embedded discs D^n in $\mathbb{R}^{2n}$.*

Proof. Arguing by contradiction, assume that such a disc D^n exists. Choose the tangent space at the origin and its orthogonal complement as coordinate n-dimensional spaces. Making D smaller, if necessary, assume that the disc is the graph of a (germ of a) C^2 smooth map $f : \mathbb{R}^n \to \mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be the domain of f.

Let $z = (u, f(u)) \in D$ where $u \in U$. The tangent space $T_z D$ is given by a linear equation $y = A(u)x - b(u)$ where $A(u)$ is an $n \times n$ matrix and $b(u)$ is a vector in $\mathbb{R}^n$, both depending on u. In terms of f, they have the following expressions. Let $f_1, \dots, f_n$ be the components of f.

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Lemma 1.1. *One has*

$$A_{ij} = \frac{\partial f_i}{\partial u_j}, \quad b_i = \sum_{k=1}^{n} \frac{\partial f_i}{\partial u_k} u_k - f_i.$$

Proof. The first statement is obvious, and the second follows from the fact that the space $y = A(u)x - b(u)$ passes through the point $z = (u, f(u))$. □

One has the next characterization of T-discs.

Lemma 1.2. *For all $u \neq v \in U$, the vector $b(u) - b(v)$ does not belong to* $\mathrm{Im}(A(u) - A(v))$.

Proof. The spaces $y = A(u)x - b(u)$ and $y = A(v)x - b(v)$ intersect if and only if $b(u) - b(v) \in \mathrm{Im}(A(u) - A(v))$. □

Lemma 1.3. *If $u \neq v$ then $b(u) \neq b(v)$ and $A(u) - A(v)$ is degenerate.*

Proof. The first claim follows from the fact that zero vector lies in any subspace, contradicting Lemma 1.2. If $A(u) - A(v)$ is nondegenerate then it is surjective, again contradicting Lemma 1.2. □

Now we compute the Jacobian of the map $b \colon U \to \mathbb{R}^n$. Denote by E the Euler vector field in $\mathbb{R}^n$:

$$E = \sum_{k=1}^{n} u_k \frac{\partial}{\partial u_k}.$$

Lemma 1.4. *One has*

$$\frac{\partial b_i}{\partial u_j} = \sum_k \frac{\partial^2 f_i}{\partial u_j \partial u_k} u_k = E(A_{ij}).$$

Proof. This follows from Lemma 1.1. □

Lemma 1.5. *For all $u \in U$, the Jacobian Jb of the map b is degenerate.*

Proof. Lemma 1.4 implies that

$$Jb = \lim_{\varepsilon \to 0} \frac{A(u + \varepsilon u) - A(u)}{\varepsilon}.$$

By Lemma 1.3 with $v = u + \varepsilon u$, the numerator is a degenerate matrix for all ε, and so is its quotient by ε. Thus Jb is a limit of degenerate matrices. Since determinant is a continuous function, the limit also has zero determinant and therefore is degenerate. □

Finally, we arrive at a contradiction. By Lemma 1.3, the map b is one-to-one, and by the invariance of domain theorem, its image has positive measure. By Lemma 1.5, every value of b is singular, and by Sard's Lemma its image has zero measure. This completes the proof of Theorem 1. □

According to Lemma 1.3, the n-parameter family of $n \times n$ matrices $A(u)$, $u \in D^n$ enjoys the property that $A(u) - A(v)$ is degenerate for all $u \neq v$. If $n = 2$, such families can be explicitly described. Assume that not all matrices $A(u)$ are zero.

Theorem 2. *The family $A(u)$ consists either of the matrices with a fixed 1-dimensional image or with a fixed 1-dimensional kernel.*

Proof. Let M_2 be the space of linear maps $\mathbb{R}^2 \to \mathbb{R}^2$. One has a non-degenerate quadratic form in M_2 given by the determinant of a matrix; this form has signature $(2, 2)$. Consider the respective dot product.

Let $V \subset M_2$ be the linear span of the family $A(u)$.

Lemma 2.1. *The subspace V is isotropic.*

Proof. It suffices to prove that $A(u) \cdot A(v) = 0$ for all u, v. If $u = v$, this means precisely that $A(u)$ is degenerate. For $u \neq v$, the matrix $A(u) - A(v)$ is degenerate, hence $(A(u) - A(v)) \cdot (A(u) - A(v)) = 0$. Using bilinearity of the dot product, it follows that $A(u) \cdot A(v) = 0$. □

Since the dot product is non-degenerate, an isotropic subspace is at most 2-dimensional.

Lemma 2.2. *A 2-dimensional isotropic subspace in M_2 consists either of the matrices with a fixed 1-dimensional image or with a fixed 1-dimensional kernel.*

Proof. Let $A \in V$ be a non-zero matrix. Choose a basis in the target space $\mathbb{R}^2$ in such a way that Im A is orthogonal to the column vector $(0, 1)$. Then

$$A = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

with $a^2 + b^2 \neq 0$. Let $B \in V$ be another matrix, not proportional to A. Then $A \cdot B = 0$, and hence

$$B = \begin{pmatrix} c & d \\ at & bt \end{pmatrix}$$

for some real c, d, t. If $t = 0$ then (c, d) is not proportional to (a, b), and the space V consists of matrices with zero second row. This is the first case of the lemma: the matrices have a fixed image spanned by the column vector $(1, 0)$.

Otherwise, $t \neq 0$. Since B is degenerate, one has: $(c, d) = s(a, b)$ for some real s. Then

$$\frac{B - sA}{t} = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$$

and the space V consists of matrices with a fixed kernel spanned by the column vector $(-b, a)$. □

Lemma 2.2 obviously implies Theorem 2. □

For $n = 2$, Theorem 2 implies the claim of Theorem 1. Indeed, assume that the Jacobi matrix Jf has a fixed 1-dimensional kernel, say, spanned by vector ξ. Then the map f has zero directional derivative along ξ, and the tangent planes to the graph of f are the same along this direction. Hence this graph is not T-embedded. Likewise, if Jf has a fixed 1-dimensional image then the transpose matrix has a fixed kernel, say, η. This implies that the function $f(u) \cdot \eta$ has zero differential, and hence the image of f is 1-dimensional. It follows that the graph of f belongs to a 3-dimensional space and therefore is not T-embedded.

Let us conclude with two examples motivated by the following erroneous attempt to prove Theorem 1: if there exists a T-embedded disc $D^n \subset \mathbb{R}^{2n}$ then its tangent spaces provide a foliation $\mathcal{F}$ of a domain in $\mathbb{R}^{2n}$ by n-dimensional affine subspaces. Then D^n is everywhere tangent to the leaves of this n-dimensional foliation and therefore must lie within a leaf. The mistake in this argument is that, no matter how smooth the embedding is, the foliation $\mathcal{F}$ may be not differentiable. This phenomenon is illustrated in the following example.

Example 1. Let γ be a smooth plane curve with positive curvature and free from vertices (extrema of curvature). Then, by the classical Kneser theorem (1912), the osculating circles to γ are pairwise disjoint and nested as illustrated in Figure 1; see, e.g., [6]. These osculating circles foliate the annulus A between the largest and smallest of them. Denote this foliation by $\mathcal{F}$. Then $\mathcal{F}$ is not C^1, namely, one has the following result.

Proposition. *Let $f : A \to \mathbb{R}$ be a differentiable function, constant on the leaves of $\mathcal{F}$. Then f is constant in A.*

Proof. Since f is constant on the leaves of $\mathcal{F}$, the differential df vanishes on any vector tangent to any leaf. Since γ is everywhere tangent to the leaves, df is zero on the tangent vectors to γ. Hence f is constant on γ. But A is the union of the leaves of $\mathcal{F}$ through the points of γ, hence f is constant in A. □

One also wonders whether $\mathbb{R}^{2n}$ can be foliated by non-parallel affine n-dimensional subspaces (clearly impossible for $n = 1$).

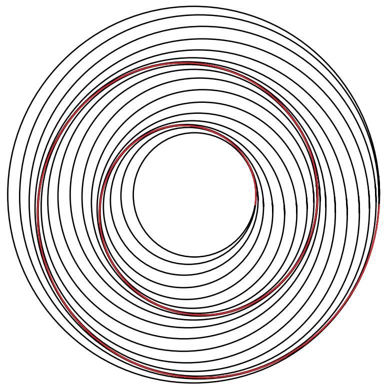

Figure 1. Osculating circles of a spiral.

Example 2. The following construction gives a foliation of $\mathbb{R}^4$ by pairwise non-parallel 2-dimensional affine subspaces. Start with partitioning 3-dimensional space into the vertical z-axis and the hyperboloids of 1 sheet

$$x^2 + y^2 = t(z^2 + 1), \quad t > 0$$

(when $t = 0$, one has the z-axis). Each hyperboloid is foliated by lines, and thus $\mathbb{R}^3$ gets foliated by lines; these lines are pairwise skew. Multiply this foliation by $\mathbb{R}^1$ to obtain the desired example.

This example, of course, is the Hopf fibration of 3-dimensional sphere by great circles, "in disguise": the radial projection of the sphere on $\mathbb{R}^3$ yields a foliation of space by pairwise skew lines. For classification of foliations of S^3 by great circles see [5].

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