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# Some Novikov rings that are von Neumann finite and knot-like groups

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**Abstract.** We show that for a finitely generated group $G$ and for every discrete character $\chi : G \to \mathbb{Z}$ any matrix ring over the Novikov ring $\widehat{\mathbb{Z}G}_\chi$ is von Neumann finite. As a corollary we obtain that if $G$ is a non-trivial discrete group with a finite $K(G, 1)$ CW-complex $Y$ of dimension $n$ and Euler characteristics zero and $N$ is a normal subgroup of $G$ of type $\mathrm{FP}_{n-1}$ containing the commutator subgroup $G'$ and such that $G/N$ is cyclic-by-finite, then $N$ is of homological type $\mathrm{FP}_n$ and $G/N$ has finite virtual cohomological dimension

$$\mathrm{vcd}(G/N) = \mathrm{cd}(G) - \mathrm{cd}(N).$$

This completes the proof of the Rapaport Strasser conjecture that for a knot-like group $G$ with a finitely generated commutator subgroup $G'$ the commutator subgroup $G'$ is always free and generalises an earlier work by the author where the case when $G'$ is residually finite was proved. Another corollary is that a finitely presentable group $G$ with $\mathrm{def}(G) > 0$ and such that $G'$ is finitely generated and perfect can be only $\mathbb{Z}$ or $\mathbb{Z}^2$, a result conjectured by A. J. Berrick and J. Hillman in [1].

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## Introduction

Let $G$ be a finitely generated group and $\chi : G \to \mathbb{R}$ a group homomorphism, i.e. a character of $G$. For any non-zero character $\chi$ there is a Novikov ring $\widehat{\mathbb{Z}G}_\chi$ containing precisely those (in general infinite sums) $\lambda = \sum_{g \in G, z_g \in \mathbb{Z}} z_g g$ such that the intersection of the support of $\lambda$ in $G$ with the set $\chi^{-1}(-\infty, j]$ is finite for any choice of a natural number $j$. The Novikov ring has strong relation with the homological $\Sigma$-invariants. More precisely a non-zero character $\chi$ represents a class of the homological invariant $\Sigma^m(G, \mathbb{Z})$ (here $G$ is of homological type $\mathrm{FP}_m$) if and only if $\mathrm{Tor}_j^{\mathbb{Z}[G]}(\widehat{\mathbb{Z}G}_\chi, \mathbb{Z}) = 0$ for all $0 \le j \le m$ [7, Thm. B.4.6], a homotopical version can

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be found in [5, Thm. 14.4, 14.5]. Detailed proofs of [7, Thm. B.4.6] can be found in [4, Appendix]. In the case when $\chi$ is a discrete character the above criterion for $m = \infty$ was rediscovered and further generalised in [17, Thm. 2 and last paragraph of the introduction], where finite domination of generally non-acyclic free complexes is considered. More generalisations in this direction can be found in [11, Section 3].

The homological invariant $\Sigma^m(G, \mathbb{Z})$ defined in [6] for groups $G$ of type $\mathrm{FP}_m$ is important in determining the homological type of a subgroup $N$ of $G$ containing the commutator subgroup. By definition $\Sigma^m(G, \mathbb{Z})$ contains some classes $[\chi] = \mathbb{R}_{>0}\chi$ of characters $\chi \in \mathrm{Hom}(G, \mathbb{R}) \setminus \{0\}$ and by [6, Thm. B] $N$ is of homological type $\mathrm{FP}_m$ if and only if for every $\chi \in \mathrm{Hom}(G, \mathbb{R}) \setminus \{0\}$ with $\chi(N) = 0$ we have $[\chi] \in \Sigma^m(G, \mathbb{Z})$.

A ring $R$ with unity is said to be von Neumann finite if whenever for some $a, b \in R$ we have $ab = 1$ this implies that $ba = 1$, i.e., every left inverse is a right inverse. In [13] the terminology used was slightly different, a ring $R$ was said to have the Kaplansky property if for every natural number $n$ the matrix ring $M_n(R)$ is von Neumann finite. The group algebra $kG$ of any group $G$ and any field $k$ of characteristic zero is von Neumann finite [12, p. 122], [16, Ch. 2, Cor. 1.9], [15]. Using the theory of von Neumann dimension of Hilbert $G$-modules we show the same result holds for some Novikov rings.

**Theorem 1.** *Let $G$ be a finitely generated group, $\chi\colon G \to \mathbb{Z}$ a non-zero discrete character of $G$. Then every matrix ring $M_n(\widehat{\mathbb{Z}G}_\chi)$ is von Neumann finite.*

The problem whether a matrix ring over $\widehat{\mathbb{Z}G}_\chi$ is von Neumann finite was first studied in [13, Thm. 3], where the case when $N = ker(\chi)$ is residually finite was treated with techniques different from the ones used in the present paper. In this paper we treat two consequences of Theorem 1 ( see Corollary 1 and Corollary 2). New applications of Theorem 1 to Poincaré duality groups can be found in [11]. The following theorem is one of the main results of [13].

**Theorem 2** ([13, Thm. 1, Cor. 1]). *Let $G$ be a non-trivial discrete group of geometric dimension $n$ with a finite $K(G, 1)$ CW-complex $Y$ of dimension $n$ such that the Euler characteristics of $Y$ is zero. Suppose that $N$ is a normal subgroup of $G$ containing the commutator subgroup such that $N$ is of homological type* $\mathrm{FP}_{n-1}$ *and $N$ is residually finite. Then*

a) *$N$ is of homological type* $\mathrm{FP}_n$;
b) *$G/N$ has finite virtual cohomological dimension*

$$\mathrm{vcd}(G/N) = \mathrm{cd}(G) - \mathrm{cd}(N).$$

*In particular either $N$ has finite index in $G$ or $N$ has cohomological dimension at most* $\mathrm{cd}(G) - 1$.

In [13] Theorem 2 was stated for $G/N \simeq \mathbb{Z}$ but the proof there requires only that $G/N$ is abelian. In this paper we will use Theorem 2 for $G/N$ cyclic-by-finite.

Furthermore the proof of Theorem 2 works in a more general setting, it requires only that every matrix ring over a Novikov ring $\widehat{\mathbb{Z}G}_\chi$ is von Neumann finite for any non-zero character of $G$ (not necessarily discrete) such that $\chi(N) = 0$. The condition that $n$ is exactly the geometric dimension of $G$ is redundant, the proof of Theorem 2 requires only that the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ has a resolution of length $n$ of free modules $F_i$ of finite rank $m_i$ where the alternating sum $\sum_{0 \le i \le n} (-1)^i m_i$ is 0. These remarks together with Theorem 1 give the following result.

**Theorem 3.** *Let $G$ be a non-trivial discrete group with a finite $K(G,1)$ CW-complex $Y$ of dimension $n$ such that the Euler characteristics of $Y$ is zero. Suppose that $N$ is a normal subgroup of $G$ containing the commutator subgroup such that $N$ is of homological type* $\mathrm{FP}_{n-1}$ *and $G/N$ is cyclic-by-finite. Then*

a) *$N$ is of homological type* $\mathrm{FP}_n$;
b) *$G/N$ has finite virtual cohomological dimension*

$$\mathrm{vcd}(G/N) = \mathrm{cd}(G) - \mathrm{cd}(N).$$

*In particular either $N$ has finite index in $G$ or $N$ has cohomological dimension* $\mathrm{cd}(G) - 1$.

To be able to apply Theorem 1, we cannot omit the assumption that $G/N$ is cyclic-by-finite, since we need every character with $\chi(N) = 0$ to be either discrete or zero. It is important to note that the condition in Theorem 3 on the Euler characteristic cannot be removed as there is an example of a group $G$ of cohomological dimension 2 and type $\mathrm{FP}_\infty$ with a finitely generated normal subgroup $N$ such that $G/N \simeq \mathbb{Z}$ but $N$ is not free [3, Thm. B and Remark 5.4]. The deficiency of this group $G$ is not 1. More examples of groups $G$ of cohomological dimension $n$ and type $\mathrm{FP}_\infty$ with normal subgroups $N$ of homological type $\mathrm{FP}_{n-1}$ but not $\mathrm{FP}_n$ such that $G/N \simeq \mathbb{Z}$ can be found in [14].

**Corollary 1.** *Let $G$ be a non-trivial discrete group with a finite $K(G,1)$ CW-complex $Y$ of dimension $n$ such that the Euler characteristics of $Y$ is zero. Suppose that $N$ is a normal subgroup of $G$ containing the commutator subgroup such that $N$ is of homological type* $\mathrm{FP}_{n-1}$. *Then $G/N$ has finite virtual cohomological dimension*

$$\mathrm{vcd}(G/N) = \mathrm{cd}(G) - \mathrm{cd}(N).$$

*and $N$ is of homological type* $\mathrm{FP}_\infty$.

To see how the above corollary follows from Theorem 3 consider a subgroup $N_0$ of $G$ such that $N \subseteq N_0$ and $G/N_0 \simeq \mathbb{Z}$ (if such $N_0$ does not exist then $N$ has finite index in $G$ and there is nothing to prove). As $N_0/N$ is a finitely generated abelian group it is of homological type $\mathrm{FP}_\infty$, so $N$ being of homological type $\mathrm{FP}_{n-1}$ forces $N_0$ to be of homological type $\mathrm{FP}_{n-1}$ [2, Exer. p. 23]. Then by Theorem 3 $\mathrm{cd}(N_0) = \mathrm{cd}(G) - 1 \leq n - 1$ and $N_0$ is of homological type $\mathrm{FP}_n$. By [13, Prop. 3] applied for the normal subgroup $N$ of $N_0$ we get $\mathrm{cd}(N) = \mathrm{cd}(N_0) - \mathrm{vcd}(N_0/N)$, hence $\mathrm{cd}(N) = \mathrm{cd}(G) - 1 - \mathrm{vcd}(N_0/N) = \mathrm{cd}(G) - \mathrm{vcd}(G/N) \leq \mathrm{cd}(G) - 1 \leq n - 1$. The last property together with the fact that $N$ is of homological type $\mathrm{FP}_{n-1}$ implies that $N$ is of type $\mathrm{FP}_\infty$.

Theorem 3 is linked to a long lasting conjecture due to E. Rapaport Strasser that for a knot-like group $G$ if the commutator subgroup $G'$ is finitely generated then $G'$ should be free [19]. A discrete group is called a knot-like group if $G/G'$ is the infinite cyclic group and $G$ is finitely presented of deficiency 1. By [13, Cor. 2] the Rapaport conjecture holds when the commutator subgroup is residually finite. The Rapaport conjecture in its general form can be deduced as a corollary of Theorem 3. Indeed by [10, Thm. 2 and Lemma 2] a knot-like group $G$ with finitely generated commutator subgroup $G'$ has geometric dimension at most 2, hence the cohomological dimension $\mathrm{cd}(G)$ is at most 2. Without loss of generality we can assume that both the cohomological and geometric dimensions of $G$ are 2, otherwise $\mathrm{cd}(G) = 1$ and by the Stallings theorem $G$ is free [20]. Finally by Theorem 3 for $N = G'$ we conclude that $\mathrm{cd}(G') = 1$, using again Stallings' result $G'$ is free.

**Corollary 2.** *Let $G$ be a knot-like group with a finitely generated commutator subgroup $G'$. Then $G'$ is free, i.e. the Rapaport conjecture holds.*

By [1, Thm. 3.11] every finitely presentable group $G$ with positive deficiency and finitely generated perfect commutator subgroup has deficiency 1, geometric dimension at most 2 and the abelianisation of $G$ is $\mathbb{Z}$ or $\mathbb{Z}^2$. Furthermore if $G'$ is of homological type $\mathrm{FP}_2$ then $G$ is either $\mathbb{Z}$ or $\mathbb{Z}^2$. As pointed out to me by J. Hillman, the above results together with Theorem 3 imply the following corollary.

**Corollary 3.** *Let $G$ be a finitely presentable group with $\mathrm{def}(G) > 0$ and such that $G'$ is finitely generated and perfect. Then $G$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$.*

Indeed let $N$ be a subgroup of $G$ containing $G'$ and such that $G/N \simeq \mathbb{Z}$. As $G'$ is finitely generated $N$ is finitely generated and by Theorem 3 $\mathrm{cd}(N) \leq 1$, so $N$ is a free group, possibly trivial. Then the subgroup $G'$ of $N$ is free and perfect, so $G'$ is the trivial group.

## 1. Matrix rings over the Novikov ring

Let $G$ be a finitely generated group and $\chi\colon G \to \mathbb{Z}$ a non-zero character. In this section we discuss the ring $M_n(\widehat{\mathbb{Z}G}_\chi)$ assuming that it is not von Neumann finite.

Denote by $N$ the kernel of $\chi$. Then $G \simeq N \rtimes \langle t\rangle$, where $\chi(t) = 1$. The definition of the Novikov ring $\widehat{\mathbb{Z}G}_\chi$ was given in the introduction for the coefficient ring $\mathbb{Z}$. In general we can define $\widehat{RG}_\chi$ for any ring with unity $R$ as a particular completion of the group ring $R[G]$ (here $G$ commutes with the elements of $R$) i.e. $\widehat{RG}_\chi$ contains precisely those (in general infinite sums) $\lambda = \sum_{g\in G, r_g\in R} r_g g$ such that the intersection of the support of $\lambda$ in $G$ with the set $\chi^{-1}(-\infty, j]$ is finite for any choice of a natural number $j$. The Novikov ring we consider here is a special case of a more general definition [18, Def. 5.1 (ii)]. Using the above definition of the Novikov ring as a completion of the group ring $R[G]$ for $R = M_n(\mathbb{Z})$ we get a natural isomorphism of rings with unity

$$M_n(\widehat{\mathbb{Z}G}_\chi) \simeq \widehat{M_n(\mathbb{Z})G}_\chi.$$

Let $\alpha, \beta$ be elements of $\widehat{M_n(\mathbb{Z})G}_\chi$ such that $\alpha\beta = 1$ but

$$\beta\alpha \neq 1.$$

Then $0 \neq 1-\beta\alpha = \delta = \sum t^i\delta_i$ (note the sum can be infinite) and $\delta_i \in M_n(\mathbb{Z}[N]) \simeq M_n(\mathbb{Z})[N]$ are not all zero. Let $i_0$ be the smallest integer such that

$$\delta_{i_0} \neq 0.$$

By the definition of the Novikov ring we have

$$\alpha = \sum_{j\geq a} \alpha_j t^j \text{ and } \beta = \sum_{j\geq b} \beta_j t^j,$$

where the sums are in general infinite, $\alpha_j, \beta_j \in M_n(\mathbb{Z}[N])$ and $a, b$ are natural numbers such that $\alpha_a \neq 0, \beta_b \neq 0$. Substituting $\alpha$ with $t^{-a}\alpha$ and $\beta$ by $\beta t^a$ we can assume that $a = 0$. As $1 = \alpha\beta = \big(\sum_{j\geq 0}\alpha_j t^j\big)\big(\sum_{j\geq b}\beta_j t^j\big) = \sum_{j\geq b}\big(\sum_{0\leq i\leq j-b}\alpha_i\beta_{j-i}\big)t^j$ the coefficient $\sum_{0\leq i\leq -b}\alpha_i\beta_{-i}$ is 1 and the index set $\{i \mid 0 \leq i \leq -b\}$ is not empty, hence $b \leq 0$. We define

$$k = -b \geq 0, \gamma = \beta t^k = \sum_{j\geq 0}\gamma_j t^j,$$

where $\gamma_j \in M_n(\mathbb{Z}[N])$ and so

$$\alpha\gamma = \alpha\beta t^k = t^k.$$

From now on let $d$ be a natural number bigger than or equal to $k + i_0 + c$, where $c = \max\{k, -i_0\} \geq 0$. Define

$$V^{(d)} = \bigoplus_{0 \leq i \leq d} t^i M_n(S),$$

where $S = \mathbb{Z}[N]$. We view $V^{(d)}$ as a free right $M_n(S)$-module and define several endomorphisms of $V^{(d)}$ as such a module:

$$T\Big(\sum_{0 \leq i \leq d} t^i \mu_i\Big) = \sum_{0 \leq i \leq d-1} t^{i+1} \mu_i,$$

$$T^*\Big(\sum_{0 \leq i \leq d} t^i \mu_i\Big) = \sum_{1 \leq i \leq d} t^{i-1} \mu_i,$$

where all $\mu_i \in M_n(S)$. For $\lambda \in M_n(S)$ we define the endomorphism $\theta(\lambda)$ on $V^{(d)}$ by

$$\theta(\lambda)\Big(\sum_{0 \leq i \leq d} t^i \mu_i\Big) = \sum_{0 \leq i \leq d} t^i (\lambda^{t^i}) \mu_i,$$

where $\lambda^{t^i} \in M_n(S) \simeq M_n(\mathbb{Z})[N]$ is the result of the conjugation (on the right) of the elements of $N$ with $t^i$. Note that for $\lambda \in M_n(S)$

$$T^* \theta(\lambda) = \theta(\lambda^t) T^* \text{ and } T\theta(\lambda) = \theta(\lambda^{t^{-1}}) T.$$

We think of $V^{(d)}$ as a subset of

$$\bigoplus_{i \in \mathbb{Z}} t^i M_n(S) = M_n(S) \rtimes \langle t \rangle \simeq M_n(\mathbb{Z})[N \rtimes \langle t \rangle] \simeq M_n(\mathbb{Z}[G])$$

and define

$$\pi^{(d)} \colon M_n(S) \rtimes \langle t \rangle \simeq \bigoplus_{i \in \mathbb{Z}} t^i M_n(S) \to V^{(d)}$$

to be the projection that is identity on $V^{(d)}$ and sends

$$\big(\bigoplus_{i<0} t^i M_n(S)\big) \oplus \big(\bigoplus_{i>d} t^i M_n(S)\big)$$

to zero. Then $\theta(\lambda)$ can be viewed as the composition $\pi^{(d)} s_\lambda i^{(d)}$, where $i^{(d)}$ is the embedding of $V^{(d)}$ in $M_n(S) \rtimes \langle t \rangle$ and $s_\lambda$ is the left multiplication with $\lambda$ in the ring $M_n(S) \rtimes \langle t \rangle$.

Now we define two endomorphisms of the free right $M_n(S)$-module $V^{(d)}$

$$\alpha^{(d)} = \sum_{0 \leq j \leq d} \theta(\alpha_j) T^j, \; \beta^{(d)} = \Big(\sum_{0 \leq j \leq d} \theta(\gamma_j) T^j\Big)(T^*)^k.$$

**Lemma 1.** *The restriction of the composition $\alpha^{(d)}\beta^{(d)}$ to $\bigoplus_{0\le i\le k-1} t^i M_n(S)$ is the zero map and the restriction of $\alpha^{(d)}\beta^{(d)}$ to $\bigoplus_{k\le i\le d} t^i M_n(S)$ is the identity map.*

*Proof.* First note that $\bigoplus_{0\le i\le k-1} t^i M_n(S)$ is the kernel of $(T^*)^k$, so

$$\alpha^{(d)}\beta^{(d)}\big(\textstyle\bigoplus_{0\le i\le k-1} t^i M_n(S)\big) = 0.$$

By the way the operators $T$, $T^*$ and $\theta(M_n(S))$ are defined we have that the operator

$$\Big(\sum_{0\le j\le d}\theta(\alpha_j)T^j\Big)\Big(\sum_{0\le j\le d}\theta(\gamma_j)T^j\Big)$$

acts on $V^{(d)}$ as the composition of the embedding $i^{(d)}$ followed by multiplication on the left in $M_n(S)\rtimes\langle t\rangle$ with

$$\Big(\sum_{0\le j\le d}\alpha_j t^j\Big)\Big(\sum_{0\le j\le d}\gamma_j t^j\Big) = t^k$$

and then applying the projection $\pi^{(d)}$ i.e. dropping out the factors that have an exponent of $t$ not in the range $0\le j\le d$. Thus the composition $\alpha^{(d)}\beta^{(d)}$ is $T^k(T^*)^k$ and the restriction of $T^k(T^*)^k$ on $\bigoplus_{k\le i\le d} t^i M_n(S)$ is the identity map. □

Note that $N$ is a normal subgroup of $G$, hence for $S=\mathbb{Z}[N]$ we have $t^{-i}St^i = S$ for every $i\in\mathbb{Z}$. For every matrix $(a_{jk})\in M_n(S)$ we define $t^{-i}(a_{jk})t^i = (t^{-i}a_{jk}t^i)$.

We remind the reader that $d\ge k+c+i_0$ and $c=\max\{k,-i_0\}\ge 0$. The following technical result will be used in the proof of Lemma 4.

**Lemma 2.** *For $c\le s\le d-k-i_0$*

$$\big(I^{(d)}-\beta^{(d)}\alpha^{(d)}\big)(t^s)\in t^{i_0+s}\delta_{i_0}^{t^s} + \textstyle\bigoplus_{i_0+s+1\le i\le d} t^i M_n(S)$$

*where $I^{(d)}$ is the identity operator of $V^{(d)}$ and $\delta_{i_0}^{t^s} = t^{-s}\delta_{i_0}t^s$ is the conjugate of $\delta_{i_0}$ by $t^s$ in $M_n(S)$ defined above.*

*Proof.* Note that

$$\begin{aligned}\beta^{(d)}\alpha^{(d)} &= \Big(\sum_{0\le j\le d}\theta(\gamma_j)T^j\Big)(T^*)^k\Big(\sum_{0\le j\le d}\theta(\alpha_j)T^j\Big)\\ &= \Big(\sum_{0\le j\le d}\theta(\gamma_j)T^j\Big)\Big(\sum_{0\le j\le d}\theta(\alpha_j^{t^k})(T^*)^k T^j\Big).\end{aligned}$$

Then using that

$$T^j(t^s)=0 \quad \text{for } d-s+1\le j$$

and that

$$(T^*)^k T^j(t^s) = T^j (T^*)^k(t^s) \text{ for } k \le s \le d-j$$

we get

$$\begin{aligned}
&\big(I^{(d)} - \beta^{(d)}\alpha^{(d)}\big)(t^s)\\
&\quad = \Big(I^{(d)} - \Big(\sum_{0\le j\le d}\theta(\gamma_j)T^j\Big)\Big(\sum_{0\le j\le d}\theta(\alpha_j^{t^k})(T^*)^k T^j)\Big)\Big)(t^s)\\
&\quad = \Big(I^{(d)} - \Big(\sum_{0\le j\le d}\theta(\gamma_j)T^j\Big)\Big(\sum_{0\le j\le d-s}\theta(\alpha_j^{t^k})(T^*)^k T^j\Big)\Big)(t^s)\\
&\qquad\quad - \Big(\sum_{0\le j\le d}\theta(\gamma_j)T^j\Big)\Big(\sum_{d-s+1\le j\le d}\theta(\alpha_j^{t^k})(T^*)^k T^j\Big)(t^s)\\
&\quad = \Big(I^{(d)} - \Big(\sum_{0\le j\le d}\theta(\gamma_j)T^j\Big)\Big(\sum_{0\le j\le d-s}\theta(\alpha_j^{t^k})(T^*)^k T^j\Big)\Big)(t^s)\\
&\quad = \Big(I^{(d)} - \Big(\sum_{0\le j\le d}\theta(\gamma_j)T^j\Big)\Big(\sum_{0\le j\le d-s}\theta(\alpha_j^{t^k})T^j (T^*)^k\Big)\Big)(t^s). \qquad (*)
\end{aligned}$$

Now we define an element $\delta^{(d)} \in M_n(\mathbb{Z})[N \rtimes \langle t\rangle] \simeq M_n(\mathbb{Z}[G])$ by

$$\begin{aligned}
\delta^{(d)} &= 1 - \Big(\sum_{0\le j\le d}\gamma_j t^j\Big)t^{-k}\Big(\sum_{0\le j\le d}\alpha_j t^j\Big)\\
&= 1 - \Big(\sum_{0\le j\le d}\gamma_j t^j\Big)\Big(\sum_{0\le j\le d}(\alpha_j^{t^k})t^j\Big)t^{-k}\\
&\in t^{i_0}\delta_{i_0} + \big(\textstyle\bigoplus_{j\ge i_0+1} t^j M_n(S)\big),
\end{aligned}$$

the last inclusion follows from the fact that $d-k \ge i_0$. As $\delta_{i_0} \ne 0$ follows that $\delta^{(d)} \ne 0$. Now let $\mu$ be the endomorphism of $V^{(d)}$ defined as the composition of $i^{(d)}$ with the left multiplication with $\delta^{(d)}$ in $M_n(\mathbb{Z})[N \rtimes \langle t\rangle]$ and then applying the projection $\pi^{(d)}$. By $(*)$

$$\begin{aligned}
&\mu(t^s) - \big(I^{(d)} - \beta^{(d)}\alpha^{(d)}\big)(t^s)\\
&\quad = \mu(t^s) - \Big(I^{(d)} - \Big(\sum_{0\le j\le d}\theta(\gamma_j)T^j\Big)\Big(\sum_{0\le j\le d-s}\theta(\alpha_j^{t^k})T^j(T^*)^k\Big)\Big)(t^s)\\
&\quad = -\Big(\sum_{0\le j\le d}\theta(\gamma_j)T^j\Big)\Big(\sum_{d-s+1\le j\le d}\theta(\alpha_j^{t^k})T^j(T^*)^k\Big)(t^s)\\
&\quad \in \textstyle\bigoplus_{s-k+(d-s+1)=d-k+1\le i\le d} t^i M_n(S)\\
&\quad \subseteq \textstyle\bigoplus_{i_0+s+1\le i\le d} t^i M_n(S).
\end{aligned}$$

The last inclusion comes from the fact that $s \le d-k-i_0$. Finally, as $0 \le i_0+s \le d$,

$$\mu(t^s) \in t^{i_0+s}\delta_{i_0}^{t^s} + \bigoplus_{i_0+s+1\le i\le d} t^i M_n(S). \qquad \square$$

**Lemma 3.** *$e = \beta^{(d)}\alpha^{(d)}$ is idempotent, i.e. $e^2 = e$.*

*Proof.* As proved in Lemma 1, $\alpha^{(d)}\beta^{(d)}$ is an idempotent of a special type. Let $v \in V^{(d)}$, $\alpha^{(d)}(v) = w_1 + w_2$, where $w_1 \in \bigoplus_{0\le i\le k-1} t^i M_n(S)$, $w_2 \in \bigoplus_{k\le i\le d} t^i M_n(S)$. Then $\alpha^{(d)}\beta^{(d)}(w_1+w_2) = w_2$ and

$$e^2(v) = \beta^{(d)}\alpha^{(d)}\beta^{(d)}\alpha^{(d)}(v) = \beta^{(d)}\alpha^{(d)}\beta^{(d)}(w_1+w_2) = \beta^{(d)}(w_2).$$

Since $(T^*)^k(w_1) = 0$ we get $\beta^{(d)}(w_1) = \big(\sum_{0\le j\le d}\theta(\gamma_j)T^j\big)(T^*)^k(w_1) = 0$ and

$$e(v) = \beta^{(d)}\alpha^{(d)}(v) = \beta^{(d)}(w_1+w_2) = \beta^{(d)}(w_1) + \beta^{(d)}(w_2) = \beta^{(d)}(w_2). \qquad \square$$

## 2. Proof of Theorem 1

We define $A_0^{(d)}$ and $B_0^{(d)}$ to be the matrices in $M_{d+1}(M_n(S)) \simeq M_{n(d+1)}(S)$ that represent the operators $\alpha^{(d)}$ and $\beta^{(d)}$ respectively. For example if $A_0^{(d)} = (a_{j,i})$, $a_{j,i} \in M_n(S)$ we have $\alpha^{(d)}(t^i) = \sum_j t^j a_{j,i}$. We remind the reader that $S = \mathbb{Z}[N]$.

By Lemma 3 $B_0^{(d)}A_0^{(d)}$ is an idempotent matrix, hence $I_{n(d+1)} - B_0^{(d)}A_0^{(d)} \in M_{n(d+1)}(\mathbb{Z}[N])$ is an idempotent matrix, so its columns generate a projective right submodule $P$ of $\mathbb{R}[N]^{n(d+1)}$, where $I_{n(d+1)}$ is the identity $n(d+1)\times n(d+1)$-matrix. We think of the elements of $\mathbb{R}[N]^{n(d+1)}$ as columns of length $n(d+1)$ and entries in $\mathbb{R}[N]$. By definition $l_2(N)$ is the Hilbert space with orthonormal basis $N$ i.e. square norm summable functions on $N$ with coefficients in $\mathbb{R}$. Then $P \otimes_{\mathbb{R}[N]} l_2(N)$ is a Hilbert $N$-submodule of $l_2(N)^{n(d+1)}$ via the multiplication of $N$ on $l_2(N)$ on the right.

**Lemma 4.** *The von Neumann dimension* $\dim_N(P \otimes_{\mathbb{R}[N]} l_2(N))$ *is $kn$.*

*Proof.* Let $Q$ be the projective right $\mathbb{R}[N]$-submodule in $\mathbb{R}[N]^{n(d+1)}$ generated by the columns of $B_0^{(d)}A_0^{(d)}$. Then $P \oplus Q = \mathbb{R}[N]^{n(d+1)}$,

$$(P \otimes_{\mathbb{R}[N]} l_2(N)) \oplus (Q \otimes_{\mathbb{R}[N]} l_2(N)) = l_2(N)^{n(d+1)}$$

and

$$\dim_N(P \otimes_{\mathbb{R}[N]} l_2(N)) + \dim_N(Q \otimes_{\mathbb{R}[N]} l_2(N)) = n(d+1).$$

Note that the matrix $B_0^{(d)}A_0^{(d)}$ defining $Q$ is an idempotent, that in general is not self-adjoint, still its von Neumann trace (the sum of Kaplansky traces of all diagonal elements) is exactly the von Neumann dimension of $Q \otimes_{\mathbb{R}[N]} l_2(N)$ (see [8, Sect. 2] for the case of matrices of size 1, the general case is exactly the same), i.e.

$$\operatorname{trace}\left(B_0^{(d)} A_0^{(d)}\right) = \dim_N(Q \otimes_{\mathbb{R}[N]} l_2(N)),$$

and similarly

$$\operatorname{trace}\left(I_{n(d+1)} - B_0^{(d)} A_0^{(d)}\right) = \dim_N(P \otimes_{\mathbb{R}[N]} l_2(N)).$$

By [9, Cor. 3.1.4]

$$\operatorname{trace}\left(B_0^{(d)} A_0^{(d)}\right) = \operatorname{trace}\left(A_0^{(d)} B_0^{(d)}\right)$$

and by Lemma 1

$$\operatorname{trace}\left(A_0^{(d)} B_0^{(d)}\right) = n(d+1-k).$$

Hence

$$\begin{aligned}
\dim_N(P \otimes_{\mathbb{R}[N]} l_2(N)) &= n(d+1) - \dim_N(Q \otimes_{\mathbb{R}[N]} l_2(N)) \\
&= n(d+1) - \operatorname{trace}\left(B_0^{(d)} A_0^{(d)}\right) \\
&= n(d+1) - \operatorname{trace}\left(A_0^{(d)} B_0^{(d)}\right) \\
&= n(d+1) - n(d+1-k) \\
&= nk. \qquad \square
\end{aligned}$$

Let $P_0$ be the submodule of $P$ generated by the columns

$$\rho_{cn+1}, \ \rho_{cn+2}, \ldots, \ \rho_{n(d+1-k-i_0)}$$

of the matrix $I_{n(d+1)} - B_0^{(d)} A_0^{(d)} \in M_{(d+1)n}(\mathbb{Z}[N])$, here $\rho_i$ is the $i$th column. Let $P_0 \otimes_{\mathbb{R}[N]} l_2(N) \to P \otimes_{\mathbb{R}[N]} l_2(N)$ be the map induced by the embedding of $P_0$ in $P$ and $W^{(d)}$ be the closure of the image of this map. Then $W^{(d)}$ is a Hilbert $N$-submodule of $l_2(N)^{n(d+1)}$ via the right $N$ action on $l_2(N)$.

For $c = \max\{k, -i_0\} \le j \le d-k-i_0$ let $e_j$ be the matrix of size $((d+1)n) \times n$ with columns $\rho_{jn+1}, \rho_{jn+2}, \ldots, \rho_{(j+1)n}$. Define square matrices $(e_j)_i \in M_n(\mathbb{Z}[N])$ for $1 \le i \le d$, where $(e_j)_i$ has as consecutive rows the $(1+in)$-th,…, $(i+1)n)$-th rows of $e_j$. By Lemma 2,

$$\text{if } c \le j \le d-k-i_0, \ i < j+i_0, \text{ then the matrix } (e_j)_i \text{ is zero;}$$

$$(e_c)_{c+i_0} = \delta_{i_0}^{t^c} \in M_n(\mathbb{Z}[N]);$$

$$\text{if } c+1 \le j \le d-k-i_0, \text{ then we have } (e_j)_{j+i_0} = (e_{j-1})^t_{j-1+i_0} = \delta_{i_0}^{t^j}, \qquad (**)$$

where upper index $t$ is conjugation on the right side with the element $t$.

For $c \leq j \leq d-k-i_0$ let $W_j^{(d)}$ be the closure of the image of

$$(P_0)_j \otimes_{\mathbb{R}[N]} l_2(N) \to P \otimes_{\mathbb{R}[N]} l_2(N) \subseteq l_2(N)^{n(d+1)},$$

where $(P_0)_j$ is the submodule of $P_0$ generated by all the columns in the matrices $e_j, \dots, e_{d-k-i_0}$.

For $i \geq 0$ define $V_i^{(d)}$ as the Hilbert $N$-submodule of $l_2(N)^{n(d+1)}$ consisting of the elements with 0 coordinates in the first $in$ positions. Then by $(**)$ for $c \leq j \leq d-k-i_0$

$$W_j^{(d)} \subseteq V_{j+i_0}^{(d)} \text{ and } W_j^{(d)} \not\subseteq V_{j+i_0+1}^{(d)}.$$

Then the composition map

$$\varphi_j^{(d)}\colon W_j^{(d)} \to V_{j+i_0}^{(d)} \to V_{j+i_0}^{(d)}/V_{j+i_0+1}^{(d)} \simeq l_2(N)^n$$

of the inclusion and the canonical projection map has $W_{j+1}^{(d)}$ in its kernel. Then the closure of the image of $\varphi_j^{(d)}$ is a quotient of $W_j^{(d)}/W_{j+1}^{(d)}$ . By $(**)$ the closure of the image of $\varphi_j^{(d)}$ is some $t$-power conjugate of the closure $\overline{Y}$ of the image $Y$ of $X \otimes_{\mathbb{R}[N]} l_2(N)$ in $\mathbb{R}[N]^n \otimes_{\mathbb{R}[N]} l_2(N) \simeq l_2(N)^n$, where $X$ is the right $\mathbb{R}[N]$-submodule of $\mathbb{R}[N]^n$ generated by the columns of $\delta_{i_0}^{t^c} \in M_n(\mathbb{R}[N])$.

**Lemma 5.** *Let $H$ be a countable group, $M$ a Hilbert $H$-submodule of $l_2(H)^m$ for some natural number $m$, $\epsilon$ an automorphism of the Hilbert space $l_2(H)$ that extends a group automorphism of $H$, and let*

$$\nu\colon l_2(H)^m \to l_2(H)^m$$

*be the isometry whose restriction on every coordinate is $\epsilon$. Then the von Neumann dimensions $\dim_H(M)$ and $\dim_H(\nu(M))$ are equal.*

*Proof.* Let $\pi_M$ and $\pi_{\nu(M)}$ be the orthogonal projections of $l_2(H)^m$ to $M$ and $\nu(M)$ respectively. Then $\pi_M$ and $\pi_{\nu(M)}$ can be thought as elements of $M_m(N(H))$ where $N(H)$ is the von Neumann algebra of $H$. By definition $\dim_H(M) = \text{trace}(\pi_M)$ and $\dim_H(\nu(M)) = \text{trace}(\pi_{\nu(M)})$. Let $x_i$ be the element of $l_2(H)^m$ with just one non-zero entry 1 on the $i$th place. Then

$$\nu(x_i) = x_i,\, \pi_{\nu(M)}(x_i) = \nu(\pi_M(x_i)).$$

Using again that $\nu$ is an isometry it follows that

$$\begin{aligned}\text{trace}(\pi_{\nu(M)}) &= \sum_{1\leq i\leq m} \langle \pi_{\nu(M)}(x_i), x_i\rangle = \sum_{1\leq i\leq m} \langle \nu(\pi_M(x_i)), x_i\rangle \\ &= \sum_{1\leq i\leq m} \langle \nu(\pi_M(x_i)), \nu(x_i)\rangle = \sum_{1\leq i\leq m} \langle \pi_M(x_i), x_i\rangle = \text{trace}(\pi_M).\end{aligned}$$

□

As noted before, $W_j^{(d)}/W_{j+1}^{(d)}$ is a non-trivial Hilbert $N$-module that has a quotient obtained from $\bar{Y}$ by applying $\nu$ from Lemma 5 for $\epsilon$ a conjugation by some power of $t$ and $H = N$. Then by Lemma 5 the von Neumann dimension of $W_j^{(d)}/W_{j+1}^{(d)}$ is at least the von Neumann dimension of $\bar{Y}$. As the von Neumann dimension of Hilbert $N$-modules is additive [9, p. 203] we deduce that

$$\dim_N \left(W^{(d)}\right) = \sum_{j=c}^{d-k-i_0} \dim_N \left(W_j^{(d)}/W_{j+1}^{(d)}\right) \geq (d+1-k-c-i_0) \dim_N(\bar{Y}).$$

As the von Neumann dimension preserves inclusion and by Lemma 4

$$kn = \dim_N(P \otimes_{\mathbb{R}[N]} l_2(N)) \geq \dim_N \left(W^{(d)}\right) \geq (d+1-k-c-i_0) \dim_N(\bar{Y}) > 0.$$

Then

$$kn/(d+1-k-c-i_0) \geq \dim_N(\bar{Y})$$

a contradiction as $\dim_N(\bar{Y})$ is a fixed positive real number, $k, n, i_0, c$ are fixed numbers, $d \geq k+c+i_0$ and $d$ can be arbitrary large.

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